# INVARIANT SUBSPACE PROBLEM AND SPECTRAL PROPERTIES OF BOUNDED LINEAR OPERATORS ON BANACH SPACES, BANACH LATTICES, AND TOPOLOGICAL VECTOR SPACES

BY

VLADIMIR G. TROITSKY

Magister Diploma with Honors in Mathematics, Novosibirsk State University, 1993

THESIS

Submitted in partial fulfillment of the requirements
for the degree of Doctor of Philosophy in Mathematics
in the Graduate College of the
University of Illinois at Urbana-Champaign, 1999

Urbana, Illinois

# Abstract

Chapter 1 deals with the Invariant Subspace Problem for Banach spaces and Banach lattices. First, we show that the celebrated Lomonosov theorem [Lom73] cannot be improved by increasing the number of commuting operators. Specifically, we prove that if $T\colon \ell_1 \to \ell_1$ is the operator without a non-trivial closed invariant subspace constructed by C. J. Read in [Read85], then there are three operators $S_1$, $S_2$ and $K$ (non-multiples of the identity) such that $T$ commutes with $S_1$, $S_1$ commutes with $S_2$, $S_2$ commutes with $K$, and $K$ is compact. We also show that the commutant of $T$ contains only series of $T$. Further, we show that the modulus of the quasinilpotent operator without an invariant subspace constructed by C. J. Read in [Read97] has an invariant subspace (and even an eigenvector). This answers a question posed by Y. Abramovich, C. Aliprantis and O. Burkinshaw in [AAB93, AAB98].

In Chapter 2 we develop a version of spectral theory for bounded linear operators on topological vector spaces. We show that the Gelfand formula for spectral radius and Neumann series can still be naturally interpreted for operators on topological vector spaces. Of course, the resulting theory has many similarities to the conventional spectral theory of bounded operators on Banach spaces, though there are several important differences. The main difference is that an operator on a topological vector space has several spectra and several spectral radii, which fit a well-organized pattern.

In Chapter 3 we use the results of Chapter 2 to prove locally-convex versions of some results on the Invariant Subspace Problem on Banach lattices obtained in [AAB93, AAB94, AAB98]. For example, we show that if $S$ and $T$ are two commuting positive continuous operators with finite spectral radii on a locally convex-solid vector lattice, $T$ is

locally quasinilpotent at a positive vector, and $S$ dominates a positive compact operator, then $S$ and $T$ have a common closed non-trivial invariant subspace.

To the memory of G. V. Troitsky.

# Acknowledgments

I would like to thank Professor Yuri Abramovich for introducing me to most of the problems I deal with in this paper and for his numerous suggestions and improvements. Our many hours of discussions have been most valuable for my understanding of mathematics and, in particular, for my work on this paper.

Many thanks are due to Professor Peter Loeb for his constant support and encouragement during my work. I also thank Professor Roko Aliprantis for his optimism and criticism and Professor Robert Kaufman for reading parts of the draft and asking some interesting questions, which resulted in new developments.

I thank Professors Victor Lomonosov, Heinrich Lotz, Michael Neumann, Joseph Rosenblatt, and Anton Schep for their support and interest in my work.

David Boyd and Kristine Baxter proofread parts of the draft and helped me in fixing some English irregularities. Dima Barsky and Igor Belyi helped me in installing and maintaining Linux, which has provided me with a perfect TEX typesetting environment.

Finally, I would like to thank my wife Svetlana for being very supportive and patient.

# Preface

The Invariant Subspace Problem is one of the famous unsolved problems in modern mathematics. Originally it was posed as follows: *Does every bounded linear operator have a (closed non-trivial) invariant subspace?* The problem was motivated by the Jordan decomposition of matrices and by the desire to understand the structure and the geometry of an arbitrary operator. The problem was solved in the affirmative for compact operators on Banach spaces by N. S. Aronszajn and K. T. Smith [AS54]. V. I. Lomonosov showed in [Lom73] that if $T$ is a bounded operator on a Banach space such that there is a "chain" of three consecutively commuting operators from $T$ to a non-zero compact operator $K$, i.e., if there exists an operator $S$ (not a multiple of the identity) such that $T$ commutes with $S$ and $S$ commutes with $K$, then $T$ has an invariant subspace. Lomonosov's theorem was a breakthrough because it covered a very large class of operators. It did not, however, cover all of them.

In the mid-seventies P. Enflo [Enf76] constructed the first counterexample of a continuous operator on a Banach space with no closed non-trivial invariant subspaces, thus answering the Invariant Subspace Problem for general Banach spaces in the negative. Later, C.J. Read produced several classes of operators without invariant subspaces. In [Read84] he presented his original example of an operator (in fact, a class of operators) on a Banach space without invariant subspaces. A year later Read published a follow-up [Read85], showing that his example can be slightly modified so that the operators would act on $\ell_1$. In [Read86] he presented a considerably simplified version of his example. Finally, in [Read97] he constructed a class of quasinilpotent operators on $\ell_1$ without invariant subspaces. Still, for many particular classes of spaces and/or operators there is a hope to solve the problem in the affirmative. The Invariant Subspace Problem is still open for operators on Hilbert spaces. Another large class of operators for which there is a hope of solving the Invariant Subspace Problem in the affirmative is the class of positive operators on Banach lattices. M.G. Krein proved in [KR48] that every positive operator on a space of continuous functions on a compact Hausdorff space has an invariant subspace. During the last several years considerable progress in the Invariant Subspace Problem for positive operators on Banach lattices has been made in the series of papers [AAB93, AAB94, AAB98], in which the existence of an invariant subspace for a positive operator was proved under various additional hypotheses. In particular, it is shown in [AAB93] that on a discrete Banach space every positive operator which commutes with a non-zero quasinilpotent operator has an invariant subspace.

There was a hope to discover more operators with invariant subspaces by increasing the length of a chain leading from a given operator to a compact operator in Lomonosov's theorem. This question was asked by Y. Abramovich and C. Aliprantis. In Section 1.1 of this thesis we show that Lomonosov's theorem cannot be extended even to chains of four operators. Namely, we present four operators $T$, $S_1$, $S_2$, and $K$ such that $T$ commutes with $S_1$, $S_1$ commutes with $S_2$, $S_2$ commutes with $K$, and $K$ is compact, but, nevertheless, $T$ has no invariant subspaces. In fact, $T$ is a Read operator here. Further, in Section 1.2 we show that the only operators that commute with a Read operator are the series of the operator.

The following question was posed in [AAB93, AAB98]: *Let $T$ be the operator without invariant subspaces, constructed in [Read85]. Does the modulus of this operator have invariant subspaces?* This question is important for the following reason. The operator $T$ is defined on $\ell_1$, which is a Banach lattice. Even though $T$ itself is not positive, it must have a modulus $|T|$ (since every operator on $\ell_1$ has a modulus), which is a positive operator. In fact, we show in Section 1.3 that Read's operator $T$ is "almost positive" in the sense that it differs from $|T|$ by a small nuclear perturbation. It seemed therefore quite plausible that $|T|$ would also have no invariant subspaces, and this would answer negatively the Invariant Subspace Problem for positive operators. Even though the operators $T$ and $|T|$ are "very close", the technique of [Read85] is not applicable to $|T|$, and the question of the existence of invariant subspaces of $|T|$ was not easy. We solve this problem in Section 1.3 using an essential spectrum technique. Namely, we prove the existence of invariant subspaces for a certain class of operators, and show that $|T|$ belongs to this class. Moreover, we show that $|T|$ has a positive eigenvector.

Another direction of the author's research is the study of the Invariant Subspace Problem in locally convex spaces. In Chapter 3 we generalized some of the results of [AAB98] to operators on locally convex spaces. It is worth mentioning, however, that even a direct extension of known results from Banach spaces to topological vector spaces is not always trivial. One major difficulty is that it is not clear which class of operators should be considered, because there are several non-equivalent ways of defining bounded operators on topological vector spaces. Another major difficulty is the lack of a readily available developed spectral theory. The spectral theory of operators on Banach spaces has been thoroughly studied for a long time, and is extensively used. Unfortunately, little has been known about spectral theory of bounded operators on general topological vector spaces, and many techniques used in Banach spaces cannot be applied for operators on topological vector spaces. In particular, the spectrum, the spectral radius, and the Neumann series are the tools which are widely used in the study of the Invariant Subspace Problem in Banach spaces, but which have not been sufficiently studied for general topological vector spaces. To overcome this obstacle we have developed a version of the spectral theory of bounded operators on general topological vector spaces and on locally convex spaces. This material is presented in Chapter 2. Some results in this direction have also been obtained by B. Gramsch [Gram66], and by F. Garibay and R. Vera [GV97, GV98, VM97].

In particular, we consider the following classification of bounded operators on a topological vector space. We call a linear operator $T$

- ***nb-bounded*** if $T$ maps some neighborhood of zero into a bounded set,
- ***nn-bounded*** if there is a base of neighborhoods of zero such that $T$ maps every neighborhood in this base into a multiple of itself, and

- ***bb-bounded*** if $T$ maps bounded sets into bounded sets.

The classes of all linear operators, of all bb-bounded operators, of all continuous operators, of all nn-bounded operators, and of all nb-bounded operators form nested algebras. The spectrum of an operator $T$ in each of these algebras is defined as usual, i.e., the set of $\lambda$'s for which $\lambda I - T$ is not invertible in this algebra. We show that the well known Gelfand formula for the spectral radius of an operator on a Banach space, $r(T) = \lim\limits_{n\to\infty} \sqrt[n]{\|T^n\|}$ can be generalized to each of the five classes of operators on topological vector spaces, and then we use this formula to define the spectral radius of an operator in each of the classes. Then in Section 2.4 we show that if $T$ is a continuous operator on a sequentially complete locally convex space and $|\lambda|$ is greater than the spectral radius of $T$ in any of the five classes, then the Neumann series $\sum_{n=0}^{\infty} \frac{T^n}{\lambda^{n+1}}$ converges in the topology of the class, and $\lambda$ does not belong to the corresponding spectrum of $T$, i.e., the spectral radius is greater than or equal to the geometrical radius of the spectrum. In Sections 2.5 and 2.6 we show that the radii are equal for nb-bounded and compact operators.

Once we have this machinery, we use it in Chapter 3 to deal with the Invariant Subspace Problem in locally convex spaces. In particular, we prove locally convex-solid versions of some theorems in [AAB98]. For example, we prove in Theorem 3.4.4 that if $S$ and $T$ are two linear operators on a locally convex-solid vector lattice such that they are either nn-bounded or continuous with finite spectral radii, $T$ is locally quasinilpotent at a non-zero vector, and $S$ dominates a positive compact operator, then $S$ and $T$ have a common closed non-trivial invariant ideal.

# Table of Contents

# List of abbreviations

| | | |
|---|---|---|
| $E_+$ | Positive cone | 3 |
| $E_u$ | Principal ideal | 3 |
| $\|T\|_{\mathrm{ess}}$ | Essential norm | 5 |
| $L(X,Y)$, $L(X)$ | Bounded operators | 4 |
| $\mathfrak{m}_{pq}$ | Mixed operator seminorm | 26 |
| $\mathcal{N}_0$ | Base of zero neighborhoods | 1 |
| $N_T$ | Null ideal | 61 |
| $\nu(T)$ | Nuclear norm | 4 |
| $\mathcal{Q}_T$ | Points of local quasinilpotence | 5, 61 |
| $r(T)$ | Spectral radius | 5, 52 |
| $r_l$, $r_{bb}$, $r_c$, $r_{nn}$, $r_{nb}$ | Spectral radii | 32 |
| $r_{\mathrm{ess}}(T)$ | Essential spectral radius | 5 |
| $(r_u)$ | Relative uniform topology | 64 |
| $\rho(T)$ | Resolvent set | 5 |
| $\rho^l$, $\rho^{bb}$, $\rho^c$, $\rho^{nn}$, $\rho^{nb}$ | Resolvent sets | 30 |
| $R_\lambda$ | Resolvent operator | 5 |
| $R_\lambda^0$ | Sum of Newmann series | 40 |
| $R_{\lambda,n}$ | Partial sum of Newmann series | 41 |
| $\sigma(T)$ | Spectrum | 5, 52 |
| $\sigma^l$, $\sigma^{bb}$, $\sigma^c$, $\sigma^{nn}$, $\sigma^{nb}$ | Spectra | 30 |
| $\lvert\sigma(T)\rvert$ | Geometrical radius of spectrum | 5 |
| $\sigma_{\mathrm{ess}}(T)$ | Essential spectrum | 5 |
| $x_\gamma \xrightarrow{U} x$ | $U$-convergence | 2 |
| $x^* \otimes y$ | Elementary tensor product | 4 |

# Chapter 0

# Preliminaries

## 0.1 Vector spaces

The symbols $X$ and $Y$ always denote topological vector spaces. A ***neighborhood*** of a point $x \in X$ is any subset of $X$ containing an open set which contains $x$. Neighborhoods of zero will often be referred to as ***zero neighborhoods***. Every zero neighborhood $V$ is ***absorbing***, i.e., $\bigcup_{n=1}^{\infty} nV = X$. In every topological vector space (over $\mathbb{R}$ or $\mathbb{C}$) there exists a base $\mathcal{N}_0$ of zero neighborhoods with the following properties:

(i) Every $V \in \mathcal{N}_0$ is ***balanced***, i.e., $\lambda V \subseteq V$ whenever $|\lambda| \leqslant 1$;

(ii) For every $V_1, V_2 \in \mathcal{N}_0$ there exists $V \in \mathcal{N}_0$ such that $V \subseteq V_1 \cap V_2$;

(iii) For every $V \in \mathcal{N}_0$ there exists $U \in \mathcal{N}_0$ such that $U + U \subseteq V$;

(iv) For every $V \in \mathcal{N}_0$ and every scalar $\lambda$ the set $\lambda V$ is in $\mathcal{N}_0$.

Whenever we mention a base zero neighborhood, we assume that the base satisfies these properties.

A topological vector space is called ***normed*** if the topology is given by a norm. In this case the collection of all balls centered at zero is a base of zero neighborhoods. A complete normed space is referred to as a ***Banach space***. See [DS58] for a detailed study of normed and Banach spaces.

A subset $A$ of a topological vector space is called ***bounded*** if it is absorbed by every zero neighborhood, i.e., for every zero neighborhood $V$ one can find $\alpha > 0$ such that $A \subseteq \alpha V$. A set $A$ in a topological vector space is said to be ***pseudo-convex*** or ***semi-convex*** if $A + A \subseteq \alpha A$ for some number $\alpha$ (for convex sets $\alpha = 2$). If $U$ is a zero neighborhood, $(x_\gamma)$ is a net in $X$, and $x \in X$, we write $x_\gamma \xrightarrow{U} x$ if for every $\varepsilon > 0$ one can find an index $\gamma_0$ such that $x_\gamma - x \in \varepsilon U$ whenever $\gamma \geqslant \gamma_0$. It is easy to see that when $U$ is pseudo-convex, this convergence determines a topology on $X$, and the set of all scalar multiples of $U$ forms a base of the topology. We denote $X$ equipped with this topology by $(X, U)$. Clearly, $(X, U)$ is Hausdorff if and only if $\bigcap_{n=1}^{\infty} \frac{1}{n} U = \{0\}$.

A topological vector space is said to be ***locally bounded*** if there exists a bounded zero neighborhood. Notice that if $U$ is a bounded zero neighborhood then it is pseudo-convex. Conversely, if $U$ is a pseudo-convex zero neighborhood, then $(X, U)$ is locally bounded. Recall that a ***quasi-norm*** is a real-valued function on a vector space which satisfies all the axioms of norm except the triangle inequality, which is substituted by $\|x+y\| \leqslant k\left(\|x\|+\|y\|\right)$ for some fixed positive constant $k$. It is known (see, e.g., [Köt60]) that a topological vector space is quasi-normable if and only if it is locally bounded and Hausdorff. A complete quasi-normed space is called ***quasi-Banach***.

If the topology of a topological vector space $X$ is given by a ***seminorm*** p, we say that $X = (X, p)$ is a ***seminormed*** space. Clearly, in this case $X = (X, U)$ where the convex set $U$ is the unit ball of $p$ and, conversely, $p$ is the Minkowski functional of $U$. A Hausdorff topological vector space is called ***locally convex*** if there is a base of convex zero neighborhoods or, equivalently, if the topology is generated by a family of seminorms (the Minkowski functionals of the convex zero neighborhoods). When dealing with locally convex spaces we will always assume that the base zero neighborhoods are convex. Similarly, a Hausdorff topological vector space is said to be ***locally pseudo-convex*** if it has a base of pseudo-convex zero neighborhoods. A complete metrizable topological vector space is usually referred to as a ***Fréchet space***.

Further details on topological vector spaces can be found in [DS58, Köt60, RR64, Edw65, Sch71, KN76]. For details on locally bounded and quasi-normed topological vector spaces we refer the reader to [Köt60, KPR84, Rol85].

A ***vector lattice*** is an ordered vector space which is a lattice with respect to the order (i.e., with every two points $x$ and $y$ it contains their supremum $x \vee y = \sup(x, y)$ and their infimum $x \wedge y = \inf(x, y)$) such that the vector structure and the order structure are compatible, namely $x \leqslant y$ implies $x + z \leqslant y + z$ and $\alpha x \leqslant \alpha y$ for all vectors $x$, $y$, and $z$, and every positive number $\alpha$. Vector lattices are sometimes also called ***Riesz spaces***. The symbols $E$ and $F$ will be used to denote vector lattices. Every element $x$ in a vector lattice has ***modulus*** $|x| = x \vee (-x)$, ***positive part*** $x^+ = x \vee 0$, and ***negative part*** $x^- = (-x) \vee 0$, and the usual identities $x = x^+ - x^-$, $|x| = x^+ + x^-$, and $x^+ \wedge x^- = 0$ hold. We say that $x$ and $y$ are ***disjoint*** if $|x| \wedge |y| = 0$. If $E$ is a vector lattice, we will denote by $E_+$ the cone of positive elements of $E$. A subset $A \subseteq E$ is said to be ***solid*** if $|y| \leqslant |x|$ implies $y \in A$ for every $x \in A$ and $y \in E$. A solid subspace of a vector lattice is referred to as an ***(order) ideal***. If $u \in E_+$, then the ideal $E_u = \{x \in E \,:\, |x| \leqslant \lambda u \text{ for some } \lambda > 0\}$ is called the ***principal ideal*** generated by $u$. A positive element $u \in E$ is called a ***(strong) order unit*** if $E_u = E$.

A (semi)norm $p$ on a vector lattice $E$ is said to be a ***lattice (semi)norm*** if $|x| \leqslant |y|$ implies $p(x) \leqslant p(y)$ for every $x, y \in E$. A locally convex space $E$ equipped with a vector lattice structure is said to be a ***locally convex-solid vector lattice*** or a ***locally convex-solid Riesz space*** if every generating seminorm is a lattice seminorm, or, equivalently, if it has a base of convex solid zero neighborhoods. Similarly, a Banach space equipped with a vector lattice structure such that the norm is a lattice norm is referred to as a ***Banach lattice***. Note that most classical Banach spaces are Banach lattices. In particular, $C(K)$, $C_0(K)$, $\ell_p$, and $L_p(\mu)$ $(1 \leqslant p \leqslant \infty)$ are Banach lattices.

A Banach lattice $E$ is called an ***abstract M-space*** or an ***AM-space*** if $\|x \vee y\| = \max\{\|x\|, \|y\|\}$ for every disjoint $x$ and $y$ in $E$. Furthermore, we say that $E$ is an ***AM-space with unit*** $u$ if $\|x\| = \inf\{\lambda > 0 \,:\, |x| \leqslant \lambda u\}$ for some order unit $u$ and every $x$ in $E$. In this case by Kakutani-Krein representation theorem (see, for instance, [AAB98, Theorem 12.28]) there exists a compact Hausdorff space $\Omega$ such that $E$ is lattice isomorphic to the Banach lattice $C(\Omega)$ of all continuous functions on $\Omega$ with sup-norm, and the element $u$ corresponds to the constant function $\mathbf{1}$ on $\Omega$.

Further details on vector and Banach lattices can be found in [Vul67, LZ71, Sch74, Zaan83, AB85]

## 0.2 Operators

By an ***operator*** we always mean a ***linear operator*** between vector spaces. We will usually use the symbols $S$ and $T$ to denote operators. Recall that an operator $T$ between normed spaces is said to be ***bounded*** if its operator norm defined by $\|T\| = \sup\{\|Tx\| : \|x\| \leqslant 1\}$ is finite. It is well known that an operator between normed spaces is bounded if and only if it is continuous. By $L(X, Y)$ we denote the collection of all bounded operators between normed spaces $X$ and $Y$. If $X$ and $Y$ are Banach spaces, then $L(X, Y)$ endowed with the operator norm is again a Banach space, while $L(X) = L(X, X)$ is a Banach algebra. Recall, that a ***Banach algebra*** is a Banach space equipped with an algebra structure such that the algebra multiplication is continuous.

An operator between two vector spaces is said to be of ***finite rank*** if the range of $T$ is finite dimensional. A continuous finite rank operator $T\colon X \to Y$ between two Banach spaces can be written in the form $T = \sum_{i=1}^{n} x_i^* \otimes y_i$ with $x_i^*$ in $X^*$ and $y_i$ in $Y$. Here, as usual, the elementary tensor $x^* \otimes y : X \to Y$ is defined by $(x^* \otimes y)(x) = x^*(x)y$.

An operator $T\colon X \to Y$ between two Banach spaces is called ***nuclear*** if it can be written in the form $T = \sum_{i=1}^{\infty} x_i^* \otimes y_i$ with $x_i^*$ in $X^*$, $y_i$ in $Y$, and $\sum_{i=0}^{\infty}\|x_i^*\|\|y_i\| < \infty$. The ***nuclear norm*** $\nu(T)$ is defined by $\nu(T) = \inf \sum_{i=0}^{\infty}\|x_i^*\|\|y_i\|$, where the infimum is taken over all nuclear representations of $T$. For a nuclear operator $T$ we have $\|T\| \leqslant \nu(T)$. It is easy to see that every nuclear operator can be approximated by finite-rank operators.

If $X$ and $Y$ are two Banach spaces and $T \in L(X, Y)$ then $T$ is said to be ***compact*** if it maps the unit ball of $X$ into a precompact subset of $Y$. We will denote by $K(X, Y)$ the collection of all compact operators from $X$ to $Y$, and let $K(X) = K(X, X)$. It is well known that $K(X,Y)$ is a closed subspace of $L(X,Y)$ containing all finite-rank and nuclear operators. Further, $K(X)$ is a two-sided algebraic ideal in $L(X)$. The ***Calkin algebra*** of $X$ is the quotient Banach algebra $C(X)=L(X)/K(X)$ equipped with the quotient norm, which is often referred to as the ***essential norm***: $\|T\|_{\mathrm{ess}}=\inf\{\|T+K\| : K\in K(X)\}$.

If $\mathcal{A}$ is a unital algebra and $a\in\mathcal{A}$, then the ***resolvent set*** of $a$ is the set $\rho(a)$ of all $\lambda\in\mathbb{C}$ such that $e-\lambda a$ is invertible in $\mathcal{A}$. The resolvent set of an element $a$ in a non-unital algebra $\mathcal{A}$ is defined as the set of all $\lambda\in\mathbb{C}$ for which $e-\lambda a$ is invertible in the unitalization $\mathcal{A}_\times$ of $\mathcal{A}$. The ***spectrum*** of an element of an algebra is defined via $\sigma(a)=\mathbb{C}\setminus\rho(a)$. It is well-known that whenever $\mathcal{A}$ is a unital Banach algebra then $\sigma(a)$ is compact and nonempty for every $a\in\mathcal{A}$. In this case the ***spectral radius*** $r(a)$ is defined via ***Gelfand formula***: $r(a)=\lim\limits_{n\to\infty}\sqrt[n]{\|a^n\|}$. It is well known that $r(a)=\bigl|\sigma(a)\bigr|$, where $\bigl|\sigma(a)\bigr|$ is the ***geometrical radius*** of $\sigma(a)$, i.e., $\bigl|\sigma(a)\bigr|=\sup\{|\lambda| : \lambda\in\sigma(a)\}$. An element $a\in A$ is said to be ***quasinilpotent*** if $\sigma(a)=\{0\}$.

If $T$ is a bounded operator on a Banach space $X$ then we will consider the spectrum $\sigma(T)$ and the resolvent set $\rho(T)$ in the sense of the Banach algebra $L(X)$. If $\lambda\in\rho(T)$ then the inverse $(I-\lambda T)^{-1}$ is called the ***resolvent operator*** and is denoted by $R(T;\lambda)$ or just $R_\lambda$. It is well known that if $\lambda\in\mathbb{C}$ satisfies $|\lambda|>r(T)$ then the ***Neumann series*** $\sum_{i=0}^{\infty}\frac{T^i}{\lambda^{i+1}}$ converges to $R_\lambda$ in operator norm. We say that $T$ is ***locally quasinilpotent*** at $x\in X$ if $\lim\limits_{n\to\infty}\sqrt[n]{\|T^nx\|}=0$. We denote by $\mathcal{Q}_T$ the set of all points at which $T$ is locally quasinilpotent.

Further, if $T$ is a bounded operator on a Banach space $X$ then the spectrum and the spectral radius of the canonical image of $T$ in the Calkin algebra $C(X)$ will be referred to as the ***essential spectrum***[1] $\sigma_{\mathrm{ess}}(T)$ and the ***essential spectral radius*** $r_{\mathrm{ess}}(T)$ respectively. It follows immediately that $r_{\mathrm{ess}}(T)=\lim\limits_{n\to\infty}\sqrt[n]{\|T^n\|_{\mathrm{ess}}}$. Further, $\|T\|_{\mathrm{ess}}\leqslant\|T\|$ yields $r_{\mathrm{ess}}(T)\leqslant r(T)$. It is known that if $\sigma_{\mathrm{ess}}(T)=\{0\}$ then every nonzero point of $\sigma(T)$ is an eigenvalue of $T$. Further details on essential spectral radius can be found in [Nus70, CPY74].

[1] There are several non-equivalent definitions of the essential spectrum of an operator. E.g. sometimes $\sigma_{\mathrm{ess}}(T)$ is defined as $\sigma(T)$ excluding the isolated eigenvalues of finite multiplicity.

An operator between two vector lattices is said to be ***positive*** if it maps positive elements to positive elements. Positive operators have many nice properties. In particular, a positive operator on a Banach lattice is automatically continuous, and the spectral radius of a positive operator always belongs to the spectrum. For properties of positive operators we refer the reader to [AB85, Sch74].

Let $S, T\colon E \to F$ be two operators between vector lattices with $T$ positive. We say that $S$ is ***dominated*** by $T$ provided $|Sx| \leqslant T|x|$ for each $x \in E$. We say that $S$ is ***polynomially dominated*** by $T$ whenever there exists a polynomial $P(t)$ with non-negative coefficients such that $S$ is dominated by $P(T)$. A positive operator $T$ on a Banach lattice $E$ is said to be ***compact-friendly*** if there exist three non-zero operators $R$, $K$, and $C$ on $E$ such that $R$ and $K$ are positive, $K$ is compact, $T$ commutes with $R$, and $C$ is dominated by both $R$ and $K$. Compact-friendly operators were first studied in [AAB94]. The class of all compact-friendly operators is rather large. In particular, every positive kernel operator is compact-friendly. Also, every positive operator in $L(\ell_p)$ for $1 \leqslant p < \infty$ is compact-friendly.

We say that an operator is ***non-scalar*** if it is not a multiple of the identity operator.

## 0.3 The Invariant Subspace Problem

Suppose that $T$ is an operator on a topological vector space $X$, and $Y$ is a linear subspace of $X$, we say that $Y$ is $T$***-invariant*** if $T(Y) \subseteq Y$. We say that $Y$ is $T$***-hyperinvariant*** if it is invariant under every continuous operator that commutes with $T$. The ***Invariant Subspace Problem*** is the problem of finding invariant subspaces of continuous operators. Here we present only some basic observations and several important results related to our work. For detailed surveys on the Invariant Subspace Problem see [RR73, AAB98].

Of course, the zero subspace and the whole space are always invariant for every operator, so we will be looking for non-trivial invariant subspaces. It is easy to see that $\operatorname{Null} T$ and $\operatorname{Range} T$ are $T$-hyperinvariant. Clearly, if $T$ has an eigenvector, then the one-dimensional subspace spanned by this eigenvector is invariant under $T$. Further, since every complex matrix has an eigenvalue, it follows that in a finite dimensional complex space every operator has an invariant subspace.

Suppose that $X$ is a Banach space, take any nonzero $x \in X$ and consider the linear span of the orbit of $x$ under $T$, i.e., $Y = \text{lin}\{x, Tx, T^2x, \dots\}$. Clearly, $Y$ is a $T$-invariant non-zero linear subspace of $X$ with at most countable Hamel basis, so that $Y \neq X$. Therefore, the Invariant Subspace Problem for Banach spaces (and for any topological vector space with uncountable Hamel basis) is trivial unless we require the subspace to be closed. Therefore, when dealing with the Invariant Subspace Problem, one would usually look for closed non-trivial subspaces. The case of closed subspaces, is, however, the most interesting and, usually, the most important.

