# THE 2-CONCAVIFICATION OF A BANACH LATTICE EQUALS THE DIAGONAL OF THE FREMLIN TENSOR SQUARE

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Abstract. We investigate the relationship between the diagonal of the Fremlin projective tensor product of a Banach lattice $E$ with itself and the 2-concavification of $E$.

## 1. Introduction and preliminaries

It is easy to see that the diagonal of the projective tensor product $\ell_p \otimes_\pi \ell_p$ is isometric to $\ell_{\frac{p}{2}}$ if $p \geqslant 2$ and to $\ell_1$ if $1 \leqslant p \leqslant 2$. In this paper, we extend this fact to all Banach lattices. It turns out that the "right" tensor product for this problem is the Fremlin projective tensor product $E \otimes_{|\pi|} F$ of Banach lattices $E$ and $F$. Given a Banach lattice $E$, we define (following [BB12]) the diagonal of $E \otimes_{|\pi|} E$ to be the quotient of $E \otimes_{|\pi|} E$ over the closed order ideal $I_{\mathrm{oc}}$ generated by the set $\{(x \otimes y) \;:\; x \perp y\}$. We study the relationship of this diagonal with the 2-concavification of $E$. In the literature, it has been observed (see, e.g., [LT79]) that the $p$-concavification $E_{(p)}$ of $E$ is again a Banach lattice when $E$ is $p$-convex. However, without the $p$-convexity assumption, $E_{(p)}$ is only a semi-normed lattice. We show that in the case when $E$ is 2-convex, the diagonal of $E \otimes_{|\pi|} E$ is lattice isometric to $E_{(2)}$ and that in general, the diagonal of $E \otimes_{|\pi|} E$ is lattice isometric to $E_{[2]}$, where $E_{[p]}$ is the completion of $E_{(p)}/\ker\|\cdot\|_{(p)}$. We also show that if $E$ satisfies the lower $p$-estimate then $E_{[p]}$ is lattice isomorphic to an AL-space. In particular, if $E$ satisfies the lower 2-estimate then the diagonal of $E \otimes_{|\pi|} E$ is lattice isomorphic to an AL-space.

We consider the special case when $E$ and $F$ are Banach lattices with (1-unconditional) bases $(e_i)$ and $(f_i)$, respectively. We show that the double sequence $(e_i \otimes f_j)$ is an unconditional basis of $E \otimes_{|\pi|} F$ (while it need not be an unconditional basis for the Banach space projective tensor product $E \otimes_\pi F$, see [KP70]). We also show that in this case

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$E_{(p)}$ is a normed lattice and the diagonal of $E \otimes_{|\pi|} E$ is lattice isometric to the completion of $E_{(2)}$ via $e_i \otimes e_i \mapsto e_i$. Moreover, if $(e_i)$ is normalized and $E$ satisfies the lower $p$-estimate then the completion of $E_{(p)}$ is lattice isomorphic to $\ell_1$. In particular, if $E$ satisfies the lower 2-estimate then the diagonal of $E \otimes_{|\pi|} E$ is lattice isometric to $\ell_1$.

In the rest of this section, we provide some background facts that are necessary for our exposition.

1.1. **Fremlin tensor product.** We refer the reader to [F72, F74] for a detailed original definition of the Fremlin tensor product $E \otimes_{|\pi|} F$ of two Banach lattices $E$ and $F$. However, we will only use a few facts about $E \otimes_{|\pi|} F$ that we describe here.