Further, if $X$ is non-separable, let $Y$ again be the linear span of the orbit of a non-zero element under $T$. Then the closure $\overline{Y}$ is a non-trivial closed $T$-invariant subspace, so that for a non-separable Banach space the problem is trivial. *Whether or not every bounded operator on a separable Banach space has a (non-trivial) closed invariant subspace* was an open question for a long time.

We mention several important advances in the history of the Invariant Subspace Problem for Banach spaces.

**Theorem 0.3.1 (M. G. Krein [KR48]).** *Every positive operator on $C(\Omega)$, where $\Omega$ is a compact Hausdorff space, has a closed invariant subspace.*

**Theorem 0.3.2 (N. Aronszajn and K. T. Smith [AS54]).** *Every compact operator on a Banach space has a closed invariant subspace.*

**Theorem 0.3.3 (V. I. Lomonosov [Lom73]).** *If $T$ is a bounded operator on a Banach space such that $T$ commutes with a non-scalar operator $S$ and $S$ commutes with a non-zero compact operator $K$, then $T$ has a closed invariant subspace*

This theorem of Lomonosov was a breakthrough in the study of the Invariant Subspace Problem because it considerably increased the class of operators with an invariant subspace. The theorem is, in fact, an immediate corollary of the following more general result of Lomonosov.

**Theorem 0.3.4.** *If $S$ is a bounded non-scalar operator on a separable Banach space commuting with a compact operator, then there exists a closed $S$-hyperinvariant subspace.*

In 1976 P. Enflo [Enf76] showed that the general Invariant Subspace Problem for continuous operators on Banach spaces is false by constructing an example of a bounded operator on a separable Banach space without closed invariant subspaces. C.J. Read presented another example of an operator (in fact, a class of operators) on a Banach space without invariant subspaces in [Read84]. A year later Read published a short follow-up paper [Read85] where he showed that his example can be slightly modified so that the operators would reside on $\ell_1$. In [Read86], a simplified version of the Read's original example was presented. Read also constructed a class of quasinilpotent operators on $\ell_1$ without invariant subspaces in [Read97]. In [Atz84], A. Atzmon presented an example of a continuous operator without invariant subspaces on a Fréchet space.

The Invariant Subspace Problem is still open for many important classes of operators or spaces. For example, it is still unknown if every bounded operator on a separable Hilbert space has a closed invariant subspace. Currently, there are several directions related to the Invariant Subspace Problem which have been intensively studied and where some progress has been made, e.g., common invariant subspaces of algebras and semigroups of operators, triangularization, cyclic and hypercyclic operators, etc.

We are primarily interested in the invariant subspace problem for ordered topological vector spaces. During the last several years there has been a noticeable increase of interest in the Invariant Subspace Problem for positive operators on Banach lattices. A rather complete and comprehensive survey on this topic is presented in [AAB98], to which we refer the reader for details and for an extensive bibliography. At this point we mention only one important result from [AAB93]:

**Theorem 0.3.5. (Abramovich, Aliprantis, and Burkinshaw [AAB93, AAB98])** *Every non-zero quasinilpotent compact-friendly operator on a Banach lattice has a non-trivial closed invariant subspace which is an ideal.*

This implies, in particular, that every quasinilpotent positive kernel operator and every positive quasinilpotent operator on $\ell_p$ $(1 \leqslant p < \infty)$ has an invariant subspace. We list more results of this type in Section 3.2. Whether or not every positive operator has an invariant subspace is still an open problem.

# Chapter 1

# The modulus and the commutant of a Read operator

## 1.1 A chain from a Read operator to a compact operator

Theorem 0.3.3 was obtained by V.I. Lomonosov in [Lom73], and it turned out to be a major step in the history of the Invariant Subspace Problem. Lomonosov's theorem says that if a continuous operator $T$ on a Banach space commutes with another non-scalar continuous operator $S$ and $S$ commutes with a non-zero compact operator $K$, then $T$ has an invariant subspace. Motivated by their study of the Invariant Subspace Problem for positive operators on Banach lattices, Y. A. Abramovich and C. D. Aliprantis have asked recently whether or not Lomonosov's theorem can be extended to chains of four or more operators. In this section we show that this cannot be done. For our initial operator $T$ we will take an operator without an invariant subspace on $\ell_1$ coming from the famous construction of C. J. Read in [Read84, Read85]. Then we produce two continuous non-scalar operators $S_1$, $S_2$, and a compact operator $K$ such that $TS_1 = S_1T$, $S_1S_2 = S_2S_1$, and $S_2K = KS_2$.

We begin with reminding the reader of the construction in [Read86], which is a simplified version of the Read's original example [Read84, Read85]. As in [Read86], we denote the standard unit vectors of $\ell_1$ by $(f_i)_{i=0}^{\infty}$. The symbol $F$ denotes the linear subspace of $\ell_1$ spanned by $f_i$'s, and thus, $F$ consists of eventually vanishing sequences.

Let $\mathbf{d} = (a_1, b_1, a_2, b_2, \dots)$ be a strictly increasing sequence of positive integers. Also let $a_0 = 1$, $v_0 = 0$, and $v_n = n(a_n + b_n)$ for $n \geqslant 1$. Then there is a unique sequence $(e_i)_{i=0}^{\infty} \subset F$ with the following properties:

0) $f_0 = e_0$;

A) if integers $r$, $n$, and $i$ satisfy $0 < r \leqslant n$, $i \in [0, v_{n-r}] + ra_n$, we have $f_i = a_{n-r}(e_i - e_{i-ra_n})$;

B) if integers $r$, $n$, and $i$ satisfy $1 \leqslant r < n$, $i \in (ra_n + v_{n-r}, (r+1)a_n)$, (respectively, $1 \leqslant n$, $i \in (v_{n-1}, a_n)$), then $f_i = 2^{(h-i)/\sqrt{a_n}} e_i$, where $h = (r + \frac{1}{2})a_n$ (respectively, $h = \frac{1}{2}a_n$);

C) if integers $r$, $n$, and $i$ satisfy $1 \leqslant r \leqslant n$, $i \in [r(a_n + b_n), na_n + rb_n]$, then $f_i = e_i - b_n e_{i-b_n}$;

D) if integers $r$, $n$, and $i$ satisfy $0 \leqslant r < n$, $i \in (na_n + rb_n, (r+1)(a_n + b_n))$, then $f_i = 2^{(h-i)/\sqrt{b_n}} e_i$, where $h = (r + \frac{1}{2})b_n$.

Indeed, since $f_i = \sum_{j=0}^{i} \lambda_{ij} e_j$ for each $i \geqslant 0$ and $\lambda_{ii}$ is always nonzero, this linear relation is invertible. Further,

$$\operatorname{lin}\{e_i : i = 1, \dots, n\} = \operatorname{lin}\{f_i : i = 1, \dots, n\} \text{ for every } n \geqslant 0. \tag{1.1}$$

In particular, all $e_i$ are linearly independent and also span $F$. Then C. J. Read defines $T\colon F \to F$ to be the unique linear map such that $Te_i = e_{i+1}$, and proves that $\|Tf_i\| \leqslant 2$ for every $i \geqslant 0$ provided $\mathbf{d}$ increases sufficiently rapidly, i. e., satisfies several conditions of the form

$$\begin{aligned} a_n &\geqslant G(n, a_0, b_0, a_1, b_1, \dots, a_{n-1}, b_{n-1}), \text{ and} \\ b_n &\geqslant H(n, a_0, b_0, a_1, b_1, \dots, a_{n-1}, b_{n-1}, a_n), \end{aligned}$$

where $G$ and $H$ are some real-valued functions. It follows that $T$ can be extended to a bounded operator on $\ell_1$. Finally, C. J. Read proves that this extension, which is still denoted by $T$, has no invariant subspaces provided $\mathbf{d}$ increases sufficiently rapidly.

Throughout this section we will assume, without loss of generality, that all integers $a_i$ and $b_i$ are even. We are going to construct non-scalar operators $S_1$, $S_2$, and $K$ such that $K$ has rank one and commutes with $S_2$, $S_2$ commutes with $S_1$, and $S_1$ commutes with $T$. In fact, we take $S_1 = T^2$, so that the equality $TS_1 = S_1T$ holds trivially. Define $S_2$ on $F$ via

$$S_2 e_i = \begin{cases} e_i & \text{if } i \text{ is even;} \\ 0 & \text{otherwise.} \end{cases}$$

We claim that

$$S_2 f_i = \begin{cases} f_i & \text{if } i \text{ is even;} \\ 0 & \text{otherwise.} \end{cases}$$

To prove this we consider all possible cases:

0) In this case $S_2 f_0 = S_2 e_0 = e_0 = f_0$;

A) Since $a_n$ is even then

$$S_2 f_i = a_{n-r}(S_2 e_i - S_2 e_{i-ra_n}) = \begin{cases} a_{n-r}(e_i - e_{i-ra_n}) = f_i & \text{if } i \text{ is even;} \\ 0 & \text{otherwise.} \end{cases}$$

B) In this case

$$S_2 f_i = 2^{(h-i)/\sqrt{a_n}} S_2 e_i = \begin{cases} 2^{(h-i)/\sqrt{a_n}} e_i = f_i & \text{if } i \text{ is even;} \\ 0 & \text{otherwise.} \end{cases}$$

C) Since $b_n$ is even, we have

$$S_2 f_i = S_2 e_i - b_n S_2 e_{i-b_n} = \begin{cases} e_i - b_n e_{i-b_n} = f_i & \text{if } i \text{ is even;} \\ 0 & \text{otherwise.} \end{cases}$$

D) Finally, in this case

$$S_2 f_i = 2^{(h-i)/\sqrt{b_n}} S_2 e_i = \begin{cases} 2^{(h-i)/\sqrt{b_n}} e_i = f_i & \text{if } i \text{ is even;} \\ 0 & \text{otherwise.} \end{cases}$$

In particular, $S_2$ is bounded on $F$ and can be extended to $\ell_1$. For every $i \geqslant 0$ we have

$$T^2 S_2 e_i = \begin{cases} T^2 e_i = e_{i+2} & \text{if } i \text{ is even;} \\ 0 & \text{otherwise.} \end{cases}$$

On the other hand

$$S_2 T^2 e_i = S_2 e_{i+2} = \begin{cases} e_{i+2} & \text{if } i \text{ is even;} \\ 0 & \text{otherwise,} \end{cases}$$

so that $T^2 S_2 x = S_2 T^2 x$ for every $x \in F$. Since $F$ is dense in $\ell_1$, it follows that $T^2$ and $S_2$ commute on $\ell_1$.

Finally, define $K$ on $\ell_1$ via $K f_0 = f_0$ and $K f_i = 0$ for all $i > 0$. Then $K$ is a bounded rank one operator on $\ell_1$, and $K$ commutes with $S_2$.

Note that if $m$ divides $a_n$ and $b_n$ for every $n$, then, in a similar manner as the previous construction, we could take for $S_1$ the operator $T^m$ instead of $T^2$. It follows now from Lomonosov's theorem that $T^m$ has an invariant subspace (confer [Read86, Lemma 6.4]).

In [Read97] C.J. Read presents as a modification of his original example a quasinilpotent operator on $\ell_1$ without closed nontrivial invariant subspaces. The same argument as above provides a chain of four commuting operators connecting this operator to a compact operator.

## 1.2 The commutant of a Read operator

Describing the commutant of an operator on a Banach space is usually a difficult problem. Clearly, the sum of every convergent power series of an operator commutes with the operator itself. But the commutant can be substantially larger than just the set of all power series of the operator. In this section we show that the commutant of a Read operator $T$ consists only of the (convergent) power series of $T$.

Recall that $T$ acts as the right shift on the sequence $(e_n)_{n=0}^{\infty}$, and this sequence spans the dense subspace $F$ of all eventually vanishing sequences in $\ell_1$. As in the previous section, $(f_n)_{n=0}^{\infty}$ is the sequence of standard unit vectors in $\ell_1$.

**Proposition 1.2.1.** *If $RT = TR$ for some $R \in L(\ell_1)$ and $Rf_0 \in F$, then $R = p(T)$ for some polynomial $p$.*

*Proof.* Since $f_0 = e_0$ and $Rf_0 \in F$, it follows that $Re_0 = p(T)e_0$ for some polynomial $p$. Therefore

$$Re_k = RT^k e_0 = T^k Re_0 = T^k p(T)e_0 = p(T)T^k e_0 = p(T)e_k,$$

so that $R$ coincides with $p(T)$ on $F$ and, therefore, on the whole space $\ell_1$. □

**Theorem 1.2.2.** *If $RT = TR$ for some $R \in L(\ell_1)$, then there exists a sequence of polynomials $(p_n)$ such that $Rx = \lim_{n\to\infty} p_n(T)x$ for every $x$ in $F$.*

*Proof.* Suppose that $RT = TR$ for some bounded operator $R$. Let $z_0 = Re_0$. Since $F$ is dense in $\ell_1$, we can find a sequence $(y_n)$ in $F$ such that $y_n \to z_0$. For each $n > 0$ there is a polynomial $p_n$ such that $y_n = p_n(T)e_0$.

Fix $k > 0$, then $Re_k = RT^k e_0 = T^k Re_0 = T^k z_0$. Further, we have $p_n(T)e_k = p_n(T)T^k e_0 = T^k p_n(T)e_0 = T^k y_n$, so that

$$\|Re_k - p_n(T)e_k\| = \|T^k z_0 - T^k y_n\| \leqslant \|T^k\| \|z_0 - y_n\| \to 0 \text{ as } n \to \infty.$$

Therefore, $p_n(T)x$ converges to $Rx$ for each $x \in F$. □

## 1.3 The modulus of a Read operator

In this section we consider the Invariant Subspace Problem for positive operators. It is an open problem if every positive operator on a Banach lattice has an invariant subspace, and there are many beautiful partial results in this direction. In particular, the following theorem was proved in [AAB93].

**Theorem 1.3.1 ([AAB93, AAB98]).** *If the modulus of a bounded operator $T\colon \ell_p \to \ell_p$ $(1 \leqslant p < \infty)$ exists and is quasinilpotent, then $T$ has a non-trivial closed invariant subspace which is an ideal.*

It follows that each positive quasinilpotent operator on $\ell_p$ $(1 \leqslant p < \infty)$ has a nontrivial closed invariant subspace. In the same papers the authors posed the following problem.

**Problem.** Does every positive operator on $\ell_1$ have an invariant subspace?

Keeping in mind that each operator on $\ell_1$ has a modulus and that Read operators without invariant subspaces [Read85, Read86, Read97] are operators on $\ell_1$, it was suggested in [AAB93, AAB98] that the modulus of some of these operators might be a natural candidate for a counterexample to the above problem. Following this suggestion, we will be dealing in this paper with the modulus of the quasinilpotent operator $T$ constructed in [Read97]. It turns out, quite surprisingly, that even though $T$ and $|T|$ are "very close", $|T|$ not only has an invariant subspace but even a positive eigenvector. This result increases the chances that every positive operator does indeed have an invariant subspace.

This section is organized as follows. After introducing some necessary notation and terminology we prove a general theorem on the existence of an invariant subspace for the modulus of a quasinilpotent operator. The rest of the section will be devoted to the verification that C. J. Read's operator, constructed in [Read97], satisfies all the hypotheses of this theorem and so its modulus does have an invariant subspace.

We will use the following important version of the Krein-Rutman theorem, which was independently established by P.P Zabreĭko and S. V.Smitskikh in [ZS79] and by R. Nussbaum in [Nus81].

**Theorem 1.3.2 ([Nus81, ZS79]).** *Let $S$ be a positive operator on a Banach lattice such that $r_{\mathrm{ess}}(S) < r(S)$, then $r(S)$ is an eigenvalue of $S$ corresponding to a positive eigenvector.*

We use this fact in the proof of the following simple but rather unexpected result.

**Theorem 1.3.3.** *Suppose that a quasinilpotent operator $S$ on $\ell_p$ has no closed nontrivial invariant ideals and $S^-$ is compact. Then $r(|S|)$ is a positive eigenvalue of $|S|$ corresponding to a positive eigenvector. In particular, $|S|$ has an invariant subspace.*

*Proof.* First observe that the operator $|S|$ cannot be quasinilpotent. Indeed, if it were, then by Theorem 1.3.1 the operator $S$ itself would have an invariant closed ideal contrary to our hypothesis. Thus, $r(|S|) > 0$.

Next we claim that $r_{\text{ess}}(|S|) = 0$. To prove this, notice that $|S| = S + 2S^-$, and so

$$|S|^n = (S + 2S^-)^n = S^n + RS^-,$$

where $R$ is some polynomial in $S$ and $S^-$. Hence $RS^-$ is compact, whence

$$\big\| |S|^n \big\|_{\text{ess}} = \big\| S^n + RS^- \big\|_{\text{ess}} \leqslant \|S^n\|,$$

and consequently

$$r_{\text{ess}}(|S|) = \lim_{n\to\infty} \sqrt[n]{\big\||S^n|\big\|_{\text{ess}}} \leqslant \lim_{n\to\infty} \sqrt[n]{\|S^n\|} = r(S) = 0.$$

An application of Theorem 1.3.2 finishes the proof. □

**Corollary 1.3.4.** *Under the hypotheses of the above theorem the operator $S^+$ also has a nontrivial closed invariant subspace.*

*Proof.* If $S^+$ is quasinilpotent, then applying Theorem 1.3.1 again, we see that $S^+$ has an invariant ideal.

Assume that $S^+$ is not quasinilpotent. Since $S^+ = S + S^-$, the same argument as in the proof of Theorem 1.3.3 shows that $r_{\text{ess}}(S^+) = 0$ and we can again apply Theorem 1.3.2. □

Theorem 1.3.3 is strong enough to enable us to prove Corollary 1.3.7 about the modulus of C. J. Read's operator. But first we would like to mention a nice generalization of Theorem 1.3.3. Mimicking the proof of Theorem 1.3.3 we can obtain the following theorem.

**Theorem 1.3.5.** *Let $S$ be a compact-friendly operator with $r_{\mathrm{ess}}(S) = 0$. Then $S$ has a nontrivial invariant subspace.*

To illustrate this theorem we mention the following result: *If $S$ is a quasinilpotent kernel operator and $S^-$ (respectively $S^+$) is compact, then $|S|$ and $S^+$ (respectively $S^-$) have invariant subspaces.*

Recall that$(f_i)_{i=0}^{\infty}$ denote the standard unit vectors of $\ell_1$. It is well known that we can consider each $S \in L(\ell_1)$ as an infinite matrix $S = (s_{ij})_{i,j=0}^{\infty}$. Let $S_{(i)}$ denote the $i$-th row of this matrix. If $x \in \ell_1$, then $(Sx)_i = \langle S_{(i)}, x\rangle$, so that $Sx = \sum_{i=0}^{\infty}\langle S_{(i)}, x\rangle f_i$. This gives a nuclear representation $S = \sum_{i=0}^{\infty} S_{(i)} \otimes f_i$, where the rows $S_{(i)}$ of $S$ are considered as linear functionals on $\ell_1$. It follows that

$$\nu(S) \leqslant \sum_{i=0}^{\infty}\|S_{(i)}\|_{\infty}\|f_i\|_1 = \sum_{i=0}^{\infty}\|S_{(i)}\|_{\infty},$$

so that $S$ is nuclear if the last sum is finite.

The construction of [Read97] is similar to the construction in [Read85] described in Section 1.1, the only difference is in the coefficients in A)-D):

0) $f_0 = e_0$;

A) if integers $r$, $n$, and $i$ satisfy $0 < r \leqslant n$, $i \in [0, v_{n-r}] + ra_n$, then $f_i = (n^{ra_n}e_i - e_{i-ra_n})(n-r)^{i-ra_n}a_{n-r}$;

B) if integers $r$, $n$, and $i$ satisfy $0 < r < n$, $i \in (ra_n + v_{n-r}, (r+1)a_n)$, (respectively, $1 \leqslant n$, $i \in (v_{n-1}, a_n)$), then $f_i = n^i 2^{(h-i)/\sqrt{a_n}}e_i$, where $h = (r+\frac{1}{2})a_n$ (respectively, $h = \frac{1}{2}a_n$);

C) if integers $r$, $n$, and $i$ satisfy $0 < r \leqslant n$, $i \in [r(a_n + b_n), na_n + rb_n]$, then $f_i = n^i e_i - b_n n^{i-b_n}e_{i-b_n}$;

D) if integers $r$, $n$, and $i$ satisfy $0 \leqslant r < n$, $i \in (na_n + rb_n, (r+1)(a_n + b_n))$, then $f_i = n^i 2^{(h-i)/\sqrt{b_n}}e_i$, where $h = (r+\frac{1}{2})b_n$.

C.J. Read proves in [Read97] that the operator constructed this way (we will use the same symbol $T$ to denote it) still has no invariant subspaces, but is in addition quasinilpotent.

Our plan is as follows: we will prove that the negative part of $T$ is nuclear, hence compact. Then Theorem 1.3.3 will imply that $r(|T|)$ is a positive eigenvalue of $|T|$, corresponding to a positive eigenvector.

**Lemma 1.3.6.** *The operator $T^-$ is nuclear, provided* **d** *increases sufficiently rapidly.*

*Proof.* Similarly to the proofs of [Read97, Lemma 5.1] and [Read86, Lemma 6.1] we study the matrices $(t_{ij})_{i,j=0}^{\infty}$ and $(t_{ij}^-)_{i,j=0}^{\infty}$ of $T$ and $T^-$ respectively. Recall that $t_{ki} = (Tf_i)_k$, so that it suffices to look at the images of the standard unit vectors under $T$. We will see that the matrix of $T$ is quite sparse and has the following structure: every entry on the diagonal right under the main diagonal is strictly positive, there are no nonnegative entries below this diagonal, and there are some entries above it. We consider consecutively all the cases mentioned above.

0) $Tf_0 = e_1 = 2^{(1-a_1/2)/\sqrt{a_1}} f_1$, so that $T^- f_0 = 0$.

A) If $i < v_{n-r} + ra_n$, i.e., $i$ is not the right end point of the interval $[ra_n, v_{n-r} + ra_n]$, then $Tf_i = (n-r)^{-1} f_{i+1}$, so that $T^- f_i = 0$. The only nontrivial case here is when $i$ is the right end of the interval, i. e. $i = v_{n-r} + ra_n$. Then we have

$$\begin{aligned} Tf_i = a_{n-r} n^{ra_n} (n-r)^{v_{n-r}} e_{1+ra_n+v_{n-r}} - a_{n-r}(n-r)^{v_{n-r}} e_{1+v_{n-r}} \\ = \varepsilon_1 f_{1+v_{n-r}+ra_n} - \varepsilon_2 f_{1+v_{n-r}}, \end{aligned}$$

where $\varepsilon_1 > 0$ and $\varepsilon_2$ is given by

$$\varepsilon_2 = (n-r+1)^{-1-v_{n-r}} 2^{(1+v_{n-r}-a_{n-r+1}/2)/\sqrt{a_{n-r+1}}} a_{n-r}(n-r)^{v_{n-r}},$$

so that

$$T^- f_{v_{n-r}+ra_n} = \varepsilon_2 f_{1+v_{n-r}}. \tag{1.2}$$

B) Similarly, if $ra_n + v_{n-r} < i < (r+1)a_n - 1$ or $v_{n-1} < i < a_n - 1$, then $Tf_i = n^{-1}2^{1/\sqrt{a_n}} f_{i+1}$, so that $T^- f_i = 0$. If $i = (r+1)a_n - 1$, then $Tf_i = n^i 2^{(1-a_n/2)/\sqrt{a_n}} e_{(r+1)a_n}$. To express this in terms of the $f_i$'s, notice that $f_{(r+1)a_n} = a_{n-r-1}(n^{(r+1)a_n} e_{(r+1)a_n} - e_0)$, which implies

$$e_{(r+1)a_n} = n^{-(r+1)a_n}(a_{n-r-1}^{-1} f_{(r+1)a_n} + f_0), \tag{1.3}$$

In this case $T^- f_i = 0$. Analogously, if $i = a_n - 1$, then $Tf_i = n^i 2^{(1-a_n/2)/\sqrt{a_n}} e_{a_n}$. It follows from $f_{a_n} = a_{n-1}(n^{a_n} e_{a_n} - e_0)$ that

$$Tf_i = n^{-1} 2^{(1-a_n/2)/\sqrt{a_n}} (a_{n-1}^{-1} f_{a_n} + f_0),$$

and again $T^- f_i = 0$. Thus, case (B) produces no nontrivial entries in $T^-$.

C) If $i$ is not the right end of the interval, i.e. $i < na_n + rb_n$, then $Tf_i = n^{-1} f_{i+1}$, so that $T^- f_i = 0$. If $i = na_n + rb_n$, then

$$\begin{aligned} Tf_i = n^{na_n + rb_n} e_{1+na_n+rb_n} - b_n n^{na_n+(r-1)b_n} e_{1+na_n+(r-1)b_n} \\ = \varepsilon_1 f_{1+na_n+rb_n} - \varepsilon_2 f_{1+na_n+(r-1)b_n}, \end{aligned}$$

where $\varepsilon_1 > 0$ and $\varepsilon_2 = b_n n^{-1} 2^{(1+na_n-b_n/2)/\sqrt{b_n}}$. It follows that

$$T^- f_{na_n+rb_n} = b_n n^{-1} 2^{(1+na_n-b_n/2)/\sqrt{b_n}} f_{1+na_n+(r-1)b_n}. \tag{1.4}$$

D) If $i < (r+1)(a_n + b_n) - 1$, then $Tf_i = n^{-1} 2^{1/\sqrt{b_n}} f_{i+1}$, so that $T^- f_i = 0$. If $i = (r+1)(a_n + b_n) - 1$ then

$$Tf_i = n^i 2^{(-a_n/2-(r+1)a_n/2+1)/\sqrt{b_n}} e_{(r+1)(a_n+b_n)}.$$

Using (C) inductively we obtain the following identity:

$$\begin{aligned} e_{(r+1)(a_n+b_n)} = n^{-(r+1)(a_n+b_n)}\{f_{(r+1)(a_n+b_n)} + b_n f_{(r+1)a_n+rb_n} + \dots \\ + n_n^r f_{(r+1)a_n+b_n}\} + b_n^{r+1} n^{-(r+1)b_n} e_{(r+1)a_n}. \end{aligned}$$

Substitute $e_{(r+1)a_n}$ from (1.3) and notice that all the the coefficients are positive and, therefore, $T^- f_i = 0$. Thus, case (D) does not produce any nontrivial entries in $T^-$.

Summarizing the calculations, the only nonzero entries of $T^-$ are given by (1.2) and (1.4):

$$t^-_{1+v_{n-r},v_{n-r}+ra_n} = (n-r+1)^{-1-v_{n-r}}2^{(1+v_{n-r}-a_{n-r+1}/2)/\sqrt{a_{n-r+1}}}a_{n-r}(n-r)^{v_{n-r}}$$

and

$$t^-_{na_n+(r-1)b_n+1,na_n+rb_n} = b_n n^{-1}2^{(1+na_n-b_n/2)/\sqrt{b_n}}$$

for all $0 < r \leqslant n$. To show that $T^-$ is nuclear it suffices to show $\sum_{k=0}^{\infty}\|T^-_{(k)}\|_\infty < \infty$. Look at the rows of $T^-$ containing non-zero entries. Notice that

$$t^-_{1+v_{n-r},v_{n-r}+ra_n} \leqslant a_{n-r}2^{(1+v_{n-r}-a_{n-r+1}/2)/\sqrt{a_{n-r+1}}} \leqslant 2^{-(1+v_{n-r})}$$

for all $0 < r \leqslant n$ provided $\mathbf{d}$ increases sufficiently rapidly. It follows that $\|T^-_{(1+v_m)}\|_\infty \leqslant 2^{-(1+v_m)}$ for every $m \geqslant 0$ and $\sum\limits_{m=0}^{\infty}\|T^-_{(1+v_m)}\|_\infty \leqslant \sum\limits_{m=0}^{\infty} 2^{-1-v_m} < 1$.

Further, the entries $t^-_{na_n+(r-1)b_n+1,na_n+rb_n}$ do not depend on $r$, and their contribution to $\sum_{k=0}^{\infty}\|T^-_{(k)}\|_\infty$ does not exceed the sum of all of them, which can be easily estimated:

$$\sum_{n=1}^{\infty}\sum_{r=1}^{n} b_n n^{-1}2^{(1+na_n-b_n/2)/\sqrt{b_n}} \leqslant \sum_{n=1}^{\infty} b_n 2^{(1+na_n-b_n/2)/\sqrt{b_n}} \leqslant \sum_{n=1}^{\infty} 2^{-n} = 1,$$

because $b_n 2^{(1+na_n-b_n/2)/\sqrt{b_n}} \leqslant 2^{-n}$ for all $n \geqslant 1$ provided $\mathbf{d}$ increases sufficiently rapidly. Thus, $\nu(T^-) \leqslant \sum_{k=0}^{\infty}\|T^-_{(k)}\|_\infty < 2$ provided $\mathbf{d}$ increases sufficiently rapidly. □

**Corollary 1.3.7.** *$T$ satisfies the following properties, provided $\mathbf{d}$ increases sufficiently rapidly:*

*(i) $|T|$, $T^+$, and $T^-$ have positive eigenvectors;*

*(ii) Neither $|T|$ nor $T^+$ has an invariant ideal.*

*Proof.* It follows from Theorem 1.3.3 and Lemma 1.3.6 that $|T|$ has a positive eigenvector. It was noticed in the proof of Lemma 1.3.6 that $T^- f_0 = 0$, so that $T^-$ also has a positive eigenvector.

To prove (ii), assume that $J$ is a closed ideal in $\ell_1$ invariant under $|T|$ or $T^+$, and that $0 \neq x \in J$, then $x_k \neq 0$ for some $k \geqslant 0$, so that $f_k \in J$. It follows from the proof of Lemma 1.3.6 that both $|T|f_i$ and $T^+ f_i$ have nonzero $(i+1)$-th component, implying $f_{k+1} \in J$. Proceeding inductively, we see that $f_i \in J$ for all $i \geqslant k$. Further, the proof of Lemma 1.3.6 also shows that $(|T|f_i)_0 \neq 0$ and $(T^+ f_i)_0 \neq 0$ for infinitely many $i$'s, so that $f_0 \in J$. It follows that $f_i \in J$ for every $i \geqslant 0$, so that $J = \ell_1$. In fact, (ii) is a manifestation of the fact that a positive operator $S$ on $\ell_p$ $(1 \leqslant p < \infty)$ has no invariant ideals if and only if there is a path between every two columns of $S$ (c.f. [AAB98, Tr]).

It follows from (ii) and Theorem 1.3.1 that $T^+$ cannot be quasinilpotent. On the other hand, since $T^+ = T + T^-$ then, analogously to the proof of Theorem 1.3.3, we have $r_{\mathrm{ess}}(T^+) = 0$. Then by Theorem 1.3.2 we conclude that $r(T^+)$ is a positive eigenvalue of $T^+$, corresponding to a positive eigenvector. □

The last statement of Corollary 1.3.7 emphasizes that the hypothesis of not having invariant ideals in Theorem 1.3.3 is weaker than not having invariant subspaces. We do not know if the moduli of the operators produced in [Read85, Read86] have invariant subspaces.

# Chapter 2

# Spectral radii of bounded operators on locally convex spaces

## 2.1 Bounded operators

There are various definitions for a bounded linear operator between two topological vector spaces. To avoid confusion, we will, of course, give different names to different types of boundedness.

**Definition 2.1.1.** Let $X$ and $Y$ be topological vector spaces. An operator $T\colon X \to Y$ is said to be

(i) ***bb-bounded*** if it maps every bounded set into a bounded set;

(ii) ***nb-bounded*** if it maps some neighborhood into a bounded set;

Further, if $X = Y$ we will say that $T\colon X \to X$ is ***nn-bounded*** if there exists a base $\mathcal{N}_0$ of zero neighborhoods such that for every $U \in \mathcal{N}_0$ there is a positive scalar $\alpha$ such that $T(U) \subseteq \alpha U$.

**Remark 2.1.2.** [Edw65] and [KN76] present (i) as the definition of a bounded operator on a topological vector space, while [RR64] and [Sch71] use (ii) for the same purpose. As we will see, these definitions are far from being equivalent.

**Proposition 2.1.3.** *Let $X$ and $Y$ be topological vector spaces. For an operator $T\colon X \to Y$ consider the following statements:*

*(i) $T$ is bb-bounded;*

*(ii) $T$ is continuous;*

*(iii) $T$ is nn-bounded;*

*(iv) $T$ is nb-bounded.*

*Then* (iv) $\Rightarrow$ (ii) $\Rightarrow$ (i). *Furthermore, if $X = Y$ then* (iv) $\Rightarrow$ (iii) $\Rightarrow$ (ii) $\Rightarrow$ (i).

*Proof.* The implications (iv) $\Rightarrow$ (ii) $\Rightarrow$ (i) are trivial. To show (iv) $\Rightarrow$ (iii) $\Rightarrow$ (ii) assume that $X = Y$ and fix a base $\mathcal{N}_0$ of zero neighborhoods. If $T$ is nb-bounded then $T(U)$ is bounded for some $U \in \mathcal{N}_0$. Note that $\widetilde{\mathcal{N}}_0 = \{\, \lambda U \cap V \;:\; \lambda > 0,\ V \in \mathcal{N}_0 \,\}$ is another base of zero neighborhoods. For each $W = \lambda U \cap V$ in $\widetilde{\mathcal{N}}_0$ we have $T(W) \subseteq \lambda T(U)$. But $T(U)$ is bounded and so $T(W) \subseteq \lambda T(U) \subseteq \lambda\alpha W$ for some positive $\alpha$, i.e., $T$ is nn-bounded.[1]

Finally, if $T$ is nn-bounded, then there is a base $\mathcal{N}_0$ such that for every zero neighborhood $U \in \mathcal{N}_0$ there is a positive scalar $\alpha$ such that $T(U) \subseteq \alpha U$. Let $V$ be an arbitrary zero neighborhood. Then there exists $U \in \mathcal{N}_0$ such that $U \subseteq V$, so that $T(U) \subseteq \alpha U \subseteq \alpha V$ for some $\alpha > 0$. Taking $W = \frac{1}{\alpha} U$ we get $T(W) \subseteq V$, hence $T$ is continuous. $\square$

**2.1.4.** It can be easily verified that if $T$ is an operator on a locally bounded space then all the statements in Lemma 2.1.3 are equivalent. In general, however, these notions are not equivalent. Obviously, the identity operator $I$ is always nn-bounded, continuous, and bb-bounded, but $I$ is nb-bounded if and only if the space is locally bounded. Every bb-bounded operator between two locally convex spaces is continuous if and only if the domain space is bornological. (Recall that a locally convex space is ***bornological*** if every balanced convex set absorbing every bounded set is a zero neighborhood, for details see [Sch71, RR64].)

[1] Note that if the topology is locally convex, then we can assume that $U$ is convex and $\mathcal{N}_0$ consists of convex neighborhoods. In this case $\widetilde{\mathcal{N}}_0$ also consists of convex neighborhoods.

**Example 2.1.5.** *A continuous but not nn-bounded operator.* Let $T$ be the left shift on the space of all real sequences $\mathbb{R}^{\mathbb{N}}$ with the topology of coordinate-wise convergence, i.e., $T\colon (x_0, x_1, x_2, \dots) \mapsto (x_1, x_2, x_3, \dots)$. Clearly $T$ is continuous. We will show that $T$ is not nn-bounded. Assume that for every zero neighborhood $U$ in some base $\mathcal{N}_0$ there is a positive scalar $\alpha$ such that $T(U) \subseteq \alpha U$. Since the set $\bigl\{x = (x_k) \,:\, |x_0| < 1\bigr\}$ is a zero neighborhood, there must be a base neighborhood $U \in \mathcal{N}_0$ such that $U \subseteq \{x : |x_0| < 1\}$. Since $T(U) \subseteq \alpha U$ for some positive $\alpha$ then $T^n(U) \subseteq \alpha^n U$, so that if $x = (x_k) \in U$ then $T^n x \in \alpha^n U$, so that $|x_n| = \bigl|(T^n x)_0\bigr| < \alpha^n$. Hence $U \subseteq \bigl\{x \,:\, |x_n| < \alpha^n \text{ for each } n > 0\bigr\}$. But this set is bounded, while the space is not locally bounded, a contradiction.

**2.1.6. Algebraic properties of bounded operators**. The sum of two bb-bounded operators is bb-bounded because the sum of two bounded sets in a topological vector space is bounded. Clearly the product of two bb-bounded operators is bb-bounded. It is well known that sums and products of continuous operators are continuous. Obviously, the product of two nn-bounded operators is nn-bounded, and it can be easily verified that the sum of two nn-bounded operators on a locally convex (or locally pseudo-convex) space is again nn-bounded. It is not difficult to see that the sum of two nb-bounded operator is nb-bounded. Indeed, suppose that $T_1$ and $T_2$ are two nb-bounded operators, then the sets $T_1(U_1)$ and $T_2(U_2)$ are bounded for some base zero neighborhoods $U_1$ and $U_2$. There exists another base zero neighborhood $U \subseteq U_1 \cap U_2$, then the sets $T_1(U)$ and $T_2(U)$ are bounded, so that $(T_1 + T_2)(U) \subseteq T_1(U) + T_2(U)$ is bounded. Finally, it is not difficult to see that the product of two nb-bounded operators is again nb-bounded. In fact, it follows immediately from Proposition 2.1.3 and the following simple observation: if we multiply an nb-bounded operator by a bb-bounded operator on the left or by an nn-bounded operator on the right, the product is nb-bounded.

Thus, the class of all bb-bounded operators, the class of all continuous operators, and the class of all nb-bounded operators are subalgebras of the algebra of all linear operators. The class of nn-bounded operators is an algebra provided that the space is locally (pseudo-)convex

### Boundedness in terms of convergence

Suppose $T\colon X \to Y$ is an operator between two topological vector spaces. It is well known that $T$ is continuous if and only if it maps convergent nets to convergent nets.

Notice that a subset of a topological vector space is unbounded if and only if it contains an unbounded sequence. Therefore, an operator is bb-bounded if and only if it maps bounded sequences (nets) to bounded sequences (respectively nets).

It is easy to see that $T$ is nn-bounded if and only if $T$ maps $U$-bounded ($U$-convergent to zero) sequences to $U$-bounded (respectively $U$-convergent to zero) sequences for every base zero neighborhood $U$ in some base of zero neighborhoods. We say that a net $(x_\gamma)$ is ***$U$-bounded*** if it is contained in $\alpha U$ for some $\alpha > 0$, and $x_\gamma \xrightarrow{U} 0$ if for every $\alpha > 0$ there exits $\gamma_0$ such that $x_\gamma \in \alpha U$ whenever $\gamma > \gamma_0$.

**2.1.7.** Suppose $T$ is nb-bounded, then $T(U)$ is bounded for some zero neighborhood $U$. Obviously $x_\gamma \xrightarrow{U} 0$ implies $Tx_\gamma \to 0$. The converse implication is also valid: if $T$ maps $U$-convergent sequences to convergent sequences, then $T$ has to be nb-bounded and the set $T(U)$ is bounded. Indeed, if $T(U)$ is unbounded, then there is a zero neighborhood $V$ in $Y$ such that $V$ does not absorb $T(U)$. Then for every $n \geqslant 1$ there exists $y_n \in T(U) \setminus nV$. Suppose $y_n = Tx_n$ for some $x_n \in U$, then $\frac{x_n}{n} \xrightarrow{U} 0$, but $T(\frac{x_n}{n}) = \frac{y_n}{n} \notin V$, so that $T(\frac{x_n}{n})$ does not converge to zero.

### Normed, quasi-normed, and semi-normed spaces

Next, we discuss bounded operators in some particular topologies. Notice that every normed, semi-normed, or quasi-normed vector space is locally bounded. Therefore bb-boundedness, continuity, nn-boundedness and nb-boundedness coincide for operators on such spaces.

**Locally convex topology**

**Definition 2.1.8.** Let $T$ be an operator on a semi-normed vector space $(X,p)$. As in the case with normed spaces, $p$ generates an ***operator semi-norm*** $p(T)$ defined by

$$p(T) = \sup_{p(x)\neq 0} \frac{p(Tx)}{p(x)}.$$

More generally, let $S\colon X \to Y$ be a linear operator between two seminormed spaces $(X,p)$ and $(Y,q)$. Then we define a ***mixed operator seminorm*** associated with $p$ and $q$ via

$$\mathfrak{m}_{pq}(S) = \sup_{p(x)\neq 0} \frac{q(Sx)}{p(x)}.$$

The semi-norm $\mathfrak{m}_{pq}(S)$ is a measure of how far in the seminorm $q$ the points of the $p$-unit ball can go under $S$. Notice, that $p(T)$ and $\mathfrak{m}_{pq}(S)$ may be infinite. Clearly, if $T$ is an operator on a semi-normed space $(X,p)$, then $\mathfrak{m}_{pp}(T) = p(T)$.

**Lemma 2.1.9.** *If $S\colon X \to Y$ is an operator between two seminormed spaces $(X,p)$ and $(Y,q)$, then*

*(i)* $\mathfrak{m}_{pq}(S) = \sup\limits_{p(x)=1} q(Sx) = \sup\limits_{p(x)\leqslant 1} q(Sx)$;

*(ii)* $q(Sx) \leqslant \mathfrak{m}_{pq}(S)p(x)$ *whenever* $\mathfrak{m}_{pq}(S) < \infty$.

*Proof.* The first equality in (i) follows immediately from the definition of $p(T)$. We obviously have

$$\sup_{p(x)=1} q(Sx) \leqslant \sup_{p(x)\leqslant 1} q(Sx).$$

In order to prove the opposite inequality, notice that if $0 < p(x) \leqslant 1$, then $q(Sx) \leqslant \frac{q(Sx)}{p(x)} \leqslant \mathfrak{m}_{pq}(S)$. Thus, it is left to show that $p(x) = 0$ implies $q(Sx) \leqslant \mathfrak{m}_{pq}(S)$. Pick any $z$ with $p(z) > 0$, then

$$p(\tfrac{z}{n}) = p(x + \tfrac{z}{n} - x) \leqslant p(x + \tfrac{z}{n}) + p(x) = p(x + \tfrac{z}{n}) \leqslant p(x) + p(\tfrac{z}{n}) = \tfrac{p(z)}{n},$$

so that $p(x+\frac{z}{n})=p(\frac{z}{n})\in(0,1)$ for $n>p(z)$. Further, since $Sx+\frac{Sz}{n}$ converges to $Sx$ we have

$$q(Sx)=\lim_{n\to\infty}q(Sx+\tfrac{Sz}{n})\leqslant\lim_{n\to\infty}\frac{q\left(S(x+\frac{z}{n})\right)}{p(x+\frac{z}{n})}\leqslant\mathfrak{m}_{pq}(S).$$

Finally, (ii) follows directly from the definition if $p(x)\neq 0$. In the case when $p(x)=0$, again pick any $z$ with $p(z)>0$, then $p(x+\frac{z}{n})\neq 0$ and

$$q(Sx)=\lim_{n\to\infty}q(Sx+\tfrac{Sz}{n})=\lim_{n\to\infty}q\left(S(x+\tfrac{z}{n})\right)\leqslant\lim_{n\to\infty}\mathfrak{m}_{pq}(S)p(x+\tfrac{z}{n})=0.$$

□

**Corollary 2.1.10.** *If $T$ is an operator on a seminormed space $(X,p)$, then*

*(i)* $p(T)=\sup\limits_{p(x)=1}p(Tx)=\sup\limits_{p(x)\leqslant 1}p(Tx)$;

*(ii)* $p(Tx)\leqslant p(T)p(x)$ *whenever* $p(T)<\infty$.

The following propositions characterize continuity, nn-boundedness, and nb-boundedness of operators on locally convex spaces in terms of operator seminorms. We assume that $X$ and $Y$ are locally convex spaces with generating families of seminorms $\mathcal{P}$ and $\mathcal{Q}$ respectively.

**Proposition 2.1.11.** *Let $S$ be an operator from $X$ to $Y$, then $S$ is continuous if and only if for every $q\in\mathcal{Q}$ there exists $p\in\mathcal{P}$ such that $\mathfrak{m}_{pq}(S)$ is finite.*

**Proposition 2.1.12.** *An operator $T$ on $X$ is nn-bounded if and only if $p(T)$ is finite for every $p\in\mathcal{P}$, or, equivalently, if $T$ maps $p$-bounded sets to $p$-bounded sets for every $p$ in some generating family $\mathcal{P}$ of seminorms.*

**Proposition 2.1.13.** *Let $S\colon X\to Y$ be a linear operator, then the following are equivalent:*

*(i)* *$S$ is nb-bounded;*

*(ii)* *$S$ maps $p$-bounded sets into bounded sets for some $p\in\mathcal{P}$;*

*(iii)* *There exists $p\in\mathcal{P}$ such that $\mathfrak{m}_{pq}(S)<\infty$ for every $q\in\mathcal{Q}$.*

**Question.** *Is there a similar characterization of bb-boundedness?*

### Operator topologies

For each of the five classes of operators, we introduce an appropriate natural operator topology. The class of all linear operators between two topological vector spaces will be usually equipped with the ***strong*** operator topology. Recall, that a sequence $(S_n)$ of operators from $X$ to $Y$ is said to ***converge strongly*** or ***pointwise*** to a map $S$ if $S_n x \to Sx$ for every $x \in X$. Clearly, $S$ will also be a linear operator.

The class of all bb-bounded operators will usually be equipped with the topology of ***uniform convergence on bounded sets***. Recall, that a sequence $(S_n)$ of operators is said to converge to zero ***uniformly*** on $A$ if for each zero neighborhood $V$ in $Y$ there exists an index $n_0$ such that $S_n(A) \subseteq V$ for all $n > n_0$. We say that $(S_n)$ converges to $S$ uniformly on bounded sets if $(S_n - S)$ converges to zero uniformly on bounded sets. Recall also that a family $\mathcal{G}$ of operators is called ***uniformly bounded*** on a set $A \subseteq X$ if the set $\bigcup_{S \in \mathcal{G}} S(A)$ is bounded in $Y$.

**Lemma 2.1.14.** *If a sequence $(S_n)$ of bb-bounded operators converges uniformly on bounded sets to an operator $S$, then $S$ is also bb-bounded.*

*Proof.* Fix a bounded set $A$. Since $S - S_n$ converges to zero uniformly on bounded sets then for every base zero neighborhood $V$ there exists an index $n_0$ such that $(S_n - S)(A) \subseteq V$ whenever $n \geqslant n_0$. This yields $S(A) \subseteq S_n(A) + V \subseteq \gamma V$ since $S_n(A)$ is bounded. Thus, $S(A)$ is bounded for every bounded set $A$, so that $S$ is bb-bounded. □

The class of all continuous operators will be usually equipped with the topology of ***equicontinuous convergence***. Recall, that a family $\mathcal{G}$ of operators from $X$ to $Y$ is called ***equicontinuous*** if for each zero neighborhood $V$ in $Y$ there is a zero neighborhood $U$ in $X$ such that $S(U) \subseteq V$ for every $S \in \mathcal{G}$. We say that a sequence $(S_n)$ converges to zero ***equicontinuously*** if for each zero neighborhood $V$ in $Y$ there is a zero neighborhood $U$ in $X$ such that for every $\varepsilon > 0$ there exists an index $n_0$ such that $S_n(U) \subseteq \varepsilon V$ for all $n > n_0$.

**Lemma 2.1.15.** *If a sequence $S_n$ of continuous operators converges equicontinuously to $S$, then $S$ is also continuous.*

*Proof.* Fix a zero neighborhood $V$, there exist zero neighborhoods $V_1$ and $U$ and an index $n_0$ such that $V_1 + V_1 \subseteq V$ and $(S_n - S)(U) \subseteq V_1$ whenever $n > n_0$. Fix $n > n_0$. The continuity of $S_n$ guarantees that there exists a zero neighborhood $W \subseteq U$ such that $S_n(W) \subseteq V_1$. Since $(S_n - S)(W) \subseteq V_1$, we get $S(W) \subseteq S_n(W) + V_1 \subseteq V_1 + V_1 \subseteq V$, which shows that $S$ is continuous. $\square$

The class of all nn-bounded operators will be usually equipped with the topology of ***nn-convergence***, defined as follows. We will call a collection $\mathcal{G}$ of operators ***uniformly nn-bounded*** if there exists a base $\mathcal{N}_0$ of zero neighborhoods such that for every $U \in \mathcal{N}_0$ there exists a positive real $\beta$ such that $S(U) \subseteq \beta U$ for each $S \in \mathcal{G}$. We say that a sequence $(S_n)$ ***nn-converges*** to zero if there is a base $\mathcal{N}_0$ of zero neighborhoods such that for every $U \in \mathcal{N}_0$ and every $\varepsilon > 0$ we have $S_n(U) \subseteq \varepsilon U$ for all sufficiently large $n$.

**Question.** *Is the class of all nn-bounded operators closed relative to nn-convergence?*

Finally, the class of all nb-bounded operators will be usually equipped with the topology of uniform convergence on a zero neighborhood.

**Example 2.1.16.** *The class of nb-bounded operators is not closed in the topology of uniform convergence on a zero neighborhood.* Let $X = \mathbb{R}^{\mathbb{N}}$, the space of all real sequences with the topology of coordinate-wise convergence. Let $T_n$ be the projection on the first $n$ components. Clearly, every $T_n$ is nb-bounded because it maps the zero neighborhood $U_n = \big\{(x_i)_{i=1}^{\infty} \;:\; |x_i| < 1 \text{ for } i = 1, \dots, n\big\}$ to a bounded set. On the other hand, $(T_n)$ converges uniformly on $X$ to the identity operator, while the identity operator on $X$ is not nb-bounded.

## 2.2 Spectra of an operator

Recall that if $T$ is a continuous operator on a Banach space, then its ***resolvent set*** $\rho(T)$ is the set of all $\lambda \in \mathbb{C}$ such that the ***resolvent operator*** $R_\lambda = (\lambda I - T)^{-1}$ exists, while the

***spectrum*** of $T$ is defined by $\sigma(T) = \mathbb{C} \setminus \rho(T)$. The Open Mapping Theorem guarantees that if $R_\lambda$ exists then it is automatically continuous. Now, if $T$ is an operator on an arbitrary topological vector space and $\lambda \in \mathbb{C}$ then the algebraic inverse $R_\lambda = (\lambda I - T)^{-1}$ may exist but not be continuous, or may be continuous but not nb-bounded, etc. In order to treat all these cases properly we introduce the following definitions.

**Definition 2.2.1.** Let $T$ be a linear operator on a topological vector space. We denote the set of all scalars $\lambda \in \mathbb{C}$ for which $\lambda I - T$ is invertible in the algebra of linear operators by $\rho^l(T)$. We say that $\lambda \in \rho^{bb}(T)$ (respectively $\rho^c(T)$ or $\rho^{nn}(T)$) if the inverse of $\lambda I - T$ is bb-bounded (respectively continuous or nn-bounded). Finally, we say that $\lambda \in \rho^{nb}(T)$ if the inverse of $\lambda I - T$ belongs to the unitalization of the algebra of nb-bounded operators, i.e., when $(\lambda I - T)^{-1} = \alpha I + S$ for a scalar $\alpha$ and an nb-bounded operator $S$.

The spectral sets $\sigma^l(T)$, $\sigma^{bb}(T)$, $\sigma^c(T)$, $\sigma^{nn}(T)$, and $\sigma^{nb}(T)$ are defined to be the complements of the resolvent sets $\rho^l(T)$, $\rho^{bb}(T)$, $\rho^c(T)$, $\rho^{nn}(T)$, and $\rho^{nb}(T)$ respectively.[2] We will denote the (left and right) inverse of $\lambda I - T$ whenever it exists by $R_\lambda$.

**2.2.2.** It follows immediately from Proposition 2.1.3 that $\sigma^l(T) \subseteq \sigma^{bb}(T) \subseteq \sigma^c(T) \subseteq \sigma^{nn}(T) \subseteq \sigma^{nb}(T)$. It follows from the Open Mapping Theorem that for a continuous operator $T$ on a Banach space all the introduced spectra coincide with the usual spectrum $\sigma(T)$. Since the Open Mapping Theorem is still valid on Fréchet spaces, we have $\sigma^l(T) = \sigma^{bb}(T) = \sigma^c(T)$ for a continuous operator $T$ on a Fréchet space.

**2.2.3.** If $T$ is an operator on a locally bounded space $(X, U)$, then by 2.1.4 bb-boundedness of $T$ is equivalent to nb-boundedness, so that $\sigma^{bb}(T) = \sigma^c(T) = \sigma^{nn}(T) = \sigma^{nb}(T)$. We will denote this set by $\sigma_U(T)$ to avoid ambiguity. Spectral theory of continuous operators on quasi-Banach spaces was developed in [Gram66].

**2.2.4.** There are several reasons why we define $\sigma^{nb}$ in a slightly different fashion than the other spectra. Namely, for $\lambda$ to be in $\rho^{nb}(T)$ we require $(\lambda I - T)^{-1}$ be not just nb-bounded, but be nb-bounded up to a multiple of the identity operator. On one hand,

[2] We use superscripts in order to avoid confusion with $\sigma_c(T)$, which is commonly used for continuous spectrum.

this is the standard way to define the spectrum of an element in a non-unital algebra, and we know that the algebra of nb-bounded operators is unital only when the space is locally bounded. On the other hand, if we defined $\rho^{nb}(T)$ as the set of all $\lambda \in \mathbb{C}$ for which $(\lambda I - T)^{-1}$ is nb-bounded, then we wouldn't have gotten any deep theory because $(\lambda I - T)^{-1}$ is almost never nb-bounded when the space is not locally bounded.

Indeed, suppose that $X$ is not locally bounded, $T$ is a bb-bounded operator on $X$, and $\lambda \in \mathbb{C}$. Then $R_\lambda = (\lambda I - T)^{-1}$ cannot be nb-bounded, because in this case $I = (\lambda I - T)R_\lambda$ would be nb-bounded by 2.1.6 as a product of a bb-bounded and an nb-bounded operators. But we know that $I$ is not nb-bounded because $X$ is not locally bounded.

We will see in Proposition 2.5.3 that in a locally convex but non locally bounded space nb-bounded operators are never invertible, which implies that in such spaces $(\lambda I - T)^{-1}$ is not nb-bounded for any linear operator $T$.

**2.2.5.** Next, let $T$ be a (norm) continuous operator on a Banach space, $\sigma(T)$ the usual spectrum of $T$, and let $\sigma^l(T)$, $\sigma^{bb}(T)$, $\sigma^c(T)$ be computed with respect to the weak topology. It is known that an operator on a Banach space is weak-to-weak continuous if and only if it is norm-to-norm continuous; therefore it follows that $\sigma^c(T) = \sigma(T)$. Furthermore, $\sigma^l(T)$ does not depend on the topology, so that it also coincides with $\sigma(T)$. Thus $\sigma^l(T) = \sigma^{bb}(T) = \sigma^c(T) = \sigma(T)$.

We would like to mention that our definition of spectra of an operator on a topological vector space is different from the one of Waelbroeck in [Wael54].

## 2.3 Spectral radii of an operator

The spectral radius of a bounded linear operator $T$ on a Banach space is usually defined via the Gelfand formula $r(T) = \lim\limits_{n\to\infty} \sqrt[n]{\|T^n\|}$. The formula involves a norm and so makes no sense in a general topological vector space. Fortunately, this formula can be rewritten without using a norm, and then generalized to topological vector spaces. Similarly to the situation with spectra, this generalization can be done in several ways, so that we will obtain various types of spectral radii for an operator on a topological vector space. We will show later that, as with the Banach space case, there are some relations between the spectral radii, the radii of the spectra, and the convergence of the Neumann series of an operator on a locally convex topological vector space. The content of this section may look technical at the beginning, but later on the reader will see that all the facts lead to a simple and natural classification. We start with an almost obvious numerical lemma.

**Lemma 2.3.1.** *If* $(t_n)$ *is a sequence in* $\mathbb{R}^+ \cup \{\infty\}$, *then*

$$\limsup_{n\to\infty} \sqrt[n]{t_n} = \inf\{\, \nu > 0 \;:\; \lim_{n\to\infty} \tfrac{t_n}{\nu^n} = 0 \,\} = \inf\{\, \nu > 0 \;:\; \limsup_{n\to\infty} \tfrac{t_n}{\nu^n} < \infty \,\}.$$

*Proof.* Suppose $\limsup\limits_{n\to\infty} \sqrt[n]{t_n} = r$. If $0 < \nu < r$, then $\sqrt[n_k]{t_{n_k}} > \mu > \nu$ for some $\mu$ and some subsequence $(t_{n_k})$, so that $\frac{t_{n_k}}{\nu^{n_k}} > \frac{\mu^{n_k}}{\nu^{n_k}} \to \infty$ as $k$ goes to infinity. It follows that $\limsup \lim_{n\to\infty} \frac{t_n}{\nu^n} = \infty$. On the other hand, if $r$ is finite and $\nu > r$ then $\sqrt[n]{t_n} < \mu < \nu$ for some $\mu$ and for all sufficiently large $n$. Then $\lim\limits_{n\to\infty} \frac{t_n}{\nu^n} \leqslant \lim\limits_{n\to\infty} \frac{\mu^n}{\nu^n} = 0$. □

This lemma implies that the spectral radius $r(T)$ of a (norm) continuous operator $T$ on a Banach space equals the infimum of all positive real scalars $\nu$ such that the sequence $\left(\frac{T^n}{\nu^n}\right)$ converges to zero (or just is bounded) in operator norm topology. This can be considered as an alternative definition of the spectral radius, and can be generalized to any topological vector space. Since for each of the five considered classes of operators on topological vector spaces we introduced appropriate concepts of convergent and bounded sequences, we arrive to the following definition.

**Definition 2.3.2.** Given a linear operator $T$ on a topological vector space $X$, define the following numbers:

$$\begin{aligned}
r_l(T) &= \inf\{\, \nu > 0 \;:\; \text{the sequence } \left(\tfrac{T^n}{\nu^n}\right) \text{ converges strongly to zero} \,\}; \\
r_{bb}(T) &= \inf\{\, \nu > 0 \;:\; \tfrac{T^n}{\nu^n} \to 0 \text{ uniformly on every bounded set} \,\}; \\
r_c(T) &= \inf\{\, \nu > 0 \;:\; \tfrac{T^n}{\nu^n} \to 0 \text{ equicontinuously } \}; \\
r_{nn}(T) &= \inf\{\, \nu > 0 \;:\; \left(\tfrac{T^n}{\nu^n}\right) \text{ nn-converges to zero}\}; \\
r_{nb}(T) &= \inf\{\, \nu > 0 \;:\; \tfrac{T^n}{\nu^n} \to 0 \text{ uniformly on some 0-neighborhood} \,\}.
\end{aligned}$$

The following proposition explains the relations between the introduced radii.

**Proposition 2.3.3.** *If $T$ is a linear operator on a topological vector space $X$, then $r_l(T) \leqslant r_{bb}(T) \leqslant r_c(T) \leqslant r_{nn}(T) \leqslant r_{nb}(T)$.*

*Proof.* Let $T$ be a linear operator on a topological vector space $X$. Since every singleton is bounded then $r_l(T) \leqslant r_{bb}(T)$. Next, assume $\nu > r_c(T)$, fix $\mu$ such that $r_c(T) < \mu < \nu$, then the sequence $(\frac{T^n}{\mu^n})$ converges to zero equicontinuously. Take a bounded set $A$ and a zero neighborhood $U$. There exists a zero neighborhood $V$ and a positive integer $N$ such that $\frac{T^n}{\mu^n}(V) \subseteq U$ whenever $n \geqslant N$. Also, $A \subseteq \alpha V$ for some $\alpha > 0$, so that $\frac{T^n}{\nu^n}(A) \subseteq \frac{\mu^n}{\nu^n}\frac{T^n}{\mu^n}(\alpha V) \subseteq \frac{\mu^n \alpha}{\nu^n} U \subseteq U$ for all sufficiently large $n$. It follows that the sequence $(\frac{T^n}{\nu^n})$ converges to zero uniformly on $A$ and, therefore, $\nu \geqslant r_{bb}(T)$. Thus, $r_{bb}(T) \leqslant r_c(T)$.

To prove the inequality $r_c(T) \leqslant r_{nn}(T)$ we let $\nu > r_{nn}(T)$. Then for some base $\mathcal{N}_0$ of zero neighborhoods and for every $V \in \mathcal{N}_0$ and $\varepsilon > 0$ there exists a positive integer $N$ such that $\frac{T^n}{\nu^n}(V) \subseteq \varepsilon V$ for every $n \geqslant N$. Given a zero neighborhood $U$, we can find $V \in \mathcal{N}_0$ such that $V \subseteq U$. Then $\frac{T^n}{\nu^n}(V) \subseteq \varepsilon V \subseteq \varepsilon U$ for every $n \geqslant N$, so that the sequence $(\frac{T^n}{\nu^n})$ converges to zero equicontinuously, and, therefore, $\nu \geqslant r_c(T)$.

Finally, we must show that $r_{nn}(T) \leqslant r_{nb}(T)$. Suppose that $\nu > r_{nb}(T)$, we claim that $\nu \geqslant r_{nn}(T)$. Take $\mu$ so that $\nu > \mu > r_{nb}(T)$. One can find a zero neighborhood $U$ such that for every zero neighborhood $V$ there is a positive integer $N$ such that $\frac{T^n}{\mu^n}(U) \subseteq V$ for every $n \geqslant N$. Fix a base $\mathcal{N}_0$ of zero neighborhoods, and define a new base $\widetilde{\mathcal{N}_0}$ of zero neighborhoods via $\widetilde{\mathcal{N}_0} = \{mU \cap W \ : \ m \in \mathbb{N}, W \in \mathcal{N}_0\}$. Let $V \in \widetilde{\mathcal{N}_0}$ and $\varepsilon > 0$. Then $V = mU \cap W$ for some positive integer $m$ and $W \in \mathcal{N}_0$. Then $\frac{T^n}{\mu^n}(V) \subseteq m\frac{T^n}{\mu^n}(U) \subseteq mV$ and for every sufficiently large $n$, so that $\frac{T^n}{\nu^n}(V) \subseteq \frac{\mu^n}{\nu^n} mV \subseteq \varepsilon V$, for each sufficiently large $n$, which implies $\nu \geqslant r_{nn}(T)$. □

The following lemma shows that, similarly to the case of Banach spaces, one can use boundedness instead of convergence when defining the spectral radii of an operator on a topological vector space. This gives alternative ways of computing the radii, which are often more convenient.

**Lemma 2.3.4.** *Let $T$ be a linear operator on a topological vector space, then* *(i)* $r_l(T) = \inf\big\{\, \nu > 0 \ :\ \big(\frac{T^n x}{\nu^n}\big) \text{ is bounded for every } x \,\big\}$;

*(ii) if $T$ is bb-bounded then*

$r_{bb}(T) = \inf\big\{\, \nu > 0 \ :\ \big(\frac{T^n}{\nu^n}\big) \text{ is uniformly bounded on every bounded set} \,\big\}$;

*(iii) if $T$ is continuous then*

$r_c(T) = \inf\big\{\, \nu > 0 \ :\ \big(\frac{T^n}{\nu^n}\big) \text{ is equicontinuous} \,\big\}$;

*(iv) if $T$ is nn-bounded then*

$r_{nn}(T) = \inf\big\{\, \nu > 0 \ :\ \big(\frac{T^n}{\nu^n}\big) \text{ is uniformly nn-bounded} \,\big\}$;

*(v) if $T$ is nb-bounded then*

$r_{nb}(T) = \inf\big\{\, \nu > 0 \ :\ \big(\frac{T^n}{\nu^n}\big) \text{ is uniformly bounded on some 0-neighborhood} \,\big\}$.

*Moreover, in each of these cases it suffices to consider any tail of the sequence $\big(\frac{T^n}{\nu^n}\big)$ instead of the whole sequence.*

*Proof.* To prove (i) let

$$r'_l(T) = \inf\big\{\, \nu > 0 \ :\ \big(\tfrac{T^n}{\nu^n}x\big) \text{ is bounded for every } x \,\big\}.$$

Since every convergent sequence is bounded, we certainly have $r_l(T) \geqslant r'_l(T)$. Conversely, suppose $\nu > r'_l(T)$, and take any positive scalar $\mu$ such that $\nu > \mu > r'_l(T)$. Then for every $x \in X$ the sequence $\frac{T^n}{\mu^n}x$ is bounded, and it follows that the sequence $\frac{T^n x}{\nu^n} = \frac{\mu^n}{\nu^n}\frac{T^n x}{\mu^n}$ converges to zero, so that $\nu \geqslant r_l(T)$ and, therefore $r'_l(T) \geqslant r_l(T)$.

To prove (ii), suppose $T$ is bb-bounded, and let

$$r'_{bb}(T) = \inf\big\{\, \nu > 0 \ :\ \big(\tfrac{T^n}{\nu^n}\big) \text{ is uniformly bounded on every bounded set} \,\big\}.$$

We'll show that $r'_{bb}(T) = r_{bb}(T)$. If $\big(\frac{T^n}{\nu^n}\big)$ converges to zero uniformly on every bounded set, then for each bounded set $A$ and for each zero neighborhood $U$ there exists a positive integer $N$ such that $\frac{T^n}{\nu^n}(A) \subseteq U$ whenever $n \geqslant N$. Also, since $T$ is bb-bounded, then for every $n < N$ we have $\frac{T^n}{\nu^n}(A) \subseteq \alpha_n U$ for some $\alpha_n > 0$. Therefore, if $\alpha = \max\{\alpha_1, \dots, \alpha_{N-1}, 1\}$, then $\frac{T^n}{\nu^n}(A) \subseteq \alpha U$ for every $n$, so that the sequence $\frac{T^n}{\nu^n}$ is uniformly bounded on $A$. Thus $\nu \geqslant r'_{bb}(T)$, so that $r'_{bb}(T) \leqslant r_{bb}(T)$.

Now suppose $\nu > r'_{bb}(T)$. There exists $\mu$ such that $\nu > \mu > r'_{bb}(T)$. The set $\bigcup_{n=1}^{\infty} \frac{T^n}{\mu^n}(A)$ is bounded for every bounded set $A$, so that for every zero neighborhood $U$ there exists a scalar $\alpha$ such that $\frac{T^n}{\mu^n}(A) \subseteq \alpha U$ for every $n \in \mathbb{N}$. Then $\frac{T^n}{\nu^n}(A) \subseteq \frac{\mu^n \alpha}{\nu^n} U \subseteq U$ for all sufficiently large $n$. This means that the sequence $\left(\frac{T^n}{\nu^n}\right)$ converges to zero uniformly on $A$, and it follows that $\nu \geqslant r_{bb}(T)$.

Further, if $T$ is bb-bounded, then any finite initial segment $(\frac{T^n}{\nu^n})_{n=0}^{N}$ is always uniformly bounded on every bounded set, so that a tail $(\frac{T^n}{\nu^n})_{n=N}^{\infty}$ is uniformly bounded on every bounded set if and only if the whole sequence $(\frac{T^n}{\nu^n})_{n=0}^{\infty}$ is uniformly bounded on every bounded set.

The statements (iii), (iv), and (v) can be proved in a similar way. $\square$

**2.3.5. Locally bounded spaces.** If $T$ is a linear operator on a locally bounded topological vector space $(X, U)$, then it follows directly from Definition 2.3.2 that $r_{bb}(T) = r_c(T) = r_{nn}(T) = r_{nb}(T)$, because the corresponding convergences are equivalent. In this case we would denote each of these radii by $r_U(T)$.

### Spectral radii via seminorms

The following proposition provides formulas for computing spectral radii of an operator on a locally convex space in terms of seminorms.

**Proposition 2.3.6.** *If $T$ is an operator on a locally convex space $X$ with a generating family of seminorms $\mathcal{P}$, then*

*(i)* $r_l(T) = \sup\limits_{p \in \mathcal{P},\ x \in X} \limsup\limits_{n \to \infty} \sqrt[n]{p(T^n x)}$;

*(ii)* $r_c(T) = \sup\limits_{q \in \mathcal{P}} \inf\limits_{p \in \mathcal{P}} \limsup\limits_{n \to \infty} \sqrt[n]{\mathfrak{m}_{pq}(T^n)}$;

*(iii)* $r_{nn}(T) = \inf\limits_{\mathcal{Q}} \sup\limits_{p \in \mathcal{Q}} \limsup\limits_{n \to \infty} \sqrt[n]{p(T^n)}$, *where the infimum is taken over all generating families of seminorms;*

*(iv)* $r_{nb}(T) = \inf\limits_{p \in \mathcal{P}} \sup\limits_{q \in \mathcal{P}} \limsup\limits_{n \to \infty} \sqrt[n]{\mathfrak{m}_{pq}(T^n)}$;

*Proof.* It follows from the definition of $r_l(T)$ and Lemma 2.3.1 that

$$r_l(T) = \inf\bigl\{\, \nu > 0 \;:\; \lim_{n\to\infty} p\bigl(\tfrac{T^n x}{\nu^n}\bigr) = 0 \text{ for every } x \in X, p \in \mathcal{P} \,\bigr\}$$
$$= \sup_{x\in X,\ p\in\mathcal{P}} \inf\bigl\{\, \nu > 0 \;:\; \lim_{n\to\infty} \tfrac{p(T^n x)}{\nu^n} = 0 \,\bigr\} = \sup_{x\in X,\ p\in\mathcal{P}} \limsup_{n\to\infty} \sqrt[n]{p(T^n x)}.$$

Let $U_p = \{\, x \in X \;:\; p(x) < 1 \,\}$ for every $p \in \mathcal{P}$. Then, rephrasing the definition of $r_c(T)$ and applying Lemma 2.3.1, we have

$$r_c(T) = \inf\bigl\{\, \nu > 0 \;:\; \forall q \in \mathcal{P}\ \exists p \in \mathcal{P}\ \forall \varepsilon > 0\ \exists N \in \mathbb{N}\ \forall n \geqslant N\ \tfrac{T^n}{\nu^n}(U_p) \subseteq \varepsilon U_q \,\bigr\}$$
$$= \sup_{q\in\mathcal{P}} \inf_{p\in\mathcal{P}} \inf\bigl\{\, \nu > 0 \;:\; \forall \varepsilon > 0\ \exists N \in \mathbb{N}\ \forall n \geqslant N\ \mathfrak{m}_{pq}\bigl(\tfrac{T^n}{\nu^n}\bigr) < \varepsilon \,\bigr\}$$
$$= \sup_{q\in\mathcal{P}} \inf_{p\in\mathcal{P}} \inf\bigl\{\, \nu > 0 \;:\; \lim_{n\to\infty} \tfrac{\mathfrak{m}_{pq}(T^n)}{\nu^n} = 0 \,\bigr\} = \sup_{q\in\mathcal{P}} \inf_{p\in\mathcal{P}} \limsup_{n\to\infty} \sqrt[n]{\mathfrak{m}_{pq}(T^n)}.$$

Similarly,

$$r_{nn}(T) = \inf\bigl\{\, \nu > 0 \;:\; \exists \mathcal{Q}\ \forall p \in \mathcal{Q}\ \forall \varepsilon > 0\ \exists N \in \mathbb{N}\ \forall n > N\ \tfrac{T^n}{\nu^n}(U_p) \subseteq \varepsilon U_p \,\bigr\}$$
$$= \inf_{\mathcal{Q}} \sup_{p\in\mathcal{Q}} \inf\bigl\{\, \nu > 0 \;:\; \lim_{n\to\infty} \tfrac{p(T^n)}{\nu^n} = 0 \,\bigr\} = \inf_{\mathcal{Q}} \sup_{p\in\mathcal{Q}} \limsup_{n\to\infty} \sqrt[n]{p(T^n)}.$$

Finally,

$$r_{nb}(T) = \inf\bigl\{\, \nu > 0 \;:\; \exists p \in \mathcal{P}\ \forall q \in \mathcal{P}\ \exists N \in \mathbb{N}\ \forall n > N\ \tfrac{T^n}{\nu^n}(U_p) \subseteq U_q \,\bigr\}$$
$$= \inf_{p\in\mathcal{P}} \sup_{q\in\mathcal{P}} \inf\bigl\{\, \nu > 0 \;:\; \limsup_{n\to\infty} \tfrac{\mathfrak{m}_{pq}(T^n)}{\nu^n} \leqslant 1 \,\bigr\} = \inf_{p\in\mathcal{P}} \sup_{q\in\mathcal{P}} \limsup_{n\to\infty} \sqrt[n]{\mathfrak{m}_{pq}(T^n)}.$$

□

**Some special properties of $r_c(T)$**

Continuity of an operator can be characterized in terms of neighborhoods (the preimage of every neighborhood contains a neighborhood) or, alternatively, in terms of convergence (every convergent net is mapped to a convergent net). Analogously, though defined in terms of neighborhoods, $r_c(T)$ can also be characterized in terms of convergent nets. This approach was used by F. Garibay and R. Vera in a series of papers [GV97, GV98, VM97]. Recall that a net $(x_\alpha)$ in a topological vector space is said to be ***ultimately***

***bounded*** if every zero neighborhood absorbs some tail of the net, i.e., for every zero neighborhood $V$ one can find an index $\alpha_0$ and a positive real $\delta > 0$ such that $x_\alpha \in \delta V$ whenever $\alpha > \alpha_0$. As far as we know, ultimately bounded sequences were first studied in [DeV71] for certain locally-convex topologies. The relationship between ultimately bounded nets and convergence of sequences of operators on locally convex spaces was studied in [GV97, GV98, VM97]. The following proposition (which is, in fact, a version of [VM97, Corollary 2.14]) shows how $r_c(T)$ can be characterized in terms of the action of powers of $T$ on ultimately bounded sequences. It also implies that $r_c(T)$ coincides with the number $\gamma(T)$ which was introduced in [GV97, GV98, VM97] for a continuous operator on locally convex spaces.

**Proposition 2.3.7.** *Let $T$ be a linear operator on a topological vector space $X$, then*

$$r_c(T) = \inf\Bigl\{\nu > 0 \,:\, \lim_{n,\alpha} \tfrac{T^n}{\nu^n} x_\alpha = 0 \textit{ whenever } (x_\alpha) \textit{ is ultimately bounded}\Bigr\}$$

$$= \inf\Bigl\{\nu > 0 \,:\, \bigl(\tfrac{T^n}{\nu^n} x_\alpha\bigr)_{n,\alpha} \textit{ is ultimately bounded whenever } (x_\alpha) \textit{ is ultimately bounded}\Bigr\}.$$

*Proof.* To prove the first equality it suffices to show that $r_c(T) < 1$ if and only if $\lim_{n,\alpha} T^n x_\alpha = 0$ whenever $(x_\alpha)$ is an ultimately bounded net. Suppose that $r_c(T) < 1$, and let $V$ be a zero neighborhood. One can find a zero neighborhood $U$ such that for every $\varepsilon > 0$ there exists $n_0 \in \mathbb{N}$ such that $T^n(U) \subseteq \varepsilon V$ for each $n > n_0$. Let $(x_\alpha)$ be an ultimately bounded net. There exists an index $\alpha_0$ and a number $\delta > 0$ such that $x_\alpha \in \delta U$ whenever $\alpha > \alpha_0$. Then for $\varepsilon = \delta^{-1}$ one can find $n_0$ such that $T^n(U) \subseteq \delta^{-1} V$ for each $n > n_0$, so that $T^n x_\alpha \in \delta T^n(U) \subseteq V$ whenever $\alpha > \alpha_0$ and $n > n_0$. This means that $\lim_{n,\alpha} T^n x_\alpha = 0$.

Conversely, suppose that $\lim_{n,\alpha} T^n x_\alpha = 0$ for each ultimately bounded net $(x_\alpha)$, and assume that $T^n$ does not converge equicontinuously to zero. Then there exists a zero neighborhood $V$ such that for every zero neighborhood $U$ one can find $\varepsilon_U$ such that for every $m \in \mathbb{N}$ there exists $n_{U,m} > m$ with $T^{n_{U,m}}(U) \nsubseteq \varepsilon_U V$. Then there exists $x_{U,m} \in U$ such that

$$T^{n_{U,m}} x_{U,m} \notin \varepsilon_U V. \tag{2.1}$$

The collection of all zero neighborhood ordered by inclusion is a directed set, so that $(x_{U,n})$ is an ultimately bounded net. Indeed, if $W$ is a zero neighborhood then $x_{U,n} \in W$ for each zero neighborhood $U \subseteq W$ and every $n \in \mathbb{N}$. But it follows from (2.1) that the net $\big(T^n x_{U,m}\big)_{n,m,U}$ does not converge to zero.

To prove the second equality, let

$$\begin{aligned} \gamma_1 &= \inf\Big\{\nu > 0 \,:\, \lim_{n,\alpha} \tfrac{T^n}{\nu^n} x_\alpha = 0 \text{ whenever } (x_\alpha) \text{ is ultimately bounded}\Big\} \text{ and} \\ \gamma_2 &= \inf\Big\{\nu > 0 \,:\, \big(\tfrac{T^n}{\nu^n} x_\alpha\big)_{n,\alpha} \text{ is ultimately bounded if } (x_\alpha) \text{ is ultimately bounded}\Big\}. \end{aligned}$$

Since every net which converges to zero is necessarily ultimately bounded, it follows that $\gamma_1 \geqslant \gamma_2$. Now let $\nu > \gamma_2$, and let $(x_\alpha)$ be an ultimately bounded sequence. Suppose that $\gamma_2 < \mu < \nu$, then $\big(\frac{T^n}{\mu^n} x_\alpha\big)_{n,\alpha}$ is ultimately bounded, that is, for each zero neighborhood $V$ there exists an indices $\alpha_0$ and $n_0$ and a positive $\varepsilon$ such that $\frac{T^n}{\mu^n} x_\alpha \in \varepsilon V$ whenever $\alpha > \alpha_0$ and $n > n_0$. It follows that $\frac{T^n}{\nu^n} x_\alpha \in \frac{\mu^n \varepsilon}{\nu^n} V \subseteq V$ for $\alpha > \alpha_0$ and all sufficiently large $n$. This implies that $\lim\limits_{n,\alpha} \frac{T^n}{\nu^n} x_\alpha = 0$ so that $\nu \geqslant \gamma_1$. □

**Question.** *Are there similar ways for computing $r_l(T)$, $r_{bb}(T)$, $r_{nn}(T)$, and $r_{nb}(T)$ in terms of nets?*

Proposition 2.3.7 enables us to prove some important properties of $r_c$. The following lemma is analogous to Lemma 3.13 of [VM97].

**Lemma 2.3.8.** *If $S$ and $T$ are two commuting linear operators on a topological vector space $X$ such that $r_c(S)$ and $r_c(T)$ are finite, then $r_c(ST) \leqslant r_c(S) r_c(T)$.*

*Proof.* Suppose $\mu > r_c(S)$ and $\nu > r_c(T)$ and let $(x_\alpha)$ be an ultimately bounded net in $X$. Then the net $(\frac{T^n x_\alpha}{\nu^n})_{n,\alpha}$ is ultimately bounded by Proposition 2.3.7. By applying Proposition 2.3.7 again we conclude that $(\frac{S^m T^n x_\alpha}{\mu^m \nu^n})_{m,n,\alpha}$ converges to zero. In particular, $\lim\limits_{n,\alpha} \frac{(ST)^n x_\alpha}{(\mu\nu)^n} = \lim\limits_{n,\alpha} \frac{S^n T^n x_\alpha}{\mu^n \nu^n} = 0$, and applying Proposition 2.3.7 one more time we get $\mu\nu > r_c(ST)$. □

**Theorem 2.3.9.** *If $S$ and $T$ are two commuting continuous operators on a locally convex space $X$ then $r_c(S+T) \leqslant r_c(S) + r_c(T)$.*

*Proof.* Assume without loss of generality that both $r_c(S)$ and $r_c(T)$ are finite. Suppose that $\eta > r_c(S) + r_c(T)$ and take $\mu > r_c(S)$ and $\nu > r_c(T)$ such that $\eta > \mu + \nu$. Let $(x_\alpha)$ be an ultimately bounded net in $X$. By Proposition 2.3.7 it suffices to show that $\lim\limits_{n,\alpha} \frac{1}{\eta^n}(S+T)^n x_\alpha = 0$. Notice that the net $\big(\frac{T^n}{\nu^n}x_\alpha\big)_{n,\alpha}$ is ultimately bounded. This implies that the net $\big(\frac{S^m}{\mu^m}\frac{T^n}{\nu^n}x_\alpha\big)_{m,n,\alpha}$ converges to zero. Fix a seminorm $p$, then there exist indices $n_0$ and $\alpha_0$ such that $p(S^m T^n x_\alpha) < \mu^m \nu^n$ whenever $m, n \geqslant n_0$ and $\alpha \geqslant \alpha_0$. Also, notice that we can split $\eta$ into a product of two terms $\eta = \eta_1\eta_2$ such that $\eta_1 > 1$ while still $\eta_2 > \mu + \nu$. Further, if $n > 2n_0$ and $\alpha \geqslant \alpha_0$ then we have

$$\begin{aligned} &p\left(\tfrac{1}{\eta^n}(S+T)^n x_\alpha\right) \leqslant \\ &\qquad \tfrac{1}{\eta^n}\sum_{k=0}^{n_0}\tbinom{n}{k}p\big(S^k T^{n-k}x_\alpha\big) + \tfrac{1}{\eta^n}\sum_{k=n_0+1}^{n-n_0}\tbinom{n}{k}p\big(S^k T^{n-k}x_\alpha\big) + \tfrac{1}{\eta^n}\sum_{k=n-n_0+1}^{n}\tbinom{n}{k}p\big(S^k T^{n-k}x_\alpha\big). \end{aligned}$$

Since $\binom{n}{k} = \frac{(n-k+1)\cdots(n-1)\cdot n}{1\cdot 2\cdots(k-1)\cdot k} \leqslant n^k$ and $\sum_{k=0}^{n}\binom{n}{k}\mu^k\nu^{n-k} = (\mu+\nu)^n$, we have

$$\begin{aligned} &p\left(\tfrac{1}{\eta^n}(S+T)^n x_\alpha\right) \leqslant \\ &\qquad \frac{n^{n_0}}{\eta^n}\sum_{k=0}^{n_0}p\big(S^k T^{n-k}x_\alpha\big) + \frac{1}{\eta^n}\sum_{k=n_0+1}^{n-n_0}\tbinom{n}{k}\mu^k\nu^{n-k} + \frac{n^{n_0}}{\eta^n}\sum_{k=n-n_0+1}^{n}p\big(S^k T^{n-k}x_\alpha\big) \\ &\qquad\qquad \leqslant \frac{n^{n_0}}{\eta_1^n}\cdot\frac{1}{\eta_2^n}\sum_{k=0}^{n_0}\Big(p\big(T^{n-k}S^k x_\alpha\big) + p\big(S^{n-k}T^k x_\alpha\big)\Big) + \frac{(\mu+\nu)^n}{\eta^n}. \end{aligned}$$

Notice that $\lim\limits_{n\to\infty}\frac{(\mu+\nu)^n}{\eta^n} = 0$ and that $\lim\limits_{n\to\infty}\frac{n^{n_0}}{\eta_1^n} = 0$. Since $T$ is continuous, the net $(T^k x_\alpha)_\alpha$ is ultimately bounded for every fixed $k$, so that $\lim\limits_{n,\alpha}\frac{1}{\eta_2^{n-k}}S^{n-k}T^k x_\alpha = 0$. It follows that for every $k$ between 0 and $n_0$ the expression $\frac{1}{\eta_2^n}p\big(S^{n-k}T^k x_\alpha\big)$ is uniformly bounded for all sufficiently large $n$ and $\alpha$. Similarly, for every $k$ between 0 and $n_0$ the expression $\frac{1}{\eta_2^n}p\big(T^{n-k}S^k x_\alpha\big)$ is uniformly bounded for all sufficiently large $n$ and $\alpha$. Therefore there exist indices $n_1$ and $\alpha_1$ such that the finite sum

$$\frac{1}{\eta_2^n}\sum_{k=0}^{n_0}\Big(p\big(T^{n-k}S^k x_\alpha\big) + p\big(S^{n-k}T^k x_\alpha\big)\Big)$$

is uniformly bounded for all $n \geqslant n_1$ and $\alpha \geqslant \alpha_1$. It follows that $\lim\limits_{n,\alpha} p\Big(\frac{1}{\eta^n}(S+T)^n x_\alpha\Big) = 0$, so that $\eta > r_c(S+T)$. $\square$

**Corollary 2.3.10.** *If $T$ is a continuous operator on a locally convex space with finite $r_c(T)$ then $r_c\big(P(T)\big)$ is finite for every polynomial $P(z)$.*

**Definition 2.3.11.** We say that a sequence $(x_n)$ in a topological vector space is ***fast null*** if $\lim_{n\to\infty} \alpha^n x_n = 0$ for every positive real $\alpha$.

**Lemma 2.3.12.** *If $T$ is a linear operator on a topological vector space with $r_c(T) < \infty$ then $(T^n x_n)$ is fast null whenever $(x_n)$ is fast null.*

*Proof.* Suppose $(x_n)$ is a fast null sequence in a topological vector space and $r_c(T) < \infty$. Let $\nu > r_c(T)$, the sequence $\nu^n \alpha^n x_n$ converges to zero, hence is ultimately bounded, then by Proposition 2.3.7 we have

$$\lim_{n\to\infty} \alpha^n T^n x_n = \lim_{n\to\infty} \tfrac{T^n}{\nu^n} \nu^n \alpha^n x_n = 0.$$

□

## 2.4 Spectra and spectral radii

It is well known that for a continuous operator $T$ on a Banach space its spectral radius $r(T)$ equals the geometrical radius of the spectrum $\big|\sigma(T)\big| = \sup\{|\lambda| \ : \ \lambda \in \sigma(T)\}$. Further, whenever $|\lambda| > r(T)$, the resolvent operator $R_\lambda = (\lambda I - T)^{-1}$ is given by the Neumann series $\sum_{i=0}^{\infty} \frac{T^i}{\lambda^{i+1}}$. We are going to show in the next five theorems that the spectral radii that we have introduced are upper bounds for the actual radii of the correspondent spectra, and that when $|\lambda|$ is greater than or equal to any of these spectral radii, then the Neumann series converges in the correspondent operator topology to the resolvent operator.

In the following Theorems 2.4.1–2.4.5 we assume that $T$ is a linear operator on a sequentially complete locally convex space, $\lambda$ is a complex number, and $R_\lambda$ is the resolvent of $T$ at $\lambda$ in the sense of Definition 2.2.1.

**Theorem 2.4.1.** *If $|\lambda| > r_l(T)$ then the Neumann series converges pointwise to a linear operator $R_\lambda^0$, and $R_\lambda^0(\lambda I - T) = I$. Moreover, if $T$ is continuous, then $R_\lambda^0 = R_\lambda$ and $\big|\sigma^l(T)\big| \leqslant r_l(T)$.*

*Proof.* For any $\lambda \in \mathbb{C}$ such that $|\lambda| > r_l(T)$ one can find $z \in \mathbb{C}$ such that $0 < |z| < 1$ and $\lambda z > r_l(T)$. Consider a point $x \in X$ and a base zero neighborhood $U$. Since by the definition of $r_l(T)$ the sequence $\left(\frac{T^n x}{(\lambda z)^n}\right)$ converges to zero, there exist a positive integer $n_0$, such that $\frac{T^n x}{(\lambda z)^n} \in U$ whenever $n \geqslant n_0$. Therefore, $\frac{T^n x}{\lambda^n} \in z^n U \subseteq |z|^n U$ because $U$ is balanced. Thus, if $n \geqslant m \geqslant n_0$, then $\sum_{i=n}^m \frac{T^i x}{\lambda^i} \in \sum_{i=n}^m |z|^i U \subseteq \left(\sum_{i=n}^m |z|^i\right) U$ because $U$ is convex. Since $|z| < 1$, we have $\sum_{i=n}^m |z|^i < 1$ for sufficiently large $m$ and $n$, and so $\sum_{i=n}^m \frac{T^i x}{\lambda^i} \in U$ because $U$ is balanced. Therefore $R_{\lambda,n} x = \frac{1}{\lambda} \sum_{i=0}^n \frac{T^i x}{\lambda^i}$ is a Cauchy sequence and hence it converges to some $R_\lambda^0 x$ because $X$ is sequentially complete.

Clearly, $R_\lambda^0$ is a linear operator. Notice that $R_{\lambda,n}(\lambda x - Tx) = x - \frac{T^{n+1}x}{\lambda^{n+1}}$ for every $x$. As $n$ goes to infinity, the left hand side of this identity converges to $R_\lambda^0(\lambda x - Tx)$, while the right hand side converges to $x$. Thus it follows that $R_\lambda^0(\lambda I - T) = I$.

Finally, notice that $R_{\lambda,n}$ commutes with $T$ for every $n$. Therefore, if $T$ is continuous, then

$$R_\lambda^0 Tx = \lim_{n\to\infty} R_{\lambda,n} Tx = \lim_{n\to\infty} TR_{\lambda,n} x = T(\lim_{n\to\infty} R_{\lambda,n} x) = TR_\lambda^0 x$$

for every $x$. This implies that $(\lambda I - T)R_\lambda^0 = R_\lambda^0(\lambda I - T) = I$, so that $R_\lambda^0$ is the (left and right) inverse of $\lambda I - T$. This means that $R_\lambda^0 = R_\lambda$ and $\lambda \in \rho^l(T)$. Thus, $\left|\sigma^l(T)\right| \leqslant r_l(T)$. $\square$

**Theorem 2.4.2.** *If $T$ is bb-bounded and $|\lambda| > r_{bb}(T)$, then the Neumann series converges uniformly on bounded sets, and its sum $R_\lambda^0$ is bb-bounded. Moreover, if $T$ is continuous, then $R_\lambda^0 = R_\lambda$ and $\left|\sigma^{bb}(T)\right| \leqslant r_{bb}(T)$.*

*Proof.* Suppose that $|\lambda| > r_{bb}(T)$, then the sum $R_\lambda^0$ of the Neumann series exists by Theorem 2.4.1. As in the proof of Theorem 2.4.1 we denote the partial sums of the Neumann series by $R_{\lambda,n}$. Fix $z \in \mathbb{C}$ such that $0 < |z| < 1$ and $\lambda z > r_{bb}(T)$, and consider a bounded set $A$ and a closed base zero neighborhood $U$. Since $\frac{T^n}{(\lambda z)^n}$ converges to zero uniformly on $A$, there exits $n_0 \in \mathbb{N}$ such that $\frac{T^n}{\lambda^n z^n}(A) \subseteq U$ for all $n > n_0$. Also, since $|z| < 1$, we can assume without loss of generality that $\sum_{i=n_0}^\infty |z|^i < |\lambda|$. Then

$$\frac{1}{\lambda} \sum_{i=n+1}^m \frac{T^i x}{\lambda^i} \in \frac{1}{\lambda}\Big(\sum_{i=n+1}^m |z|^i\Big) U \subseteq U$$

whenever $x \in A$ and $m > n > n_0$. Since $U$ is closed, we have

$$R^0_\lambda x - R_{\lambda,n}x = \lim_{m\to\infty} \tfrac{1}{\lambda} \sum_{i=n+1}^{m} \tfrac{T^i x}{\lambda^i} \in U$$

for each $x \in A$ and $n > n_0$, so that $(R^0_\lambda - R_{\lambda,n})(A) \subseteq U$ whenever $n > n_0$. This shows that $R_{\lambda,n}$ converges to $R^0_\lambda$ uniformly on bounded sets. By Lemma 2.1.14 this implies that $R^0_\lambda$ is bb-bounded.

Further, if $T$ is continuous, then by Theorem 2.4.1 we have $R_\lambda = R^0_\lambda$, so that $\lambda \in \rho^{bb}(T)$, whence it follows that $\left|\sigma^{bb}(T)\right| \leqslant r_{bb}(T)$. $\square$

The next theorem is similar to Theorem 2.18 of [VM97].

**Theorem 2.4.3.** *If $T$ is a continuous and $|\lambda| > r_c(T)$, then the Neumann series converges equicontinuously to $R_\lambda$, and $R_\lambda$ is continuous. In particular, $\left|\sigma^c(T)\right| \leqslant r_c(T)$ holds.*

*Proof.* Let $|\lambda| > r_c(T)$. It follows from Theorem 2.4.1 that the Neumann series converges to $R_\lambda$. Again, we denote the partial sums of the Neumann series by $R_{\lambda,n}$. Let $z \in \mathbb{C}$ be such that $0 < |z| < 1$ and $\lambda z > r_c(T)$. For a fixed closed zero neighborhood $U$ there exists a zero neighborhood $V$ such that $\frac{T^n}{\lambda^n z^n}(V) \subseteq U$ for every $n \geqslant 0$. Let $\varepsilon > 0$, then $\sum_{i=n_0}^{\infty} |z|^i < \varepsilon|\lambda|$ for some $n_0$. Then

$$\tfrac{1}{\lambda} \sum_{i=n+1}^{m} \tfrac{T^i x}{\lambda^i} \in \tfrac{1}{\lambda}\Big(\sum_{i=n+1}^{m} |z|^i\Big)\varepsilon U \subseteq U$$

whenever $x \in V$ and $m > n > n_0$. Since $U$ is closed, we have

$$R_\lambda x - R_{\lambda,n}x = \lim_{m\to\infty} \tfrac{1}{\lambda} \sum_{i=n+1}^{m} \tfrac{T^i x}{\lambda^i} \in \varepsilon U$$

for each $x \in V$ and $n > n_0$, so that $(R_\lambda - R_{\lambda,n})(V) \subseteq \varepsilon U$ whenever $n > n_0$. This shows that $R_{\lambda,n}$ converges to $R_\lambda$ equicontinuously, and Lemma 2.1.15 yields that $R_\lambda$ is continuous.

$\square$

**Theorem 2.4.4.** *If $T$ is nn-bounded and $|\lambda| > r_{nn}(T)$, then the Neumann series nn-converges to $R_\lambda$ and $R_\lambda$ is nn-bounded. In particular, $\left|\sigma^{nn}(T)\right| \leqslant r_{nn}(T)$ holds.*

*Proof.* Let $|\lambda| > r_{nn}(T)$. By Theorem 2.4.1 the Neumann series $\sum_{i=0}^{\infty} \frac{T^i}{\lambda^{i+1}}$ converges to $R_\lambda$. Again, we denote the partial sums of the Neumann series by $R_{\lambda,n}$. Fix some $z$ such that $0 < |z| < 1$ and $\lambda z > r_{nn}(T)$. There exists a base $\mathcal{N}_0$ of closed convex zero neighborhoods such that for every $U \in \mathcal{N}_0$ there is a scalar $\beta > 0$ such that $\frac{T^n}{(\lambda z)^n}(U) \subseteq \beta U$ for all $n \geqslant 0$. Fix $U \in \mathcal{N}_0$, then for each $n \geqslant 0$ we have $\frac{T^n}{\lambda^n z^n}(U) \subseteq \beta U$ for some $\beta > 0$, so that $\frac{T^n x}{\lambda^n} \in |z|^n \beta U$ whenever $x \in U$. It follows that

$$R_{\lambda,n}x = \tfrac{1}{\lambda}\sum_{i=0}^{n} \tfrac{T^i x}{\lambda^i} \in \tfrac{\beta}{\lambda}\Big(\sum_{i=0}^{n}|z|^i\Big)U.$$

Then $R_\lambda x \in \frac{\beta}{\lambda(1-|z|)}U$, so that $R_\lambda(U) \subseteq \frac{\beta}{\lambda(1-|z|)}U$, which implies that $R_\lambda$ is nn-bounded, and, therefore, $\left|\sigma^{nn}(T)\right| \leqslant r_{nn}(T)$ holds.

Fix $\varepsilon > 0$. Then $\sum_{i=N}^{\infty}|z|^i < |\lambda|$ for some $N$. Then for every $U \in \mathcal{N}_0$ we have

$$\tfrac{1}{\lambda}\sum_{i=n+1}^{m} \tfrac{T^i x}{\lambda^i} \in \tfrac{1}{\lambda}\Big(\sum_{i=n+1}^{m}|z|^i\Big)\varepsilon U \subseteq U$$

whenever $x \in U$ and $N < n < m$. Since $U$ is closed, we have

$$R_\lambda x - R_{\lambda,n}x = \lim_{m\to\infty} \tfrac{1}{\lambda}\sum_{i=n+1}^{m} \tfrac{T^i x}{\lambda^i} \in \varepsilon U$$

for each $x \in U$ and $n > N$, so that $(R_\lambda - R_{\lambda,n})(U) \subseteq \varepsilon U$ whenever $N < n$. This shows that $R_{\lambda,n}$ nn-converges to $R_\lambda$. ∎

**Theorem 2.4.5.** *If $T$ is nb-bounded and $|\lambda| > r_{nb}(T)$, then the Neumann series converges to $R_\lambda$ uniformly on a zero neighborhood. Further, $\left|\sigma^{nb}(T)\right| \leqslant r_{nb}(T)$ holds.*

*Proof.* Let $|\lambda| > r_{nb}(T)$. By Theorem 2.4.1 the Neumann series $\sum_{i=0}^{\infty} \frac{T^i}{\lambda^{i+1}}$ converges to $R_\lambda$. Since $r_{bb}(T) \leqslant r_{nb}(T)$ then $R_\lambda$ is bb-bounded by Theorem 2.4.2. But then $\sum_{i=0}^{\infty} \frac{T^i}{\lambda^{i+1}} = \frac{1}{\lambda}I + \frac{1}{\lambda}R_\lambda T$. Notice that $R_\lambda T$ is nb-bounded as a product of a bb-bounded and an nb-bounded operators (see 2.1.6).

Suppose that $|\lambda| > r_{nb}(T)$. Fix $z \in \mathbb{C}$ such that $0 < |z| < 1$ and $\lambda z > r_{nb}(T)$, then the sequence $\left(\frac{T^n}{\lambda^n z^n}\right)$ converges to zero uniformly on some base zero neighborhood $U$.

We will show that the Neumann series converges uniformly on $U$. As in the proof of Theorem 2.4.1, we denote the partial sums of the Neumann series by $R_{\lambda,n}$. Fix a closed base zero neighborhood $V$. Since $\left(\frac{T^n}{\lambda^n z^n}\right)$ converges to zero uniformly on $U$, there exits $n_0 \in \mathbb{N}$ such that $\frac{T^n}{\lambda^n z^n}(U) \subseteq V$ for all $n > n_0$. Also, since $|z| < 1$, we can assume without loss of generality that $\sum_{i=n_0}^{\infty} |z|^i < |\lambda|$. Then

$$\frac{1}{\lambda} \sum_{i=n+1}^{m} \frac{T^i x}{\lambda^i} \in \frac{1}{\lambda}\Big(\sum_{i=n+1}^{m} |z|^i\Big) V \subseteq V$$

whenever $x \in A$ and $m > n > n_0$. Since $V$ is closed, we have

$$R_\lambda x - R_{\lambda,n} x = \lim_{m\to\infty} \frac{1}{\lambda} \sum_{i=n+1}^{m} \frac{T^i x}{\lambda^i} \in V$$

for each $x \in U$ and $n > n_0$, so that $(R_\lambda - R_{\lambda,n})(U) \subseteq V$ whenever $n > n_0$. □

In the rest of this section we discuss the conditions of sequential completeness and the local convexity assumed in Theorems 2.4.1–2.4.5, and consider several examples and special cases.

Clearly, if $X$ is a Banach space, then the norm topology on $X$ and the weak* topology on $X^*$ are sequentially complete. The weak topology of $X$ is sequentially complete if $X$ is reflexive. Also, it is known that the weak topologies of $\ell_1$ and of $L_1[0,1]$ are sequentially complete. Since all these topologies are locally convex, Theorems 2.4.1–2.4.5 are applicable to each of them.

**2.4.6. Monotone convergence property.** Notice that if $T$ is a positive operator on a locally convex-solid vector lattice then we can substitute the sequential completeness condition in Theorems 2.4.1–2.4.5 by a weaker condition called ***sequential monotone completeness property***: a locally convex-solid vector lattice is said to satisfy the sequential monotone completeness property if every monotone Cauchy sequence converges in the topology of $X$. For details, see [AB78]. Indeed, we used the sequential completeness at just one point — we used it in the proof of Theorem 2.4.1 to claim that since $R_{\lambda,n}x = \frac{1}{\lambda}\sum_{i=0}^{n} \frac{T^i x}{\lambda^i}$ is a Cauchy sequence, then it converges to some $R_\lambda x$. But if $T$ is positive, then $R_{\lambda,n}x^+$ and $R_{\lambda,n}x^-$ are increasing sequences, and the sequential monotone completeness property ensures the convergence.

**2.4.7. Pointwise convergence.** It can be easily verified that the space of continuous functions on $[0,1]$ with pointwise convergence topology is not sequentially complete, the sequence $x_n(t) = t^n$ is a counterexample. The same counterexample shows that this space does not have the monotone convergence property either.

Consider the sequence spaces $\ell_p$ for $0 < p \leqslant \infty$, $c$, $c_0$, and $c_{00}$ (the space of eventually vanishing sequences). None of these spaces is sequentially complete in the topology of coordinate-wise convergence: take the following sequence for a counterexample:

$$x_n(i) = \begin{cases} i & \text{if } i < n; \\ 0 & \text{otherwise.} \end{cases}$$

The same example shows that these spaces do not have the monotone convergence property either. Therefore neither of Theorems 2.4.1–2.4.5 or 2.4.6 can be applied.

**Example 2.4.8.** *Theorems 2.4.1–2.4.5 fail without sequential completeness.* Consider the space $c_0$ with the topology of coordinate-wise convergence. Let $T$ be the forward shift operator on $c_0$, that is, $Te_k = e_{k+1}$, where $e_k$ is the $k$-th unit vector of $c_0$. Let $V$ be any base zero neighborhood, we can assume without loss of generality that $V = \{x \in c_0 : |x_{i_1}| < 1, \dots, |x_{i_k}| < 1\}$ where $i_1 < i_2 < \dots < i_k$ are positive integers. If $x \in U$ then $T^n x$ has zero components 1 through $n$, in particular for every positive $\nu$ we have $\frac{T^n x}{\nu^n} \in V$ whenever $n > i_k$. Therefore $\left(\frac{T^n}{\nu^n}\right)$ converges uniformly on $c_0$ for every $\nu > 0$, so that $r_{nb}(T) = 0$. It follows from Proposition 2.3.3 that $r_l(T) = r_{bb}(T) = r_c(T) = r_{nn}(T) = 0$. On the other hand, $\sum_{n=1}^{\infty} T^n e_1$ diverges in $c_0$. Since $T$ is obviously continuous, this shows that Theorems 2.4.1–2.4.5, do not hold in $c_0$. Thus, sequential completeness condition is essential in the theorems.

**2.4.9. Banach spaces.** If $T$ is a (norm) continuous operator on a Banach space, then it follows from 2.2.2 and 2.3.5 that $\sigma^l(T) = \sigma^{bb}(T) = \sigma^c(T) = \sigma^{nn}(T) = \sigma^{nb}(T) = \sigma(T)$ and $r_{bb}(T) = r_c(T) = r_{nn}(T) = r_{nb}(T) = r(T)$, where $\sigma(T)$ and $r(T)$ are the usual spectrum and the spectral radius of $T$. Further, it follows from Lemma 2.3.3 that $r_l(T) \leqslant r(T)$. On the other hand, since $r(T) = \left|\sigma(T)\right|$, then $r(T) \leqslant r_l(T)$ by Theorem 2.4.1, so that $r_l(T) = r(T)$.

**2.4.10.** The following argument is a counterpart to 2.2.5. Let $T$ be a (norm) continuous operator on a Banach space $X$ and $r(T)$ the usual spectral radius of $T$, while $r_l(T)$ and $r_{bb}(T)$ be computed with respect to the weak topology of $X$. We claim that if the weak topology of $X$ is sequentially complete, then $r_l(T) = r_{bb}(T) = r(T)$. Indeed, $r(T) \leqslant r_l(T)$ by 2.2.5 and Theorem 2.4.1 because $\sigma(T) = \sigma^l(T)$. In view of Proposition 2.3.3 it suffices to show that $r_{bb}(T) \leqslant r(T)$. Let $\nu > r(T)$, and let $A$ be a weakly bounded subset of $X$. Then $A$ is norm bounded, so that the sequence $\frac{T^n}{\nu^n}$ converges to zero uniformly on $A$ in the norm topology. In particular, the set $\bigcup_{n=0}^{\infty} \frac{T^n}{\nu^n}(A)$ is norm bounded, hence weakly bounded, so that $\nu > r_{bb}(T)$.

### Quasinilpotence

Recall that a norm continuous operator $T$ on a Banach space $X$ is said to be ***quasinilpotent*** if $r(T) = 0$ or, equivalently, if $\sigma(T) = \{0\}$. Quasinilpotent operators on Banach spaces have some nice properties, therefore in the framework of topological vector spaces it is interesting to study operators having some of their spectra trivial or some of their spectral radii being zero. Notice, for example, that it follows from Proposition 2.3.6 that if $T$ is an operator on a locally convex topological vector space, then $r_l(T) = 0$ if and only if $\lim\limits_{n\to\infty} \sqrt[n]{p(T^n x)} = 0$ for every seminorm $p$ in a generating family of seminorms and for every $x \in X$. Further, if the space is in addition sequentially complete, then for such an operator we would have $\sigma^l(T) = \{0\}$ by Theorem 2.4.1.

Recall also that a norm continuous operator $T$ on a Banach space $X$ is said to be ***locally quasinilpotent at a point*** $x \in X$ if $\lim\limits_{n\to\infty} \sqrt[n]{\lVert T^n x\rVert} = 0$. Using Lemma 2.3.1, the concept of local quasinilpotence can be naturally generalized to topological vector spaces: an operator $T$ on a topological vector space $X$ is said to be ***locally quasinilpotent at a point*** $x \in X$ if $\lim\limits_{n\to\infty} \frac{T^n x}{\nu^n} = 0$ for every $\nu > 0$. It follows immediately from the definition of $r_l(T)$ that $r_l(T) = 0$ if and only if $T$ is locally quasinilpotent at every $x \in X$. It is known that a continuous operator on a Banach space is quasinilpotent if and only if it is locally quasinilpotent at every point. We see now that this is just a corollary of 2.4.9.

The following example shows that a similar result for general topological vector spaces is not valid, that is, $r_l(T)$ may be equal to zero without the other radii be equal to zero.

**Example 2.4.11.** *A continuous operator with $r_l(T) = 0$ but $r_{bb}(T) = r_c(T) = r_{nn}(T) = r_{nb}(T) = \infty$.* Consider the space of all bounded real sequences $\ell_\infty = \big\{x = (x_1, x_2, \dots) : \sup |x_k| < \infty\big\}$ with the topology of coordinate-wise convergence. This topology can be generated by the family of coordinate seminorms $\{p_m\}_{m=1}^{\infty}$ where $p_m(x) = |x_m|$. Let $e_k$ denote the $k$-th unit vector in $\ell_\infty$.

Define an operator $T\colon \ell_\infty \to \ell_\infty$ via $Te_k = \frac{(k-1)^{k-1}}{k^k} e_{k-1}$ if $k > 1$, and $Te_1 = 0$. Then $T^n e_k = \frac{(k-n)^{k-n}}{k^k} e_{k-n}$ if $n < k$ and zero otherwise. Clearly $T$ is continuous. In order to show that $r_l(T) = 0$ fix a positive real number $\nu$ and $x \in \ell_\infty$, then

$$\Big|\Big(\tfrac{T^n x}{\nu^n}\Big)_m\Big| = \Big|\tfrac{m^m}{(m+n)^{m+n}\nu^n} x_{n+m}\Big| \leqslant \sup_n \tfrac{m^m}{(m+n)^{m+n}\nu^n} \cdot \sup_n |x_n| < \infty$$

It follows from Lemma 2.3.4(i) that $r_l(T) = 0$.

Now we show that $r_{bb}(T) = \infty$ by presenting a bounded set $A$ in $\ell_\infty$ such that the sequence $\big(\frac{T^n}{\nu^n}\big)$ in not uniformly bounded on $A$ for every positive $\nu$. Let

$$A = \big\{\, x \in \ell_\infty \ :\ x_n \leqslant (2n)^{2n} \text{ for all } n \geqslant 0 \,\big\}.$$

Then $(2n)^{2n} e_n \in A$ for each $n > 0$ and $\big(\frac{T^{n-1}}{\nu^{n-1}}(2n)^{2n} e_n\big)_1 = \frac{(2n)^{2n}}{n^n \nu^n}$ is unbounded. Then by Lemma 2.3.4(ii) we have $r_{bb}(T) = \infty$, and it follows from Proposition 2.3.3 that $r_c(T) = r_{nn}(T) = r_{nb}(T) = \infty$.

It is not difficult to show that $\sigma^l(T) = \{0\}$, while $\sigma^c(T) = \sigma^{nn}(T) = \sigma^{nb}(T) = \mathbb{C}$.

### Non-locally convex spaces

We proved the key Theorems 2.4.1–2.4.5 for locally convex spaces, but they are still valid for locally pseudo-convex spaces. The local convexity of $X$ was used only once in the proof of Theorem 2.4.1, while Theorems 2.4.2–2.4.5 used Theorem 2.4.1. Hence it would suffice to modify the proof of Theorem 2.4.1 in such a way that it would work for locally pseudo-convex spaces instead of locally convex. Local convexity was used in the proof of Theorem 2.4.1 to show that if $\frac{T^n x}{(\lambda z)^n} \in U$ for all $n > n_0$ and some $n_0 \in \mathbb{N}$, then there exists

$m_0 \in \mathbb{N}$ such that $\sum_{i=n}^{m} \frac{T^i x}{\lambda^i} \in U$ for all $m, n > m_0$. (Recall that $T$ is a linear operator, $\lambda, z \in \mathbb{C}$ such that $0 < |z| < 1$ and $\lambda z > r_l(T)$, $x \in X$, and $U$ is a base zero neighborhood in $X$.) If $X$ is locally pseudo-convex, then we can assume that $U + U \subseteq \alpha U$ for some $\alpha > 0$, so that $(X, U)$ is a locally bounded space. Let $\|\cdot\|$ be the Minkowski functional of $U$, then (see [KPR84, pages 3 and 6]) for any $x_1, x_2, \dots, x_n$ in $X$ we have

$$\|x_1 + ... + x_k\| \leqslant 4^{\frac{1}{p}} \left(\|x_1\|^p + \dots + \|x_k\|^p\right)^{\frac{1}{p}},$$

where $2^{\frac{1}{p}} = \alpha$. Notice that $\|\frac{T^n x}{\lambda^n}\| \leqslant |z|^n$ for all $n > n_0$. Since $|z| < 1$, then there exists $m_0$ such that $\sum_{i=n}^{m} |z|^{ip} < \frac{1}{4}$ whenever $n, m > m_0$. But then

$$\Bigl\|\sum_{i=n}^{m} \tfrac{T^i x}{\lambda^i}\Bigr\| \leqslant 4^{\frac{1}{p}} \Bigl(\sum_{i=n}^{m} \|\tfrac{T^i x}{\lambda^i}\|^p\Bigr)^{\frac{1}{p}} \leqslant 4^{\frac{1}{p}} \Bigl(\sum_{i=n}^{m} |z|^{ip}\Bigr)^{\frac{1}{p}} < 1,$$

so that $\sum\limits_{i=n}^{m} \frac{T^i x}{\lambda^i} \in U$.

The following example shows that Theorems 2.4.1, 2.4.2, and 2.4.3 fail if we assume no convexity conditions at all.

**Example 2.4.12.** *An operator on a complete non locally pseudo-convex space, whose spectral radii are 1, and whose Neumann series nevertheless diverges at $\lambda = 2$.* Let $X$ be the space of all measurable functions on $[0, 1]$ with the topology of convergence in measure (which is not pseudo-convex). We identify the endpoints 0 and 1 and consider the interval as a circle. Fix an irrational $\alpha$ and define a linear operator $T$ on $X$ as the translation by $\alpha$, i.e., $(Tf)(t) = f(t - \alpha)$. It is easy to see that $\frac{T^n f}{\nu^n}$ converges in measure to zero for every $f \in X$ if and only if $\nu > 1$. We conclude, therefore, that $r_l(T) = 1$. Moreover, since the sets of the form $W_{\varepsilon,\delta} = \bigl\{f \in X \ : \ \mu(f > \varepsilon) < \delta\bigr\}$ form a zero neighborhood base for the topology of convergence in measure, and $T(W_{\varepsilon,\delta}) \subseteq W_{\varepsilon,\delta}$, it follows that $r_{nn}(T) \leqslant 1$. Then by Proposition 2.3.3 we have $r_l(T) = r_{bb}(T) = r_c(T) = r_{nn}(T) = 1$. Nevertheless, we are going to present a function $h \in X$ such that the Neumann series $\sum_{n=0}^{\infty} \frac{T^n h}{2^n}$ does not converge in measure, which means that the conclusions of Theorems 2.4.1–2.4.5 do not hold for this space

For each $n = 1, 2, 3, \dots$ one can find a positive integer $M_n$ such that the intervals $[k\alpha, k\alpha + \frac{1}{n}]$ (mod 1) for $k = 1, \dots, M_n$ cover the circle. Let $s_n = \sum_{i=1}^{n} M_i$, and let $h$

be the step function taking value $2^{s_n}$ on the interval $(\frac{1}{n+1}, \frac{1}{n}]$. If $s_{n-1} < k \leqslant s_n$ for some positive integers $n$ and $k$, then on $[0, \frac{1}{n}]$ we have $h \geqslant 2^{s_n} \geqslant 2^k$, so that $\frac{h}{2^k} \geqslant 1$ on $[0, \frac{1}{n}]$, and it follows that $\frac{T^k h}{2^k} \geqslant 1$ on $[k\alpha, k\alpha + \frac{1}{n}]$.

Now, given any positive integer $N$, we have $N \leqslant s_{n-1}$ for some $n$. Then for each $k = s_{n-1}+1, \dots, s_n$ we have $\frac{T^k h}{2^k} \geqslant 1$ on the interval $[k\alpha, k\alpha + \frac{1}{n}]$. It follows that

$$\sum_{k=s_{n-1}+1}^{s_n} \frac{T^k h}{2^k} \geqslant 1 \quad \text{on the set} \quad \bigcup_{k=s_{n-1}+1}^{s_n} [k\alpha, k\alpha + \tfrac{1}{n}] = s_{n-1}\alpha + \bigcup_{k=1}^{M_n} [k\alpha, k\alpha + \tfrac{1}{n}] = [0,1],$$

so that the series $\sum_{n=0}^{\infty} \frac{T^n h}{2^n}$ does not converge in measure.

## 2.5 nb-bounded operators

Since nb-boundedness is the strongest of the boundedness conditions we have introduced, it is natural to expect that stronger results can be obtained for nb-bounded operators.

**2.5.1.** The following argument is often useful when dealing with nb-bounded operators. Suppose that $X$ and $Y$ are topological vector spaces and $T\colon X \to Y$ is nb-bounded, then $T(U)$ is a bounded set in $Y$ for some base zero neighborhood $U$. We claim that if $Y$ is Hausdorff, then $\bigcap_{n=1}^{\infty} \frac{1}{n}U \subseteq \operatorname{Null} T$. Indeed, it suffices to show that if $x \in \frac{1}{n}U$ for every $n \geqslant 1$ then $Tx$ belongs to every zero neighborhood $V$ of $Y$. But $T(U) \subseteq \alpha V$ for some positive $\alpha$ (depending on $V$), and hence $Tx \in \frac{1}{n}T(U) \subseteq \frac{\alpha}{n}V \subseteq V$ whenever $n \geqslant \alpha$.

It follows that if $T$ is one-to-one, then $U$ cannot contain any nontrivial linear subspaces. In particular, if $U$ is convex then the locally bounded space $(X, U)$ is Hausdorff, hence quasinormable. In this case $T$ is a continuous operator from $(X, U)$ to $Y$, and, moreover, if $X = Y$, then $T$ is continuous as an operator from $(X, U)$ to $(X, U)$.

In fact, many "classical" topological vector spaces have the property that every zero neighborhood contains a nontrivial linear subspace, e.g., topologies of pointwise or coordinate-wise convergence, weak topologies, etc.

**Example 2.5.2.** *A topological vector space in which no base zero neighborhood contains a nontrivial linear subspace.* Let X be the space of all analytic functions on $\mathbb{C}$ equipped with the topology of uniform convergence on compact subsets of $\mathbb{C}$. The sets

$$U_{n,\varepsilon} = \big\{ f \in X \;:\; |f(z)| < \varepsilon \text{ whenever } |z| \leqslant n \big\} \quad (n \geqslant 0 \text{ and } \varepsilon > 0)$$

form a zero neighborhood base of this topology. Clearly, no $U_{n,\varepsilon}$ contains a non-trivial linear subspace. Indeed, if there is a function $f$ in $X$ and a zero neighborhood $U_{n,\varepsilon}$ such that $\lambda f \in U_{n,\varepsilon}$ for every scalar $\lambda$, then $f(z) = 0$ whenever $|z| < n$, and it follows that $f$ is identically zero on $\mathbb{C}$. Note that this topology is generated by the countable sequence of seminorms $\|f\|_n = \sup\limits_{|z| \leqslant n} \big|f(z)\big|$; clearly $\|\cdot\|_n$ is the Minkowski functional of $U_{n,1}$.

**Proposition 2.5.3.** *If $X$ is a complete locally convex space then $X$ is locally bounded if and only if $X$ admits an nb-bounded bijection.*

*Proof.* If $X$ is locally bounded then the identity map is an nb-bounded bijection. Suppose that $T$ is an nb-bounded bijection on $X$. Then there exists a closed base zero neighborhood $U$ in $X$ such that $T(U)$ is bounded. Let $A = \overline{T(U)}$, then $A$ is convex, bounded, balanced, and absorbing. It follows that the space $(X, A)$ is a locally convex and locally bounded, denote it by $X_A$. Notice also that the topology of $X_A$ is finer than the original topology on $X$ because $A$ is bounded. In particular, $X_A$ is Hausdorff.

We claim that $X_A$ is complete. Indeed, if $(x_n)$ is a Cauchy sequence in $X_A$, then it is also Cauchy in the original topology of $X$, which is complete, so that $x_n$ converges to some $x$. Fix $\varepsilon > 0$, then there exists $n_0$ such that $x_n - x_m \in \varepsilon A$ whenever $n, m \geqslant n_0$. Let $m \to \infty$, since $A$ is closed we have $x_n - x \in \varepsilon A$, i.e., $x_n \to x$ in $X_A$. Thus, $X_A$ is complete, hence Banach.

Since $A$ is bounded, we can find $m$ such that $A \subseteq mU$. Then $T(A) \subseteq T(mU) \subseteq mA$, so that $T$ is bounded in $X_A$. Then $T^{-1}$ is also bounded in $X_A$ by the Banach Theorem, so that $U = T^{-1}\big(T(U)\big) \subseteq T^{-1}(A) \subseteq nA$ for some $n > 0$, hence $U$ is bounded. $\square$

**Proposition 2.5.4.** *Let $T\colon X \to Y$ be an nb-bounded operator between Hausdorff topological vector spaces such that $X$ is not locally bounded. If*

*(i) every zero neighborhood in $X$ contains a non-trivial linear subspace, or*

*(ii) both $X$ and $Y$ are Fréchet spaces,*

*then $T$ is not a bijection.*

*Proof.* If every zero neighborhood of $X$ contains a non-trivial linear subspace, then $T$ cannot be one-to-one by 2.5.1. Suppose now that $X$ and $Y$ are Fréchet and assume that $T$ is a bijection. Let $S\colon Y \to X$ be the linear inverse of $T$. The Open Mapping Theorem implies that $S$ is continuous and hence bb-bounded. It follows that the identity operator of $X$ is nb-bounded being the composition of the nb-bounded operator $T$ and the bb-bounded operator $S$. But the identity operator is nb-bounded if and only if the space is locally bounded, a contradiction. □

**Weak topologies**

We are going to show that every operator which is nb-bounded relative to a weak topology has to be of finite rank. In order to prove this we need the following well-known lemma. For completeness we provide a simple proof of it.

**Lemma 2.5.5.** *Let $T$ be a linear operator on a vector space $L$, and let $f_1, \dots, f_n$ be linear functionals on $L$ such that $Tx = 0$ whenever $f_i(x) = 0$ for every $i = 1, \dots, n$. Then $T$ is a finite rank operator of rank at most $n$.*

*Proof.* Define a linear map $\pi$ from $L$ to $\mathbb{R}^n$ via $\pi(x) = (f_1(x), \dots, f_n(x))$. Then the dimension of the range $\pi(L)$ is at most $n$. Define also a linear map $\varphi$ from $\pi(L)$ to $L$ via $\varphi(\pi(x)) = Tx$. It can be easily verified that $\varphi$ is well-defined. Then the range of $T$ coincides with the range $\varphi(\pi(L))$, which is of dimension at most $n$. □

**Proposition 2.5.6.** *Let $X$ be a locally convex space, and $T$ an operator on $X$ such that $T$ is nb-bounded with respect to the weak topology of $X$. Then $T$ is of finite rank.*

*Proof.* Suppose $T$ maps some weak base zero neighborhood $U = \big\{x \in X \,:\, |f_i(x)| < 1,$ $i = 1, \dots, n\big\}$ $(f_1, \dots, f_n \in X')$, to a weakly bounded set. Since the weak topology is Hausdorff, it follows from 2.5.1 that $\bigcap_{n=1}^{\infty} \frac{1}{n}U \subseteq \ker T$. In particular, $Tx = 0$ whenever $f_i(x) = 0$ for every $i = 1, \dots, n$. Then Lemma 2.5.5 implies that $T$ is a finite rank operator. □

**Spectra and spectral radii of nb-bounded operators**

**Proposition 2.5.7.** *If $T$ is an nb-bounded operator on a topological vector space then $\sigma^{bb}(T) = \sigma^{c}(T) = \sigma^{nn}(T) = \sigma^{nb}(T)$.*

*Proof.* If $X$ is locally bounded then the result is trivial by 2.2.3. Suppose that $X$ is not locally bounded, then, in view of 2.2.2, it suffices to show that $\rho^{bb}(T) \subseteq \rho^{nb}(T)$. Let $\lambda \in \rho^{bb}(T)$, then $R_\lambda$ is bb-bounded. If $\lambda \neq 0$, then it follows from $R_\lambda(\lambda I - T) = I$ that $R_\lambda = \frac{1}{\lambda} R_\lambda T + \frac{1}{\lambda} I$. Thus, $R_\lambda$ is a sum of an nb-bounded operator and a multiple of the identity operator, which yields $\lambda \in \rho^{nb}(T)$. To finish the proof, it suffices to show that $\lambda = 0$ necessarily belongs to $\sigma^{bb}(T)$ (and, therefore, to $\sigma^{c}(T)$, $\sigma^{nn}(T)$, and $\sigma^{nb}(T)$). Indeed, if the resolvent $R_\lambda = T^{-1}$ were bb-bounded, then $I = T^{-1}T$ would be nb-bounded, which is impossible in a non-locally bounded space, a contradiction. □

**Proposition 2.5.8.** *If $T$ is an nb-bounded operator on a topological vector space, then $r_{bb}(T) = r_c(T) = r_{nn}(T) = r_{nb}(T)$.*

*Proof.* By Proposition 2.3.3 it suffices to show that $r_{bb}(T) \geqslant r_{nb}(T)$. Since $T$ is nb-bounded, then $T(U)$ is a bounded set for some zero neighborhood $U$. Let $\nu > r_{bb}(T)$ and fix a zero neighborhood $V$. Then $\nu V$ is again a zero neighborhood. In particular, since the sequence $\frac{T^n}{\nu^n}$ converges to zero uniformly on bounded sets, we have $\frac{T^n}{\nu^n}\big(T(U)\big) \subseteq \nu V$ for all sufficiently large $n$. Then $\frac{T^{n+1}}{\nu^{n+1}}(U) \subseteq V$, so that $\frac{T^n}{\nu^n}$ converges to zero uniformly on $U$. Therefore $\nu \geqslant r_{nb}(T)$, so that $r_{bb}(T) \geqslant r_{nb}(T)$. □

**2.5.9.** In view of Propositions 2.5.7 and 2.5.8 we can write $\sigma(T)$ instead of $\sigma^{bb}(T)$, $\sigma^{c}(T)$, $\sigma^{nn}(T)$, and $\sigma^{nb}(T)$ and $r(T)$ instead of $r_{bb}(T)$, $r_c(T)$, $r_{nn}(T)$, and $r_{nb}(T)$.

We have established in Theorems 2.4.1–2.4.5 that under some conditions the spectral radii of a linear operator are upper bounds for the geometrical radii of the corresponding spectra. Of course we would like to know when the equalities hold. It is well known that the equality $\big|\sigma(T)\big| = r(T)$ holds for every continuous operator on a Banach space. Moreover, it was shown in [Gram66] that this equality also holds for every continuous operator on a quasi-Banach space (a complete quasi-normed space). Further, by means of Proposition 2.3.7 the main result of [GV98] is equivalent to the following statement: $r(T) = |\sigma(T)|$ *for every nb-bounded operator* $T$ *on a complete locally convex space*. Here we present a direct proof of this. Our proof is a simplified version of the proof of [GV98].

**Theorem 2.5.10.** *If* $T$ *is an nb-bounded linear operator on a sequentially complete locally convex space, then* $|\sigma(T)| = r(T)$.

*Proof.* Suppose $T(U)$ is bounded for some base zero neighborhood $U$. It follows from Propositions 2.5.7, 2.5.8, and 2.3.3, and Theorem 2.4.5 that it suffices to show that $|\sigma^{nn}(T)| \geqslant r_{nb}(T)$. We are going to show that $T$ induces a continuous operator $\widetilde{T}$ on some Banach space such that $\sigma(\widetilde{T}) \subseteq \sigma^{nn}(T) \cup \{0\}$ while $r(\widetilde{T}) \geqslant r_{nb}(T)$, and then appeal to the fact that the spectral radius of a continuous operator on a Banach space equals the geometrical radius of the spectrum.

Consider $T$ as a continuous operator on the locally bounded space $X_U = (X, U)$. Then $\sigma_U(T)$ is defined by 2.2.3 and $r_U(T)$ is defined by 2.3.5. We claim that $r_U(T) \geqslant r_{nb}(T)$. To see this, suppose $r_U(T) < \nu$, then $\frac{T^n}{\nu^n}(U) \subseteq U$ for all sufficiently large $n$. Let $V$ be a base zero neighborhood, then $T(U) \subseteq \alpha V$ for some $\alpha > 0$, so that $\frac{T^n}{\nu^n}(U) = \frac{T}{\nu}\frac{T^{n-1}}{\nu^{n-1}}(U) \subseteq \frac{1}{\nu}T(U) \subseteq \frac{\alpha}{\nu}V$ for sufficiently large $n$. This implies that $\nu \geqslant r_{nb}(T)$, and it follows that $r_U(T) \geqslant r_{nb}(T)$.

On the other hand, we claim that $\sigma_U(T) \subseteq \sigma^{nn}(T)$. Suppose $\lambda \in \rho^{nn}(T)$, then $R_\lambda$ is nn-bounded with respect to some base $\mathcal{N}_0$ of zero neighborhood. We can assume without loss of generality that $U \in \mathcal{N}_0$, so that $R_\lambda(U) \subseteq \beta U$ for some $\beta > 0$. It follows that $\lambda \in \rho_U(T)$.

Since $U$ is convex, the the space $X_U$ is, in fact, a seminormed space. We can assume without loss of generality that it is a normed space, because otherwise we can consider the quotient space $X_U/(\operatorname{Null} T)$ and the quotient operator $\widehat{T}$ on this quotient space instead of $T$. Indeed, since $\bigcap_{n=1}^{\infty} \frac{1}{n}U \subseteq \operatorname{Null} T$ by 2.5.1, we conclude that the quotient space $X_U/(\operatorname{Null} T)$ is Hausdorff. It follows then that $X_U/(\operatorname{Null} T)$ is a normed space, and $\widehat{T}$ is norm bounded. The spectrum $\sigma_U(T)$ becomes even smaller when we substitute $T$ with $\widehat{T}$. Indeed, suppose $\lambda \in \rho_U(T)$, then the resolvent $R_\lambda$ exists in $X_U$ and is continuous.

If $x \in \ker T$, then $x = R_\lambda(\lambda I - T)x = \lambda R_\lambda x$, so that $R_\lambda$ leaves $\ker T$ invariant, and, therefore, induces a quotient operator $\widehat{R_\lambda}$ on $X_U/\ker T$ via $\widehat{R_\lambda}([x]) = [R_\lambda x]$. Clearly, $\widehat{R_\lambda}$ is continuous: if $[x_n] \to [x]$ in $X_U/\ker T$ then $x_n - z_n \to x$ in $X_U$ for some $(z_n)_{n=1}^\infty$ in $\ker T$, so that $[R_\lambda x_n] = [R_\lambda(x_n - z_n)] \to [R_\lambda x]$. On the other hand, $r_U(\widehat{T}) \geqslant r_U(T)$, because if $\nu > r_U(\widehat{T})$ then $\frac{\widehat{T}^n}{\nu^n}([U]) \subseteq [U]$ for all sufficiently large $n$, then $\frac{T^n}{\nu^n}(U) \subseteq U + \ker T$, so that $\frac{T^{n+1}}{\nu^{n+1}}(U) \subseteq \frac{1}{\nu}T(U) \subseteq \frac{\alpha}{\nu}U$ for some $\alpha > 0$. It follows that $\nu \geqslant r_U(T)$ and, therefore, $r_U(\widehat{T}) \geqslant r_U(T)$.

Finally, we consider the completion $\widetilde{X}_U$ of $X_U$, and extend $T$ to a continuous linear operator $\widetilde{T}$ on the completion. The spectrum of $\widetilde{T}$ is smaller that the spectrum of $T$, because if $\lambda \in \rho_U(T)$ then the resolvent $R_\lambda$ can be extended by continuity to $\widetilde{R_\lambda}$ on $\widetilde{X}$, and $\widetilde{R_\lambda}$ is a continuous inverse to $\lambda I - \widetilde{T}$, so that $\lambda \in \rho(\widetilde{T})$. On the other hand, $r(\widetilde{T}) \geqslant r_U(T)$ because if $\nu > r(\widetilde{T})$ then $\frac{\widetilde{T}^n}{\nu^n}(\widetilde{U}) \subseteq \widetilde{U}$ for all sufficiently large $n$, which implies $\frac{T^n}{\nu^n}(U) \subseteq U$ since $T$ is a restriction of $\widetilde{T}$ on $X$. $\square$

## 2.6 Compact operators

As with bounded operators, there is more than one way to define compact operators on an arbitrary topological vector space. A subset of a topological vector space is called ***precompact*** if its closure is compact. Given a linear operator $T$ on a topological vector space, we will say that $T$ is ***b-compact*** if it maps every bounded set into a precompact set. Similarly, $T$ is ***compact***[3] if it maps some neighborhood into a precompact set. Obviously, every compact operator is b-compact and nb-bounded (hence continuous); every b-compact operator is bb-bounded.

**2.6.1.** If $T$ is compact or b-compact, then sequential completeness is not needed in Theorems 2.4.1–2.4.5. Indeed, we used sequential completeness just once, namely, in the proof of Theorem 2.4.1 to justify the convergence of the sequence $R_{\lambda,n}x = \frac{1}{\lambda}\sum_{i=0}^n \frac{T^i x}{\lambda^i}$. But since the sequence $(R_{\lambda,n}x)_n$ is Cauchy and, therefore, bounded, the sequence $(TR_{\lambda,n}x)_n$

[3]To be consistent, we should have probably called these operators "n-compact", but following the convention we will refer to them as "compact".

has a convergent subsequence whenever $T$ is compact or b-compact. Furthermore, it follows from $R_{\lambda,n+1}x = \frac{1}{\lambda}(I + TR_{\lambda,n})x$ that $(R_{\lambda,n}x)_n$ has a convergent subsequence hence converges.

Let $K$ be a compact operator on an arbitrary topological vector space, and let $\sigma(K)$ and $r(K)$ be as in 2.5.9. It was proved in [Pech91] that $\sigma(K) = \{0\}$ implies $r_l(K) = 0$. In the following theorem we use the technique of [Pech91] to improve this result by showing that in general $r(K) \leqslant \lvert\sigma(K)\rvert$.

**Theorem 2.6.2.** *If $K$ is a compact operator on a Hausdorff topological vector space $X$, then $r(K) \leqslant \lvert\sigma(K)\rvert$.*

*Proof.* Assume that $\lvert\sigma(K)\rvert < r(K)$. Without loss of generality (by scaling $K$) we can assume that $\lvert\sigma(K)\rvert < 1 < r(K)$. Since $K$ is compact, there is a closed base zero neighborhood $U$ such that $\overline{K(U)}$ is compact. In particular $\overline{K(U)}$ is bounded, so that $\overline{K(U)} \subseteq \eta U$ for some $\eta > 0$. We can assume without loss of generality that $\eta > 1$. We define the following subsets of $U$:

$$U_1 = \overline{K(U)} \cap U, \quad U_{n+1} = K(U_n) \cap U \quad (n = 1, 2, \dots), \quad \text{and} \quad U_0 = \bigcap_{n=1}^{\infty} U_n.$$

Notice, that $U_1$ is compact because $\overline{K(U)}$ is compact and $U$ is closed. Also, if $U_n$ is compact, then $K(U_n)$ is compact as the image of a compact set under a continuous operator. Therefore, every $U_n$ for $n \geqslant 1$ is compact. Using induction, we can show that the sequence $(U_n)$ is decreasing. Indeed, $U_1 \subseteq U$ by definition, $U_2 = K(U_1) \cap U \subseteq K(U) \cap U \subseteq U_1$, and if $U_n \subseteq U_{n-1}$, then $U_{n+1} = K(U_n) \cap U \subseteq K(U_{n-1}) \cap U = U_n$. It follows also that $U_0$ is compact and contains zero.

Notice that $K$ maps every balanced set to a balanced set. Since $U$ is balanced, $U_n$ is balanced for each $n \geqslant 0$. If $A$ is a balanced subset of $U$, then obviously $A \subseteq (\eta A) \cap U$, and when we apply the same reasoning to $\frac{1}{\eta}K(A)$ instead of $A$ (which is also a balanced subset of $U$), we get $\frac{1}{\eta}K(A) \subseteq K(A) \cap U$. We use this to show by induction that $\frac{1}{\eta^n}K^n(U) \subseteq U_n$ for every $n \geqslant 1$. Indeed, for $n = 1$ we have $\frac{1}{\eta}K(U) \subset K(U) \cap U \subseteq U_1$.

Suppose $\frac{1}{\eta^n}K^n(U) \subseteq U_n$ for some $n \geqslant 1$, then

$$\tfrac{1}{\eta^{n+1}}K^{n+1}(U) \subseteq \tfrac{1}{\eta}K(U_n) \subseteq K(U_n) \cap U = U_{n+1},$$

which proves the induction step.

Next, we claim that there exists an open zero neighborhood $V$ and an increasing sequence of positive integers $(n_j)$ such that $U_{n_j} \setminus V$ is nonempty for every $j \geqslant 1$. Assume for the sake of contradiction that for every open zero neighborhood $V$ we have $U_n \subseteq V$ for all sufficiently large $n$. Since $\frac{1}{2}U$ contains an open zero neighborhood, then there exists a positive integer $N$ such that $U_n \subseteq \frac{1}{2}U$ whenever $n \geqslant N$. This implies that $U_{N+m} = K^m(U_N)$ for all $m \geqslant 0$. Indeed, this holds trivially for $m = 0$. Suppose that $U_{N+m} = K^m(U_N)$ for some $m \geqslant 0$. Then $U_{N+m+1} = K(U_{N+m}) \cap U = K^{m+1}(U_N) \cap U$, and this implies that $U_{N+m+1} = K^{m+1}(U_N)$ because $U_{N+m+1} \subseteq \frac{1}{2}U$. Now take any open zero neighborhood $V$, then $\frac{1}{\eta^N}V$ is again a zero neighborhood, and by assumption there exists a positive integer $M$ such that $U_n \subseteq \frac{1}{\eta^N}V$ whenever $n \geqslant M$. Let $n \geqslant \max\{M, N\}$, then

$$V \supseteq \eta^N U_n = \eta^N K^{n-N}(U_N) \supseteq \eta^N K^{n-N}\left(\tfrac{1}{\eta^N}K^N(U)\right) = K^n(U),$$

which contradicts the hypothesis $r_{nb}(K) = r(K) > 1$.

It follows from $U_{n_j} \setminus V \neq \emptyset$ for every $j \geqslant 1$ that $U_n \setminus V \neq \emptyset$ for all sufficiently large $n$ because $U_n$ is a decreasing sequence. Since $U_n \setminus V$ is a decreasing sequence of nonempty compact sets, then $U_0 \setminus V = \bigcap_{n=1}^{\infty}(U_n \setminus V) \neq \emptyset$, so that $U_0 \neq \{0\}$.

For every $n \geqslant 1$ we have $U_0 \subseteq U_n$, it follows that $K(U_0) \subseteq K(U_n)$ and, therefore, $K(U_0) \subseteq \bigcap_{n=1}^{\infty} K(U_n)$. Actually, the reverse inclusion also holds. To see this, let $y \in \bigcap_{n=1}^{\infty} K(U_n)$. Then $y = Kx_n$, where $x_n \in U_n \subseteq U_1$. Since $U_1$ is compact, the sequence $(x_n)$ has a cluster point, i.e., $x_{n_j} \to x$ for some subsequence $(x_{n_j})$ and some $x$. Since $K$ is continuous we have $y = Kx$. On the other hand, since every $U_{n_j}$ is closed we have $x \in U_{n_j}$, so that $x \in \bigcap_{n=1}^{\infty} U_{n_j} = U_0$. Thus $K(U_0) = \bigcap_{n=1}^{\infty} K(U_n)$.

Next, we claim that $U_0 \subseteq K(U_0) \subseteq \eta U_0$. Indeed,

$$U_0 = \bigcap_{n=2}^{\infty} U_n = \bigcap_{n=2}^{\infty} \big[K(U_{n-1}) \cap U\big] \subseteq \bigcap_{n=2}^{\infty} K(U_{n-1}) = K(U_0).$$

On the other hand, since $U_n$ are decreasing and $\eta > 1$, we have $K(U_n) \subseteq K(U_{n-1}) \subseteq \eta K(U_{n-1})$ and $K(U_n) \subseteq K(U) \subseteq \eta U$, so that $K(U_n) \subseteq \eta K(U_{n-1}) \cap \eta U = \eta U_n$, and this implies $K(U_0) \subseteq K(U_n) \subseteq \eta U_n$ for every $n$. Thus $K(U_0) \subseteq \eta U_0$.

Since $\overline{K(U)}$ is compact, hence bounded, then $\overline{K(U)} + \overline{K(U)}$ is also bounded. Then there is a positive constant $\gamma$ such that $\overline{K(U)} + \overline{K(U)} \subseteq \gamma U$. Without loss of generality we can assume $\gamma \geqslant 2$. It follows that

$$U_1 + U_1 = \overline{K(U)} \cap U + \overline{K(U)} \cap U \subseteq \overline{K(U)} + \overline{K(U)} \subseteq \gamma U.$$

We use induction to show that $U_n + U_n \subseteq \gamma U_{n-1}$. Indeed, since $A \cap B + C \cap D \subseteq (A + C) \cap (B + D)$ for any four sets $A$, $B$, $C$, and $D$, then

$$\begin{aligned} U_{n+1} + U_{n+1} &= K(U_n) \cap U_n + K(U_n) \cap U_n \\ &\subseteq \big[K(U_n) + K(U_n)\big] \cap (U_n + U_n) \subseteq K(U_n + U_n) \cap (U_n + U_n) \\ &\subseteq K(\gamma U_{n-1}) \cap \gamma U_{n-1} = \gamma\big[K(U_{n-1}) \cap U_{n-1}\big] = \gamma U_n. \end{aligned}$$

Finally, $U_0 + U_0 \subseteq \bigcap_{n=1}^{\infty}(U_n + U_n) \subseteq \bigcap_{n=1}^{\infty} \gamma U_n = \gamma U_0$.

Next, consider the set $F = \bigcup_{n=1}^{\infty} nU_0$. This set is closed under multiplication by a scalar, and $U_0 + U_0 \subseteq \gamma U_0$ implies that $F$ is a linear subspace of $X$. We consider the locally bounded topological vector space $(F, U_0)$ with multiples of $U_0$ as the base of zero neighborhoods. Since $U_0$ is balanced by definition, this topology is linear, and it is Hausdorff because $U_0$ is compact. Also, it is finer than the topology on $F$ inherited from $X$ because $U_0$ is compact and, therefore, bounded in $X$.

We claim that $(F, U_0)$ is complete. Indeed, if $(x_n)$ is a Cauchy sequence in $(F, U_0)$ then there exists $k > 0$ such that $x_n \in kU_0$ for each $n > 0$. Since $U_0$ is compact, the sequence $(x_n)$ has a subsequence which converges to some $x \in kU_0$ in the topology of $X$. Moreover, $\lim\limits_{n\to\infty} x_n = x$ because the sequence $(x_n)$ is Cauchy in $X$. Fix $\varepsilon > 0$, then there exists $n_0$ such that $x_n - x_m \in \varepsilon U_0$ whenever $n, m \geqslant n_0$. Let $m \to \infty$, since $U_0$ is is closed we have $x_n - x \in \varepsilon U_0$, i.e., $x_n \to x$ in $(F, U_0)$. Thus, $(F, U_0)$ is complete and, therefore, quasi-Banach.

It follows from $U_0 \subseteq K(U_0) \subseteq \eta U_0$ that $F$ is invariant under $K$ and the restriction $\widetilde{K} = K|_F$ is continuous. We claim that $\sigma(\widetilde{K}) \subseteq \sigma(K) \cup \{0\}$. Suppose that $\lambda \in \rho(K)$

and $\lambda \neq 0$, then $(\lambda I - K)$ is a homeomorphism, so that $(\lambda I - K)(U)$ is a closed zero neighborhood, and $\alpha U_1 \subseteq (\lambda I - K)(U)$ for some positive real $\alpha$ because $U_1$ is bounded. Further, $\alpha K(U_1) \subseteq K(\lambda I - K)(U) \subseteq (\lambda I - K)K(U)$. Therefore

$$\alpha U_2 \subseteq \alpha K(U_1) \cap \alpha U_1 \subseteq (\lambda I - K)K(U) \cap (\lambda I - K)(U),$$

and since $\lambda I - K$ is one-to-one we get $\alpha U_2 \subseteq (\lambda I - K)(K(U) \cap U) \subseteq (\lambda I - K)(U_1)$. Similarly, we obtain $\alpha U_{n+1} \subseteq (\lambda I - K)(U_n)$ for all $n \geqslant 1$, and then $\alpha U_0 \subseteq (\lambda I - K)(U_0)$. This implies that the restriction of $\lambda I - K$ to $F$ is onto, invertible, and the inverse is continuous. Thus, $\lambda \in \rho(\widetilde{K})$.

In particular this implies that $\left|\sigma(\widetilde{K})\right| \leqslant \left|\sigma(K)\right| < 1$. On the other hand, it follows from $U_0 \subseteq K(U_0)$ that $U_0 \subseteq \widetilde{K}^n(U_0)$ for all $n \geqslant 0$, so that $\widetilde{K}^n$ does not converge to zero uniformly on $U_0$, whence $r(\widetilde{K}) = r_{bb}(\widetilde{K}) \geqslant 1$. This produces a contradiction because it was proved in [Gram66] that the spectral radius of a continuous operator on a quasi-Banach space equals the radius of the spectrum. $\square$

**Corollary 2.6.3.** *If $K$ is a compact operator on a locally convex (or pseudo-convex) space, then $r(K) = \left|\sigma(K)\right|$.*

# Chapter 3

# The Invariant Subspace Problem for locally convex-solid lattices

## 3.1 Basic invariant subspace observations

In this section we present some cases where invariant subspaces of an operator can be easily found. A simple example of an invariant subspace is an eigenspace. Further, the linear span of an eigenvector is a one-dimensional closed invariant subspace, and the linear span of any collection of eigenvectors corresponding to the same eigenvalue is an invariant subspace. Therefore, regarding the Invariant Subspace Problem we can usually assume without loss of generality that the operator under consideration has no eigenvalues. The following proposition shows that we can also assume that the operator is one-to-one and has a dense range.

**Proposition 3.1.1.** *If $T\colon X \to X$ is a linear operator on a topological vector space, then its null-space and range are $T$-hyperinvariant subspaces.*

*Proof.* Suppose $S$ is another linear operator on $X$ such that $ST = TS$. If $x \in \operatorname{Null} T$ then $TSx = STx = 0$, so that $Sx \in \operatorname{Null} T$. It follows that $S(\operatorname{Null} T) \subseteq \operatorname{Null} T$, i.e., $\operatorname{Null} T$ is a $T$-hyperinvariant subspace. Further, for each $x \in X$ it follows from $STx = TSx$ that $STx \in \operatorname{Range} T$, so that $\operatorname{Range} T$ is also $T$-invariant. $\square$

**Corollary 3.1.2.** *If an operator $T$ on a topological vector space commutes with a continuous operator which is either not one-to-one or fails to have a dense range, then $T$ has a non-trivial closed invariant subspace.*

**Corollary 3.1.3.** *If $T$ is an nb-bounded operator on a topological vector space in which every zero neighborhood contains a linear subspace, then $T$ has a non-trivial closed invariant subspace.*

*Proof.* By 2.5.1 we know $T$ has a non-trivial null-space. $\square$

It looks quite plausible that every nb-bounded operator $T$ on a complete locally convex but not locally bounded space has a closed non-trivial invariant subspace. In [GV97] in Remark 15 and Theorem 16 the authors claim that this hypothesis is true. In order to prove it, the authors note that $T(U)$ is bounded for some zero neighborhood $U$, and so the null space of the Minkowski functional $p_U$ of $U$ is a closed $T$-invariant subspace. However, as our Example 2.5.2 demonstrates, $\operatorname{Null} p_U$ may be trivial, and, hence, the proof in [GV97] is not valid.

It follows from the following proposition that when dealing with the Invariant Subspace Problem for a continuous operator on a locally-convex space, not only can we assume that $T$ has no eigenvectors, but also that the dual operator has no eigenvectors.

**Proposition 3.1.4.** *Let $T$ be a continuous non-scalar operator on a locally convex topological space $X$. If either $T$ or its adjoint $T'$ has an eigenvector, then $T$ has a non-trivial closed hyperinvariant subspace.*

*Proof.* Clearly, if $Tx = \lambda x$ for some $x \neq 0$, then the eigenspace of $\lambda$ is closed, non-trivial, and $T$-hyperinvariant. Assume that $T'f = \lambda f$ for some scalar $\lambda$ and some non-zero $f \in X'$. Since $T$ is non-scalar, the range of $T - \lambda I$ is non-trivial. Since $f \neq 0$ and $f(Tx - \lambda x) = \langle (T' - \lambda I)f, x\rangle = 0$ for every $x \in X$, we see that the range of $T - \lambda I$ is not dense in $X$. To verify that $\operatorname{Range}(T - \lambda I)$ is $T$-hyperinvariant, take a continuous operator $S$ on $X$ such that $ST = TS$. Then for each $x \in X$ we have $S(T - \lambda I)x = (T - \lambda I)Sx \in \operatorname{Range}(T - \lambda I)$, so that $S\big(\operatorname{Range}(T - \lambda I)\big) \subseteq \operatorname{Range}(T - \lambda I)$.

Thus, the closure of Range$(T - \lambda I)$ is a non-trivial closed $T$-hyperinvariant subspace of $X$. $\square$

**3.1.5.** Recall that an operator $T$ on a topological vector space $X$ is said to be ***locally quasinilpotent*** at a point $x \in X$ if $\lim\limits_{n\to\infty} \frac{T^n x}{\nu^n} = 0$ for every real $\nu > 0$. The set of all points at which $T$ is locally quasinilpotent is denoted by $\mathcal{Q}_T$. Clearly, $\mathcal{Q}_T$ is a linear subspace of $X$. We claim that $\mathcal{Q}_T$ is $T$-hyperinvariant. Indeed, if $S$ is a continuous operator such that $ST = TS$ and $x \in \mathcal{Q}_T$, then

$$\lim_{n\to\infty} \tfrac{T^n(Sx)}{\nu^n} = S\bigl(\lim_{n\to\infty} \tfrac{T^n x}{\nu^n}\bigr) = 0$$

for every $\nu > 0$, so that $Sx \in \mathcal{Q}_T$. Therefore, if $\overline{\mathcal{Q}_T}$ is a closed $T$-hyperinvariant subspace (although it may be trivial).

**3.1.6.** Next, suppose that $E$ is a vector lattice and $T$ is a positive operator on $E$. The ***null ideal*** of $T$ is defined by the following formula: $N_T = \bigl\{x \in E \ : \ T\bigl(\lvert x\rvert\bigr) = 0\bigr\}$. Clearly, $N_T$ is an (order) ideal and a subspace of the null-space of $T$. Further, $N_T$ is invariant under every positive operator that commutes with $T$. Indeed, suppose that $S$ is another positive operator on $E$ and $x \in N_T$, then

$$T\bigl(\lvert Sx\rvert\bigr) \leqslant TS\bigl(\lvert x\rvert\bigr) = ST\bigl(\lvert x\rvert\bigr) = 0.$$

If $E$ is a locally convex-solid vector lattice, and $T$ is positive and continuous, then $N_T$ is closed.

**3.1.7.** Let $J$ be an ideal in a locally convex-solid vector lattice $E$. If $J$ is invariant under some positive operator $T\colon E \to E$, then $J$ is also invariant under every operator $S$ which is polynomially dominated by $T$. Indeed, let $P(t)$ be a polynomial with non-negative coefficients such that $P(T)$ dominates $S$. If $x \in J$, then $\lvert Sx\rvert \leqslant P(T)\bigl(\lvert x\rvert\bigr) \in J$ implies that $Sx \in J$, that is, $J$ is $S$-invariant.

**3.1.8.** For a positive operator $T$ on a vector lattice $E$ and for every integer $n \geqslant 0$ let $A_n = \sum_{i=0}^n T^i$. If $x$ is a positive element of $E$, then $x_n = A_n x$ is an increasing sequence in $E$, so that $(E_{x_n})$ is an increasing sequence of principal ideals. Then $J_x = \bigcup_{n=0}^{\infty} E_{x_n}$ is a non-zero $T$-invariant ideal in $E$. Indeed, $J_x$ is non-zero because $x \in J_x$. Further, if $y \in J_x$ then $|y| \leqslant \lambda x_n$ for some positive $\lambda$ and $n$. Then

$$|Ty| \leqslant T|y| \leqslant \lambda T x_n \leqslant \lambda \sum_{i=0}^{n} T^{i+1} x \leqslant \lambda x_{n+1},$$

so that $Ty \in E_{x_{n+1}} \subseteq J_x$.

Sometimes (e.g., when $T$ is nb-bounded but $E$ is not locally bounded, and we want $A_n$ to be nb-bounded) it is more convenient to define $A_n$'s and $J_x$ via $A_n = \sum_{i=1}^{n} T^i$ for all $n \geqslant 1$ and $J_x = \bigcup_{n=1}^{\infty} E_{A_n x}$. Then $J_x$ is still a $T$-invariant ideal; $J_x = \{0\}$ for some $x \in E_+$ if and only if the null ideal $N_T$ is non-trivial, so that when dealing with the Invariant Subspace Problem, we can usually assume without loss of generality that $J_x$ is non-zero.

## 3.2 Known results on Banach lattices

Here we list several major results on the Invariant Subspace Problem for positive operators on Banach lattices. Complete proofs of these and other related results can be found in [AAB93, AAB94, AAB98]. In Sections 3.4 and 3.5 we generalize the following results to positive operators on locally convex-solid lattices.

**Theorem 3.2.1 ([AAB98]).** *Let $S, T \colon E \to E$ be two positive operators on a Banach lattice such that*

*(i) $ST \leqslant TS$ (in particular, this holds if $S$ commutes with $T$),*

*(ii) $S$ is locally quasinilpotent at some $x > 0$, and*

*(iii) $S$ dominates a non-zero compact operator.*

*Then the operator $T$ has a non-trivial closed invariant subspace. Moreover, we can choose it to be the closure of a principal ideal in $E$.*

**Theorem 3.2.2 ([AAB98]).** *Let $S, T \colon E \to E$ be two positive operators on a Banach lattice such that*

*(i) $ST \geqslant TS$ (in particular, this holds if $S$ commutes with $T$),*

*(ii) $S$ is quasinilpotent, and*

*(iii) $S$ dominates a non-zero compact operator.*

*Then the operator $T$ has a non-trivial closed invariant subspace. Moreover, we can choose it to be the closure of a principal ideal in $E$.*

**Theorem 3.2.3 ([AAB98]).** *Let $S, T \colon E \to E$ be two positive commuting operators on a Banach lattice. If one of them is quasinilpotent at a non-zero positive vector, and the other dominates a non-zero compact operator, then $T$ and $S$ have a common non-trivial closed invariant ideal.*

**Theorem 3.2.4 ([AAB98]).** *If a positive operator $T \colon E \to E$ on a Banach lattice is compact-friendly and locally quasinilpotent at some $x > 0$, then $T$ has a non-trivial closed invariant ideal. Moreover, if another positive operator $S$ commutes with $T$, then $S$ and $T$ have a common non-trivial closed invariant ideal.*

## 3.3 Cube theorem and relative uniform topology

In order to generalize the results of Section 3.2, we will need locally convex-solid versions of some tools which Y. Abramovich, C. Aliprantis, and O. Burkinshaw used in [AAB98] when dealing with the Invariant Subspace Problem in Banach lattices.

### Cube theorem

The following important theorem originally appeared in [AB80]. The proof can be also found in [AB85]:

**Theorem 3.3.1.** *If in the scheme of continuous operators* $E \xrightarrow{M_1} F \xrightarrow{M_2} G \xrightarrow{M_3} H$ *between Banach lattices each operator* $M_i$ *is dominated by a compact positive operator, then the operator* $M_3M_2M_1$ *is compact.*

We will use a locally convex-solid version of this result, which was proved by G. Lepkes ([Lep93], Corollary 4.3.5):

**Theorem 3.3.2.** *Let* $E_j$ $(1 \leqslant j \leqslant 4)$ *be locally convex-solid vector lattices and* $T_j$, $S_j$ *be positive linear operators from* $E_j$ *to* $E_{j+1}$ *with* $0 \leqslant S_j \leqslant T_j$ *and* $T_j$ *compact,* $0 \leqslant j \leqslant 3$. *Then the operator* $S_3S_2S_1 \colon E_1 \to E_4$ *is compact.*

### Relative uniform topology

**Definition 3.3.3.** A net $(x_\alpha)$ in a vector lattice $E$ ***relatively uniformly converges*** (or ***$(r_u)$-converges***) to $x \in E$ if there exists $u \in E_+$ such that for every positive $\varepsilon$ there exists an index $\alpha_0$ such that $|x - x_\alpha| \leqslant \varepsilon u$ holds for all $\alpha \geqslant \alpha_0$. Analogously, we say that $(x_\alpha)$ is ***$(r_u)$-Cauchy*** if for all $\varepsilon > 0$ there exists an index $\alpha_0$ such that $|x_\alpha - x_\beta| \leqslant \varepsilon u$ for all $\alpha, \beta \geqslant \alpha_0$. A vector lattice is called ***$(r_u)$-complete*** if every $(r_u)$-Cauchy net $(r_u)$-converges.

The element $u$ in the above definition is referred to as a ***regulator of convergence***. It can be easily verified that $(r_u)$-convergence is compatible with vector lattice operations. In an Archimedean vector lattice $(r_u)$-convergence implies order convergence. In a locally convex-solid vector lattice $(r_u)$-convergence implies topological convergence because $|x - x_\alpha| \leqslant \varepsilon u$ implies $p(x - x_\alpha) \leqslant \varepsilon p(u)$ for every lattice seminorm $p$. Therefore $(r_u)$-completeness is a weaker condition than topological or order completeness.

We will need the following well known lemma.

**Lemma 3.3.4.** *Let* $E$ *be a vector lattice* $(r_u)$*-complete with respect to an order unit* $u$. *Then* $E$ *endowed with the norm*

$$\|y\| = \inf\{\lambda > 0 \ : \ |y| \leqslant \lambda u\}$$

*is an AM-space with unit* $u$*, whose closed unit ball is the order interval* $\big[-|u|, |u|\big]$.

*Proof.* It can be easily verified that $\|\cdot\|$ is really a lattice norm on $E$, having $\big[-|u|,|u|\big]$ as its closed unit ball. Now, notice that $\|\cdot\|$ convergence is the same as $(r_u)$-convergence, and a net $(x_\alpha)$ in $E$ is $\|\cdot\|$-Cauchy if and only if it is $(r_u)$-Cauchy, because $\|x_\alpha - x_\beta\| \leqslant \varepsilon$ is equivalent to $|x_\alpha - x_\beta| \leqslant \varepsilon u$. Since $E$ is $(r_u)$-complete, there exists $y \in E$ such that $(x_\alpha)$ is $(r_u)$-convergent to $y$. Finally, $(r_u)$-convergence of $(x_\alpha)$ to $y$ implies convergence in norm, so that $\big(E, \|\cdot\|\big)$ is a Banach space. □

The following lemma is analogous to Lemma 9.3 of [AAB98].

**Lemma 3.3.5.** *Let $E$ be a vector lattice, $(r_u)$-complete with respect to some order unit $u$. Then*

*(i) For every non-zero element $y \in E$ there exists a linear operator $V$ on $E$ such that $Vy > 0$ and $|Vx| \leqslant |x|$ for all $x \in E$;*

*(ii) For all elements $v, w \in E$ satisfying $0 \leqslant v \leqslant w$ there exists a linear operator $U$ on $E$ such that $Uw = v$ and $|Ux| \leqslant |x|$ for all $x \in L$.*

*Proof.* By the Lemma 3.3.4 we know that $E$ with the norm given by $\|y\| = \inf\{\lambda > 0 : |y| \leqslant \lambda u\}$ is an AM-space with unit $u$. By the Kakutani-Krein representation theorem, there exists a compact Hausdorff space $\Omega$ such that $E$ is lattice isomorphic to the space $C(\Omega)$ of all continuous functions on $\Omega$ with the sup-norm $\|y\|_\infty = \max\{|y(\omega)| : \omega \in \Omega\}$, and the element $u$ corresponds to the constant function $\mathbf{1}$ on $\Omega$.

(i) Now fix $y \in E$ with $y \neq 0$ and view $y$ as a continuous function on $\Omega$. By scaling appropriately, we can suppose that $\|y\|_\infty = 1$. Now consider the function $\overline{y} \in C(\Omega)$ (the complex conjugate of $y$) and denote by $V$ the multiplication operator on $C(\Omega)$ (and hence on $E$) defined by $Vx = \overline{y}x$. Clearly, $Vy = |y|^2 > 0$ and $|Vx| \leqslant |x|$ holds for each $x \in E$.

(ii) Again, as above, we view $v$ and $w$ as continuous functions on $\Omega$ and consider the function

$$h(t) = \begin{cases} \frac{v(t)}{w(t)} & \text{if } w(t) \neq 0; \\ 0 & \text{otherwise.} \end{cases}$$

Since $0 \leqslant v \leqslant w$ then $v = hw$. Let $U$ be the multiplication operator on $C(\Omega)$ (and hence on $E$) defined by $h$, i. e. $Ux = hx$ for each $x \in C(\Omega)$. Clearly, $Uw = v$ and $|Ux| \leqslant |x|$ for each $x \in E$. □

Recall that a positive element $u$ in a locally convex-solid vector lattice $E$ is said to be a ***quasi-interior point*** if the ideal $E_u$ generated by $u$ is dense in $E$.

**Corollary 3.3.6.** *Let $E$ be a complete locally convex-solid vector lattice and $u \in E_+$ a quasi-interior point in $E$. Then the following properties hold:*

*(i) For every non-zero element $y \in E_u$ there exists a continuous operator $V$ on $E$ such that $Vy > 0$ and $|Vx| \leqslant |x|$ for all $x \in E$;*

*(ii) For every element $v \in E$ satisfying $0 \leqslant v \leqslant u$ there exists a continuous operator $U$ on $E$ such that $Uu = v$ and $|Ux| \leqslant |x|$ for all $x \in E$.*

*Proof.* Notice first that $E_u$ with the norm defined via

$$\|y\| = \inf\{\lambda > 0 \;:\; |y| \leqslant \lambda u\} \tag{3.1}$$

is a normed lattice with order unit $u$. Let $F$ be the closure of $E_u$ in the norm $\|\cdot\|$ or, equivalently, in $(r_u)$-convergence. Since every point in $F$ is a $(r_u)$-limit of points of $E_u$ and since $(r_u)$-convergence implies convergence in the topology of $E$, we have $E_u \subseteq F \subseteq E$. Further, $F$ is a Banach lattice with order unit $u$ and the norm given by (3.1), so that $(F, \|\cdot\|)$ is an AM-space by Lemma 3.3.4. Now Lemma 3.3.5 guarantees the existence of operators $V$ and $U$ on $F$ with the required properties. Since $|Vx| \leqslant |x|$ and $|Ux| \leqslant |x|$ for all $x \in F$, it follows that $V$ and $U$ are continuous on $F$ in the topology induced from $E$. Since $E_u$ is dense in $E$ then $V$ and $U$ can be extended to continuous operators on $E$ satisfying the required properties. □

## 3.4 Semi-commuting operators

We start with a locally convex version of Theorem 3.2.1.

**Theorem 3.4.1.** *Let $T$ and $S$ be two positive operators on a locally convex-solid vector lattice $E$ such that*

*(i)* $ST \leqslant TS$;

*(ii) either $T$ is nn-bounded*[1] *or $T$ is continuous with $r_c(T) < \infty$;*

*(iii) $S$ is locally quasinilpotent at some $x_0 > 0$;*

*(iv) $S$ dominates a non-zero compact operator.*

*Then the operator $T$ has a non-trivial closed invariant ideal.*

*Proof.* Let $T$, $S$, and $x_0$ satisfy the hypotheses of the theorem, and let $K$ be a non-zero compact operator dominated by $S$. Similarly to 3.1.8, let $A_n = \sum_{i=0}^{n} T^i$ for each positive integer $n$, and $J_x = \bigcup_{n=0}^{\infty} E_{A_n x}$. Then the ideal $J_x$ is non-zero and $T$-invariant for every $x > 0$. The proof will be finished if we show that $J_x$ is not dense in $E$ for some positive $x$. Assume, for the sake of contradiction, that $\overline{J_x} = E$ for every $x > 0$.

We claim that without loss of generality we can suppose $Kx_0 \neq 0$. Indeed, if $Ky = 0$ for each $0 < y \in J_{x_0}$, then $K = 0$ on $J_{x_0}$. However, since $J_{x_0}$ is dense in $E$, we conclude that $K = 0$, which contradicts the hypothesis. Therefore, $Ky_0 \neq 0$ for some $0 < y_0 \in J_{x_0}$. Since $S$ is locally quasinilpotent at $x_0$ and $0 < y_0 < \lambda \sum_{i=0}^{m} T^i x_0$ for some $m$ and $\lambda > 0$, for any positive number $\nu$ we have

$$0 \leqslant \nu^n S^n y_0 \leqslant \lambda \sum_{i=1}^{m} T^i \nu^n S^n x_0 \to 0 \text{ as } n \to \infty.$$

This implies that $S$ is also locally quasinilpotent at $y_0$. Now, replacing $x_0$ by $y_0$ if necessary, we can assume $Kx_0 \neq 0$.

There exists an open solid base zero neighborhood $V$ such that $K(V)$ is pre-compact, $x_0 \notin \overline{V}$, and $Kx_0 \notin \overline{V}$.[2] We can assume without loss of generality that $K(V) \subseteq V$, this

[1] With respect to some base of convex solid zero neighborhoods.

[2] Indeed, for each of these conditions one can find a zero neighborhood satisfying the condition. Take the intersection of these neighborhoods, then there is a base zero neighborhood inside the intersection, it would satisfy all the conditions.

can be done by scaling $K$ and $S$ if necessary. If while scaling we loose the condition $Kx_0 \notin \overline{V}$, it can be recaptured be scaling $x_0$, and we still can assume that $x_0 \notin \overline{V}$. Now consider the set $U = x_0 + V$. Then $K(U)$ is precompact, $0 \notin \overline{U}$ and $0 \notin \overline{K(U)}$.

By assumption, $\overline{J_z} = E$ for each $z > 0$. Fix positive $x$ and $z$ in $E$, then $(x+V) \cap J_z$ is nonempty. let $v \in (x+V) \cap J_z$. Then $v \in E_{A_{m_0}z}$ for some $m_0$. Consider $u = v^+ \wedge x$, then obviously $u \in E_{A_{m_0}z}$ and $0 \leqslant u \leqslant x$. Since $|u - x| \leqslant |v - x|$ and $v - x \in V$, we conclude that $u - x \in V$. Pick any $k_0$ such that $u \leqslant k_0 A_{m_0} z$, then $kA_m z \geqslant k_0 A_m z \geqslant k_0 A_{m_0} z$ for every $k \geqslant k_0$ and for every $m \geqslant m_0$, so that

$$0 \leqslant x - x \wedge kA_m z \leqslant x - x \wedge k_0 A_{m_0} z \leqslant x - u.$$

This yields $x - x \wedge kA_m z \in V$ holds for all $k \geqslant k_0$ and $m \geqslant m_0$.

Thus, for every $z \neq 0$ there exist positive integers $k$ and $m$ such that $x_0 \wedge kA_m|z| \in x_0 + V = U$. Since the function $z \mapsto x_0 \wedge kA_m|z|$ is continuous, we see that the sets $O_{k,m} = \{z \in E \,:\, x_0 \wedge kA_m|z| \in U\}$ are open and cover $E \setminus \{0\}$. In view of the condition $0 \notin \overline{K(U)}$, the above argument guarantees that $\overline{K(U)} \subseteq \bigcup_{k,m=1}^{\infty} O_{k,m}$. Since the sets $O_{k,m}$ are increasing as $k$ and $m$ increase, the compactness of $\overline{K(U)}$ implies that $\overline{K(U)} \subseteq O_{k,m}$ for some fixed $k$ and $m$. In other words, there exist $k$ and $m$ such that $z \in \overline{K(U)}$ implies $x_0 \wedge kA_m|z| \in U$.

In particular, we have $x_1 = x_0 \wedge kA_m|Kx_0| \in U$. Since $K(x_1) \in K(U)$, it follows that $x_2 = x_0 \wedge kA_m|Kx_1| \in U$. Proceeding inductively, we obtain a sequence $(x_n)$ of positive vectors in $U$ defined by $x_{n+1} = x_0 \wedge kA_m|Kx_n|$. We claim that $0 \leqslant x_n \leqslant k^n A_m^n S^n x_0$ holds for each $n$. The proof is by induction. For $n = 1$, we have the inequality $x_1 = x_0 \wedge kA_m|Kx_0| \leqslant kA_m S x_0$. For the induction step, notice that $ST \leqslant TS$ implies $SA_m^n \leqslant A_m^n S$ and

$$0 \leqslant x_{n+1} \leqslant kA_m|Kx_n| \leqslant kA_m S x_n \leqslant kA_m S(k^n A_m^n S^n x_0) \leqslant k^{n+1} A_m^{n+1} S^{n+1} x_0.$$

Thus we have $0 \leqslant x_n \leqslant k^n A_m^n S^n x_0$. We claim that $\lim\limits_{n\to\infty} x_n = 0$. Indeed, suppose that $T$ is nn-bounded, then $A_m$ is also nn-bounded, and by Proposition 2.1.12 we have $p(A_m) < \infty$ for every generating seminorm $p$ on $E$. Then $\lim\limits_{n\to\infty} p(x_n) \leqslant \lim\limits_{n\to\infty} k^n p(A_m)^n p(S^n x_0) = 0$

because $S$ is locally quasinilpotent at $x_0$, so that $x_n \to 0$. If $T$ is continuous and $r_c(T)$ is finite then $r_c(A_m)$ is finite by Theorem 2.3.10, and since the sequence $(S^n x_0)$ is fast null it follows from Lemma 2.3.12 that $\lim_{n\to\infty} k^n A_m^n S^n x_0 = 0$, so that again $\lim_{n\to\infty} x_n = 0$. But this contradicts the assumptions that $x_n \in U$ and $0 \notin \overline{U}$. □

The following theorem is another modification of this result. The proof is similar, but we present it for completeness.

**Theorem 3.4.2.** *Let $T$ and $S$ be two positive operators on a locally convex-solid vector lattice $E$ such that*

*(i) $ST \geqslant TS$;*

*(ii) $S$ and $T$ are either nn-bounded, or continuous with finite $r_c(S)$ and $r_c(T)$;*

*(iii) $T$ is locally quasinilpotent at some $x_0 > 0$;*

*(iv) $S$ dominates a non-zero compact operator.*

*Then the operator $T$ has a non-trivial closed invariant ideal.*

*Proof.* Let $T$, $S$, and $x_0$ satisfy the hypotheses of the theorem, and let $K$ be a non-zero compact operator dominated by $S$. As in 3.1.8, let $A_m = \sum_{i=1}^{n} T^i$ for each positive integer $n$, and $J_x = \bigcup_{m=1}^{\infty} E_{A_m x}$, then either the null ideal $N_T$ is a closed nontrivial $T$-invariant ideal, or $J_x$ is non-zero and $T$-invariant for every $x > 0$. The proof will be finished if we show that $J_x$ is not dense in $E$ for some positive $x$. Assume for the sake of contradiction, that $\overline{J_x} = E$ for every $x > 0$.

We claim that without loss of generality we can suppose $Kx_0 \neq 0$. Indeed, if $K(y) = 0$ for each $0 < y \in J_{x_0}$, then $K = 0$ on $J_{x_0}$ which is dense in $E$. This implies $K = 0$, which contradicts the hypothesis. Therefore, $Ky_0 \neq 0$ for some $0 < y_0 \in J_{x_0}$. Since $T$ is locally quasinilpotent at $x_0$, and $0 < y_0 \leqslant \lambda \sum_{i=1}^{m} T^i x_0$ for some $m$ and $\lambda > 0$, for any positive number $\nu$ we have

$$0 \leqslant \nu^n T^n y_0 \leqslant \nu^n T^n \lambda \sum_{i=1}^{m} T^i x_0 \leqslant \lambda \sum_{i=1}^{m} \nu^n T^{n+i} x_0 \to 0 \text{ as } n \to \infty.$$

This implies that $T$ is also locally quasinilpotent at $y_0$. Now replacing $x_0$ by $y_0$ if necessary, we can assume $Kx_0 \neq 0$.

There exists an open solid base zero neighborhood $V$ such that $K(V)$ is pre-compact, $x_0 \notin \overline{V}$, $K(V) \subseteq V$, and $Kx_0 \notin \overline{V}$. Let $U = x_0 + V$, then $K(U)$ is precompact, $0 \notin \overline{U}$ and $0 \notin \overline{K(U)}$. As in the proof of Theorem 3.4.1 we produce a sequence $(x_n)$ of points in $U$ via $x_{n+1} = x_0 \wedge kA_m|Kx_n|$ for some positive integer $k$ and $m$. Furthermore, we claim that $0 \leqslant x_n \leqslant k^n S^n A_m^n x_0$ holds for each $n$. The proof is by induction. By hypothesis, $TS \leqslant ST$, so that $A_m^n S \leqslant SA_m^n$. For $n = 1$, we have $0 \leqslant x_1 = x_0 \wedge kA_m|Kx_0| \leqslant kA_m Sx_0$. For the induction step, notice that if $0 \leqslant x_n \leqslant k^n S^n A_m^n x_0$ holds for some $n$, then

$$0 \leqslant x_{n+1} = x_0 \wedge kA_m|Kx_n| \leqslant kA_m Sx_n \leqslant kA_m S(k^n S^n A_m^n x_0) \leqslant k^{n+1} S^{n+1} A_m^{n+1} x_0.$$

Thus we have $0 \leqslant x_n \leqslant k^n S^n A_m^n x_0$. We claim that $\lim\limits_{n\to\infty} x_n = 0$. Notice that $A_m = RT = TR$ where $R = I + T + \cdots + T^{m-1}$. If both $S$ and $T$ are nn-bounded, then we can assume without loss of generality that $p(S)$ and $p(T)$ are finite for every generating seminorm $p$. It follows that $p(R)$ is finite and

$$p(x_n) \leqslant p(k^n S^n A_m^n x_0) \leqslant k^n p(S)^n p(R)^n p(T^n x_0) \to 0,$$

so that $\lim\limits_{n\to\infty} x_n = 0$. If $S$ and $T$ are continuous and $r_c(S)$ and $r_c(R)$ are finite, then by Theorem 2.3.10 we have $r_c(R) < \infty$, and by Lemma 2.3.12 we have $k^n S^n A_m^n x_0 = k^n S^n R^n T^n x_0 \to 0$, so that again $\lim\limits_{n\to\infty} x_n = 0$. But this contradicts the assumptions that $x_n \in U$ and $0 \notin \overline{U}$. $\square$

And yet another modification of the same idea.

**Theorem 3.4.3.** *Let $T$ and $S$ be two positive operators on a locally convex-solid vector lattice $E$ such that*

*(i) $ST \geqslant TS$;*

*(ii) $r_c(S) = 0$ and $T$ is continuous with $r_c(T) < \infty$;*

*(iii) $S$ dominates a non-zero compact operator.*

*Then the operator $T$ has a non-trivial closed invariant ideal.*

*Proof.* Let $T$ and $S$ satisfy the properties stated in the theorem, and let $K$ be a non-zero compact operator dominated by $S$. Since $K$ is non-zero, there exists a positive $x_0$ such that $Kx_0 \neq 0$. Again, let $A_m = \sum_{i=0}^{n} T^i$ for each positive integer $n$, and $J_x = \bigcup_{m=0}^{\infty} E_{A_m x}$, then $J_x$ is non-zero and $T$-invariant for every $x > 0$. The proof will be finished if we show that $J_x$ is not dense in $E$ for some positive $x$. Assume for the sake of contradiction that $\overline{J_x} = E$ for every $x > 0$.

There exists an open solid base zero neighborhood $V$ such that $K(V)$ is pre-compact, $x_0 \notin \overline{V}$, $K(V) \subseteq V$, and $Kx_0 \notin \overline{V}$. Let $U = x_0 + V$, then $K(U)$ is precompact, $0 \notin \overline{U}$ and $0 \notin \overline{K(U)}$. As in the proof of Theorem 3.4.2 we produce a sequence $(x_n)$ of points in $U$ via $x_{n+1} = x_0 \wedge kA_m|Kx_n|$ for some positive integer $k$ and $m$, then $0 \leqslant x_n \leqslant k^n S^n A_m^n x_0$ for every $n > 0$. Again, we are going to show that $\lim\limits_{n\to\infty} x_n = 0$ and thus obtain a contradiction.

Since $r_c(T)$ is finite then $r_c(A_m)$ is also finite by Theorem 2.3.10. Let $\nu > r_c(A_m)$, then the sequence $\frac{A_m^n x_0}{\nu^n}$ is ultimately bounded by Proposition 2.3.7. Let $\mu = \frac{1}{k\nu}$, then $\mu > 0 = r_c(S)$ and

$$x_n \leqslant k^n S^n A_m^n x_0 = \frac{S^n}{\mu^n} \frac{A_m^n x_0}{\nu^n} \to 0$$

by Proposition 2.3.7. □

Next, we present a generalization of Theorem 3.2.3, which guarantees the existence of a common invariant ideal for two positive operators on a Banach lattice, one of which is locally quasinilpotent, and the other dominates a compact operator. In a locally convex-solid case we can do this assuming additionally that the compact operator is positive.

**Theorem 3.4.4.** *Let $T$ and $S$ be two commuting positive operators on a locally convex-solid vector lattice $E$ such that $S$ and $T$ are either nn-bounded or continuous with $r_c(S)$ and $r_c(T)$ finite. If $T$ is locally quasinilpotent at a positive vector and $S$ dominates a non-zero positive compact operator, then the operators $S$ and $T$ have a common non-trivial closed invariant ideal.*

*Proof.* Assume that $T$ is quasinilpotent at some point $x_0 > 0$, and that $S$ dominates a positive non-zero compact operator $K$. If $T$ is not strictly positive, then the null-ideal $N_T$ is the desired non-trivial common closed invariant ideal. So, suppose that $T$ is strictly positive. Consider the sequence of operators $A_m = \sum_{i=0}^{m}(S+T)^i$. Then by 3.1.8 the ideal $J_{x_0} = \bigcup_{m=1}^{\infty} E_{A_m x_0}$ is $(S+T)$-invariant. Since $0 \leqslant S, T \leqslant S+T$, 3.1.7 implies immediately that this ideal is invariant under both $S$ and $T$.

Assume $\overline{J_{x_0}} = E$, then there exists some $0 < y_0 \in J_{x_0}$ such that $Ky_0 > 0$ because $K$ is non-zero by assumption. Notice that $T$ is still locally quasinilpotent at $y_0$: since $y_0 \in J_{x_0}$, then there is some positive real $\lambda$ and positive integer $m$ such that $0 < y_0 \leqslant \lambda A_m x_0$. Then

$$0 \leqslant \alpha^n T^n y_0 \leqslant \alpha^n T^n \lambda \sum_{i=0}^{m} (S+T)^i x \leqslant \lambda \sum_{i=0}^{m} (S+T)^i \alpha^n T^n x,$$

which converges to zero as $n$ approaches infinity for every $\alpha \geqslant 0$.

Since $T$ is strictly positive, we have $TKy_0 > 0$, and so $TK$ is non-zero. Also, we have $TK \leqslant TS$. Thus, the positive continuous operator $TS$ dominates the non-zero compact operator $TK$. Show that $TS$ is locally quasinilpotent at $y_0$. Indeed, if $S$ is nn-bounded, then for every generating seminorm and every positive $\nu$ we have

$$\lim_{n\to\infty} p\Big(\tfrac{(ST)^n}{\nu^n} y_0\Big) \leqslant \lim_{n\to\infty} \tfrac{p(S)^n}{\nu^n} p(T^n y_0) = 0.$$

If $S$ and $T$ are continuous and $r_c(S)$ and $r_c(T)$ are finite, then the fact that $(T^n y_0)$ is a fast null sequence implies that the sequence $(S^n T^n y_0)$ is still fast null by Lemma 2.3.12. This means that $TS$ is locally quasinilpotent at $y_0$. Finally, notice that $TS$ commutes with the positive operator $S+T$ and that if $S$ and $T$ are nn-bounded or continuous with $r_c(S)$ and $r_c(T)$ finite, then by 2.1.6, Lemma 2.3.8, and Theorem 2.3.9 so are $ST$ and $S+T$. Then, by Theorem 3.4.1, there exists a non-trivial closed $(S+T)$-invariant ideal. Clearly, this ideal is invariant under both $S$ and $T$. □

In view of Lemma 3.3.4 and Krein-Kakutani Theorem, the next theorem is a generalization of Theorem 3.2.3 to a locally convex-solid topology on a $C(\Omega)$ space. Here we do not assume the compact operator to be positive.

**Theorem 3.4.5.** *Let $T$ and $S$ be two commuting positive operators on a locally convex-solid vector lattice $E$ such that $E$ is $(r_u)$-complete for some order unit $u \in E_+$ and either $S$ and $T$ are nn-bounded, or they are continuous with $r_c(S)$ and $r_c(T)$ finite. If $T$ is locally quasinilpotent at a positive vector and $S$ dominates a non-zero compact operator, then the operators $S$ and $T$ have a common non-trivial closed invariant ideal.*

*Proof.* Assume that $T$ is quasinilpotent at some point $x_0 > 0$ and that $S$ dominates a positive non-zero compact operator $K$. If $T$ is not strictly positive, then the null-ideal $N_T$ is the desired non-trivial common closed invariant subspace. So, suppose that $T$ is strictly positive. Consider the sequence of operators $A_m = \sum_{i=0}^{m}(S+T)^i$, then by 3.1.8 the ideal $J_{x_0} = \bigcup_{m=1}^{\infty} E_{A_m x_0}$ is $(S+T)$-invariant. Since $0 \leqslant S, T \leqslant S+T$, 3.1.7 implies immediately that this ideal is invariant under both $S$ and $T$.

Assume $\overline{J_{x_0}} = E$, then there exists some $0 < y_0 \in J_{x_0}$ such that $Ky_0 \neq 0$ because $K$ is non-zero by assumption. Notice that $T$ is still locally quasinilpotent at $y_0$: since $y_0 \in J_{x_0}$, then there is some positive real $\lambda$ and positive integer $m$ such that $0 < y_0 \leqslant \lambda A_m x_0$. Then

$$0 \leqslant \alpha^n T^n y_0 \leqslant \alpha^n T^n \lambda \sum_{i=0}^{m}(S+T)^i x \leqslant \lambda \sum_{i=0}^{m}(S+T)^i \alpha^n T^n x,$$

which converges to zero as $n$ approaches infinity for every $\alpha \geqslant 0$.

By Lemma 3.3.5(i) there exists a linear operator $V$ on $E$ such that $V$ is dominated by the identity operator and $VKy_0 > 0$. Since $|Vx| \leqslant |x|$ for every $x \in E$ it follows that $V$ is continuous. Further, $TVKy_0 > 0$ because $T$ is strictly positive, so that $TVK$ is a nonzero compact operator. We have $|TVKx| \leqslant T|VKx| \leqslant TS|x|$ for each $x \in E$, so that $TVK$ is dominated by $TS$.

We now show that $TS$ is locally quasinilpotent at $y_0$. If $S$ is nn-bounded, then for each generating seminorm $p$ and for every $\nu > 0$ we have

$$\lim_{n\to\infty} p\Big(\tfrac{(ST)^n}{\nu^n} y_0\Big) \leqslant \lim_{n\to\infty} \tfrac{p(S)^n}{\nu^n} p(T^n y_0) = 0.$$

On the other hand, if $S$ and $T$ are continuous and $r_c(S)$ and $r_c(T)$ are finite, then the fact that $(T^n y_0)$ is a fast null sequence implies that the sequence $(S^n T^n y_0)$ is still fast null by Lemma 2.3.12. This means that $TS$ is locally quasinilpotent at $y_0$.

Finally, notice that $TS$ commutes with the positive operator $S+T$ and that if $S$ and $T$ are nn-bounded or continuous with $r_c(S)$ and $r_c(T)$ finite, then by 2.1.6, Lemma 2.3.8, and Theorem 2.3.9 so are $ST$ and $S+T$. Then, by Theorem 3.4.1, there exists a non-trivial closed $(S+T)$-invariant ideal. Clearly, this ideal is invariant under both $S$ and $T$. $\square$

## 3.5 Compact friendly operators

Theorem 3.2.4, which appeared in [AAB98], says that every locally quasinilpotent compact-friendly positive operator on a Banach lattice has a closed non-trivial invariant ideal. We are going to generalize this result to operators on locally convex-solid vector lattices. Recall, that if $T$ is an operator on a topological vector space with $r_c(T) < 1$ then by Theorem 2.4.3 the operator $I-T$ is invertible and the inverse is given by the equicontinuously convergent series $(I-T)^{-1} = I + T + T^2 + \dots$.

**Theorem 3.5.1.** *Suppose that $T$ is a non-zero continuous positive operator on a complete locally convex-solid vector lattice $E$ such that $r_c(T)$ and $r_c\big((I-T)^{-1}\big)$ are finite. Suppose also that there are three non-zero continuous operators $R$, $C$, and $K$ such that $R$ and $K$ are positive, $r_c(R) < \infty$, $K$ is compact, $T$ commutes with $R$, and $C$ is dominated by both $R$ and $K$. If $T$ is locally quasinilpotent at some $x_0$ then $T$ has a non-trivial closed invariant ideal.*

*Proof.* By Theorem 2.4.3 we know that $(I-T)^{-1} = \sum_{i=0}^{\infty} T^i$ is a continuous operator, and we will denote it by $A$. Clearly, $A$ is positive, commutes with $T$ and $R$, and satisfies $Ax \geqslant x$ for each $x \geqslant 0$. Also, for each $x > 0$ the ideal $E_{Ax}$ is $T$-invariant. Therefore we can assume $\overline{E_{Ax}} = E$, so that $Ax$ is a quasi-interior point for every $x > 0$.

Since $C \neq 0$, there exists some $x_1 > 0$ such that $Cx_1 \neq 0$. Since $A|Cx_1|$ is a quasi-interior point and $Cx_1 \leqslant A|Cx_1|$, it follows from Corollary 3.3.6(i) that there exists a continuous operator $V_1$ on $E$ dominated by the identity operator such that $x_2 = V_1Cx_1 > 0$. Put $M_1 = V_1C$ and note that $M_1$ is dominated both $K$ and $R$.

Since $\overline{E_{Ax_2}} = E$ and $C \neq 0$ we see that there exists some $0 < y \leqslant Ax_2$, such that $Cy \neq 0$. Since $Ax_2$ is a quasi-interior point, it follows from Corollary 3.3.6(ii) that there exists a continuous operator $U$ on $E$, dominated by the identity operator such that $UAx_2 = y$. Now, the element $A|Cy|$ is quasi-interior and $|Cy| \leqslant A|Cy|$, so that by Corollary 3.3.6(i) it follows that there exists another operator $V_2$ on $E$ dominated by the identity operator such that $x_3 = V_2Cy = V_2CUAx_2 > 0$. Let $M_2 = V_2CUA$ and note that since

$$|M_2x| = |V_2CUAx| \leqslant |CUAx| \leqslant R|UAx| \leqslant R|Ax| \leqslant RA|x|$$

for every $x \in E$ then $M_2$ is dominated by the continuous positive operator $RA$. It can be shown in the same fashion that $M_2$ is dominated by the compact operator $KA$.

If we repeat the preceding argument with the vector $x_2$ replaced by $x_3$, then we obtain one more operator $M_3$ on $E$ which satisfies $M_3x_3 > 0$ and which is dominated by both $KA$ and $RA$.

From $M_3M_2M_1x_1 = M_3x_3 > 0$, we see that $M_3M_2M_1$ is a non-zero operator which, by Theorem 3.3.2 is also compact. Moreover,

$$|M_3M_2M_1x| \leqslant RARAR|x|$$

for each $x \in E$.

Now consider the operator $S = RARAR$. Note that $S$ is non-zero, positive, continuous, and $r_c(S) < \infty$. Further, $S$ commutes with $T$ and dominates the non-zero compact operator $M_3M_2M_1$. Therefore $T$ has a non-trivial closed invariant ideal by Theorem 3.4.2. □

In view of Lemma 3.3.4 and Krein-Kakutani Theorem, the next theorem is a generalization of Theorem 3.2.4 to a locally convex-solid topology on a $C(\Omega)$ space.

**Theorem 3.5.2.** *Suppose that $T$ is a non-zero continuous positive operator on a locally convex-solid vector lattice $E$ such that $E$ is $(r_u)$-complete with respect to an order unit $u \in E_+$, $r_c(T) < \infty$, and $T$ is locally quasinilpotent at $x_0 \in E_+$. Suppose also that there are three non-zero continuous operators $R$, $C$, and $K$ such that $R$ and $K$ are positive, $r_c(R) < \infty$, $K$ is compact, $T$ commutes with $R$, and $C$ is dominated by both $R$ and $K$.*

*Then $T$ has a non-trivial closed invariant ideal. Moreover, if another continuous positive operator $S$ commutes with $T$ and $R$, and $r_c(S) < \infty$, then $S$ and $T$ have a common closed non-trivial invariant ideal.*

*Proof.* For each $m \in \mathbb{N}$ consider the operator $A_m = \sum_{i=0}^{m}(S+T)^i$. Clearly, $A_m$ commutes with $S$, $T$, and $R$ for every $m$ and $A_m x \geqslant x$ for every $x \in E_+$. By 3.1.8 the ideal $J_x = \bigcup_{m=0}^{\infty} E_{A_m x}$ is $(S+T)$-invariant. Since $0 \leqslant S, T \leqslant S+T$, it follows from 3.1.7 that this ideal is invariant under both $S$ and $T$. Therefore we can assume that $\overline{J_x} = E$ for each $x > 0$.

Since $C \neq 0$, there exists some $x_1 > 0$ such that $Cx_1 \neq 0$. It follows from Lemma 3.3.5(i) that there exists a linear operator $V_1$ on $E$ dominated by the identity operator (hence continuous) such that $x_2 = V_1 C x_1 > 0$. Put $M_1 = V_1 C$ and note that $M_1$ is dominated by both $K$ and $R$.

From $\overline{J_{x_2}} = E$ and $C \neq 0$ we see that there exists some positive $y \leqslant A_{m_2} x_2$ such that $Cy \neq 0$. It follows from Lemma 3.3.5(ii) that there exists a linear operator $U$ on $E$, dominated by the identity operator (hence continuous) such that $UA_{m_2}x_2 = y$. Again, by Lemma 3.3.5(i) there exists another operator $V_2$ on $E$, such that $V_2$ is dominated by the identity (hence continuous) and $x_3 = V_2 C y = V_2 C U A_{m_2} x_2 > 0$. Let $M_2 = V_2 C U A_{m_2}$, and note that $M_2$ is dominated by the positive compact operator $KA_{m_2}$ and by the operator $RA_{m_2}$.

If we repeat the preceding argument with $x_2$ replaced by $x_3$, we obtain one more continuous operator $M_3$ on $E$ dominated by the positive compact operator $KA_{m_3}$ and by the operator $RA_{m_3}$ and such that $M_3 x_3 > 0$. It follows from $M_3 M_2 M_1 x_1 = M_3 x_3 > 0$ that $M_3 M_2 M_1$ is non-zero. Further, it is compact by Theorem 3.3.2. Moreover, we see that

$$|M_3 M_2 M_1 x| \leqslant R A_{m_3} R A_{m_2} R |x|$$

for each $x \in E$.

Now consider the operator $P = RA_{m_3} RA_{m_2} R + S$. Note that $P$ is non-zero, positive, continuous, and $r_c(P) < \infty$. Further, $P$ commutes with $T$ and dominates the non-zero compact operator $M_3 M_2 M_1$. Therefore $P$ and $T$ have a common non-trivial closed invariant ideal by Theorem 3.4.2. Finally, since $S \leqslant P$, this ideal is invariant under $S$. $\square$

# Bibliography

[AAB93] Y. A. Abramovich, C. D. Aliprantis, and O. Burkinshaw. Invariant subspaces of operators on $l_p$-spaces. *J. Funct. Anal.*, 115(2):418–424, 1993. MR94h:47009.

[AAB94] Y. A. Abramovich, C. D. Aliprantis, and O. Burkinshaw. Invariant subspace theorems for positive operators. *J. Funct. Anal.*, 124(1):95–111, 1994. MR95e:47006.

[AAB98] Y. A. Abramovich, C. D. Aliprantis, and O. Burkinshaw. The invariant subspace problem: Some recent advances. *Rend. Inst. Mat. Univ. Trieste*, XXIX Supplemento:3–79, 1998.

[AB78] Charalambos D. Aliprantis and Owen Burkinshaw. *Locally solid Riesz spaces.* Academic Press [Harcourt Brace Jovanovich Publishers], New York, 1978. Pure and Applied Mathematics, Vol. 76, MR58:12271.

[AB80] Charalambos D. Aliprantis and Owen Burkinshaw. Positive compact operators on Banach lattices. *Math. Z.*, 174(3):289–298, 1980. MR81m:47053.

[AB85] Charalambos D. Aliprantis and Owen Burkinshaw. *Positive operators.* Academic Press Inc., Orlando, Fla., 1985. MR87h:47086.

[AS54] N. Aronszajn and K. T. Smith. Invariant subspaces of completely continuous operators. *Ann. of Math. (2)*, 60:345–350, 1954. MR16:488b.

[Atz84] Aharon Atzmon. An operator on a Fréchet space with no common invariant subspace with its inverse. *J. Funct. Anal.*, 55(1):68–77, 1984. MR86b:47005.

[CPY74] S. R. Caradus, W. E. Pfaffenberger, and Bertram Yood. *Calkin algebras and algebras of operators on Banach spaces*. Marcel Dekker, Inc., New York, 1974. Lecture Notes in Pure and Applied Mathematics, Vol. 9, MR54:3434.

[DeV71] Carl L. DeVito. On Alaoglu's theorem, bornological spaces and the Mackey-Ulam theorem. *Math. Ann.*, 192:83–89, 1971. MR44:2014.

[DS58] Nelson Dunford and Jacob T. Schwartz. *Linear Operators. I. General Theory.* Interscience Publishers, Inc., New York, 1958. With the assistance of W. G. Bade and R. G. Bartle. Pure and Applied Mathematics, Vol. 7, MR22:8302.

[Edw65] R. E. Edwards. *Functional analysis. Theory and applications.* Holt, Rinehart and Winston, New York, 1965. MR36:4308.

[Enf76] P. Enflo. On the invariant subspace problem in Banach spaces. In *Séminaire Maurey–Schwartz (1975–1976) Espaces $L^p$, applications radonifiantes et géométrie des espaces de Banach, Exp. Nos. 14-15*, pages 1–7. Centre Math., École Polytech., Palaiseau, 1976. MR57:13530.

[GV97] Fernando Garibay Bonales and Rigoberto Vera Mendoza. A formula to calculate the spectral radius of a compact linear operator. *Internat. J. Math. Math. Sci.*, 20(3):585–588, 1997. MR98d:47013.

[GV98] Fernando Garibay Bonales and Rigoberto Vera Mendoza. Extending the formula to calculate the spectral radius of an operator. *Proc. Amer. Math. Soc.*, 126(1):97–103, 1998. MR98m:47002.

[Gram66] Bernhard Gramsch. Integration und holomorphe Funktionen in lokalbeschränkten Räumen. *Math. Ann.*, 162:190–210, 1965/1966. MR33:563.

[KN76] John L. Kelley and Isaac Namioka. *Linear topological spaces*. Springer-Verlag, New York, 1976. With the collaboration of W. F. Donoghue, Jr., Kenneth R. Lucas, B. J. Pettis, Ebbe Thue Poulsen, G. Baley Price, Wendy Robertson, W.

R. Scott, and Kennan T. Smith, Second corrected printing, Graduate Texts in Mathematics, No. 36, MR52:14890.

[Köt60] Gottfried Köthe. *Topologische lineare Räume. I.* Springer-Verlag, Berlin, 1960. Die Grundlehren der mathematischen Wissenschaften, Bd. 107, MR24:A411.

[KPR84] N. J. Kalton, N. T. Peck, and James W. Roberts. *An F-space sampler.* Cambridge University Press, Cambridge, 1984. MR87c:46002.

[KR48] M. G. Kreĭn and M. A. Rutman. Linear operators leaving invariant a cone in a Banach space. *Uspehi Matem. Nauk (N. S.)*, 3(1(23)):3–95, 1948. MR10:256c.

[Lep93] Guido Lepkes. Kompakte Operatoren auf lokal konvex-soliden Riesz-Räumen. In *Schriftenreihe des Mathematischen Instituts und des Graduiertenkollegs der Universität Münster. 3. Serie, Heft 8.* Univ. Münster, Münster, 1993. MR94i:47057.

[Lom73] V. I. Lomonosov. Invariant subspaces of the family of operators that commute with a completely continuous operator. *Funkcional. Anal. i Priložen.*, 7(3):55–56, 1973. MR54:8319.

[LZ71] W. A. J. Luxemburg and A. C. Zaanen. *Riesz spaces. Vol. I.* North-Holland Publishing Co., Amsterdam-London, 1971. North-Holland Mathematical Library, MR58:23483.

[Nus70] Roger D. Nussbaum. The radius of the essential spectrum. *Duke Math. J.*, 37:473–478, 1970. MR41:9028.

[Nus81] Roger D. Nussbaum. Eigenvectors of nonlinear positive operators and the linear Kreĭn-Rutman theorem. In *Fixed point theory (Sherbrooke, Que., 1980)*, pages 309–330. Springer, Berlin-New York, 1981. MR83b:47068.

[Pech91] G. Pech. Hyperinvariante Teilräume kompakter Operatoren in topologischen Vektorräumen. *J. Reine Angew. Math.*, 417:129–134, 1991. MR92d:47010.

[PS74] Carl Pearcy and Allen L. Shields. A survey of the Lomonosov technique in the theory of invariant subspaces. pages 219–229. Math. Surveys, No. 13, 1974. MR50:8113.

[Read84] C. J. Read. A solution to the invariant subspace problem. *Bull. London Math. Soc.*, 16(4):337–401, 1984. MR86f:47005.

[Read85] C. J. Read. A solution to the invariant subspace problem on the space $l_1$. *Bull. London Math. Soc.*, 17(4):305–317, 1985. MR87e:47013.

[Read86] C. J. Read. A short proof concerning the invariant subspace problem. *J. London Math. Soc. (2)*, 34(2):335–348, 1986. MR87m:47020.

[Read97] C. J. Read. Quasinilpotent operators and the invariant subspace problem. *J. London Math. Soc. (2)*, 56(3):595–606, 1997. MR98m:47004.

[Rol85] Stefan Rolewicz. *Metric linear spaces.* D. Reidel Publishing Co., Dordrecht, second edition, 1985. MR88i:46004b.

[RR64] A. P. Robertson and W. J. Robertson. *Topological vector spaces.* Cambridge University Press, New York, 1964. Cambridge Tracts in Mathematics and Mathematical Physics, No. 53 MR28:5318.

[RR73] Heydar Radjavi and Peter Rosenthal. *Invariant subspaces.* Springer-Verlag, New York, 1973. Ergebnisse der Mathematik und ihrer Grenzgebiete, Band 77, MR51:3924.

[Sch71] Helmut H. Schaefer. *Topological vector spaces.* Springer-Verlag, New York, 1971. Third printing corrected, Graduate Texts in Mathematics, Vol. 3, MR49:7722.

[Sch74] Helmut H. Schaefer. *Banach lattices and positive operators.* Springer-Verlag, New York, 1974. Die Grundlehren der mathematischen Wissenschaften, Band 215, MR54:11023.

[Tr] V. G. Troitsky. Paths and invariant ideals. Preprint.

[Tr99] Vladimir G. Troitsky. Lomonosov's theorem cannot be extended to chains of four operators. *Proc. Amer. Math. Soc.* to appear.

[Tr98] V. G. Troitsky. On the modulus of C. J. Read's operator. *Positivity*, 2(3):257–264, 1998.

[VM97] Rigoberto Vera Mendoza. The $(\gamma, t)$-topology on $L(E, E)$ and the spectrum of a bounded linear operator on a locally convex topological vector space. *Bol. Soc. Mat. Mexicana (3)*, 3(1):151–164, 1997.

[Vul67] B. Z. Vulikh. *Introduction to the theory of partially ordered spaces.* Wolters-Noordhoff Scientific Publications, Ltd., Groningen, 1967. Translated from the Russian by Leo F. Boron, with the editorial collaboration of Adriaan C. Zaanen and Kiyoshi Iséki, MR37:121.

[Wael54] L. Waelbroeck. Le calcul symbolique dans les algèbres commutatives. *J. Math. Pures Appl. (9)*, 33:147–186, 1954. MR17:513c.

[Zaan83] A. C. Zaanen. *Riesz spaces. II.* North-Holland Publishing Co., Amsterdam-New York, 1983. MR86b:46001.

[ZS79] P. P. Zabreĭko and S. V. Smickih. A theorem of M. G. Kreĭn and M. A. Rutman. *Funktsional. Anal. i Prilozhen.*, 13(3):81–82, 1979. MR80j:47050.

# Vita

Vladimir Troitsky was born on February 6, 1969 in Yuzhno-Sakhalinsk, Sakhalin Island, Russia. In 1983 he won the third prize at the Mathematical Olimpiad of USSR and was admitted to the College of Physics and Mathematics at Novosibirsk, Russia. After graduating from this college in 1985, Vladimir entered Novosibirsk State University. In 1987 Vladimir was called to serve for two years in the army as a computer programmer.

Vladimir started mathematical research during his senior years at Novosibirsk State University, where he worked with A.G. Kusraev, S.S. Kutateladze, and A.E. Gutman on problems in Nonstandard Analysis and Measure Theory. He defended his Master Thesis in 1993.

In 1993, Vladimir entered the Ph.D. program at the University of Illinois at Urbana-Champaign under the supervision of P. Loeb. In 1995 Vladimir also became involved into working on the Invariant Subspace Problem for operators on Banach Lattices under the supervision of Y. Abramovich.

While at the University of Illinois, Vladimir held on different occasions several Teaching or Research Assistant positions and several Fellowships. He was included in the *"Incomplete List of Instructors Ranked as Excellent by Their Students"* in the Fall of 1998 and in the Spring of 1999. He was also nominated for the Departmental Teaching Award three times.

At the time of this writing, Vladimir has four papers published in several respected mathematical journals, namely, in *Siberian Mathematical Journal*, *Positivity*, and *Proceedings of the AMS*. He has presented his results at the *Third Conference on Function Spaces* at Edwardsville (Illinois), *Positivity* Conference at Ankara (Turkey), *Wabash Modern Analysis* Miniconference at Indianapolis, and the 941st AMS Meeting at Urbana (Illinois). He has given a colloquium talk at Kent State University and seminar talks at Texas A&M University and Miami University.