Suppose $E$ and $F$ are two Banach lattices. We write $E \otimes F$ for their algebraic tensor product; for $x \in E$ and $y \in F$ we write $x \otimes y$ for the corresponding elementary tensor in $E \otimes F$. Every element of $E \otimes F$ is a linear combination of elementary tensors. Let $G$ be another Banach lattice and $\varphi\colon E \times F \to G$ be a bilinear map. Then $\varphi$ induces a map $\hat{\varphi}\colon E \otimes F \to G$ such that $\hat{\varphi}(x \otimes y) = \varphi(x, y)$ for all $x \in E$ and $y \in F$. We say that $\varphi$ is continuous if its norm, defined by

$$\|\varphi\| = \sup\bigl\{\bigl\|\varphi(x,y)\bigr\| \;:\; \|x\| \leqslant 1,\ \|y\| \leqslant 1\bigr\},$$

is finite. We say that $\varphi$ is ***positive*** if $\varphi(x, y) \geqslant 0$ whenever $x, y \geqslant 0$ and that $\varphi$ is a ***lattice bimorphism*** if $\bigl|\varphi(x, y)\bigr| = \varphi\bigl(|x|, |y|\bigr)$ for all $x \in E$ and $y \in F$. We say that $\varphi$ is ***orthosymmetric*** if $\varphi(x, y) = 0$ whenever $x \perp y$.

For $u \in E \otimes F$, put

$$\|u\|_{|\pi|} = \sup\|\hat{\varphi}(u)\|, \tag{1}$$

where the supremum is taken over all Banach lattices $G$ and all positive bilinear maps $\varphi$ from $E \times F$ to $G$ with $\|\varphi\| \leqslant 1$. Theorem 1E in [F74, p. 89] proves that $\|\cdot\|_{|\pi|}$ is a norm on $E \otimes F$, and the completion of $E \otimes F$ with respect to this norm is again a Banach lattice. We will write $E \otimes_{|\pi|} F$ for this space and call it the ***Fremlin tensor product*** of $E$ and $F$. The Fremlin tensor norm is a cross norm, i.e., $\|x \otimes y\|_{|\pi|} = \|x\| \cdot \|y\|$ whenever $x \in E$ and $y \in F$.

**Remark 1.** (See 1E(iii) and 1F in [F74, p. 92].) Let $E$, $F$, and $G$ be Banach lattices. There is a one-to-one norm preserving correspondence between continuous positive bilinear maps $\varphi\colon E \times F \to G$ and positive operators $T\colon E \otimes_{|\pi|} F \to G$ such that $T(x \otimes y) = \varphi(x, y)$ for all $x \in E$ and $y \in F$. We will denote $T = \varphi^{\otimes}$. Furthermore, $\varphi$ is a lattice bimorphism if and only if $T$ is a lattice homomorphism.

There is an alternative definition of $E \otimes_{|\pi|} F$, cf. [F74, 1I] and [S80, pp. 203-204]. Recall that, being a dual Banach lattice, $F^*$ is Dedekind complete by [AB06, Theorem 3.49], so that the space of regular operators $L^r(E, F^*)$ is a Banach lattice with respect to the regular norm $\|\cdot\|_r$, see [AB06, p. 255].

**Proposition 2.** *If $E$ and $F$ are Banach lattices then $E \otimes_{|\pi|} F$ can be identified with a closed sublattice of $L^r(E, F^*)^*$ such that $\langle x \otimes y, T\rangle = \langle Tx, y\rangle$ for $x \in E$, $y \in F$, and $T \in L^r(E, F^*)$.*

*Proof.* Consider the map $\alpha\colon h \in (E \otimes_{|\pi|} F)^* \mapsto T \in L(E, F^*)$ via $\langle Tx, y\rangle = h(x \otimes y)$. It is easy to see that $\alpha$ is one-to one and $T \geqslant 0$ whenever $h \geqslant 0$. It follows that $\alpha(h)$ is regular for every $h$.

Suppose that $0 \leqslant T\colon E \to F^*$. The map $\varphi$ defined by $\varphi(x, y) = \langle Tx, y\rangle$ is a positive bilinear functional on $E \times F$. Also,

$$\|T\| = \sup\bigl\{\bigl|\langle Tx, y\rangle\bigr| : \|x\| \leqslant 1,\ \|y\| \leqslant 1\bigr\} = \sup\bigl\{\bigl|\varphi(x, y)\bigr| : \|x\| \leqslant 1,\ \|y\| \leqslant 1\bigr\} = \|\varphi\|.$$

By Remark 1, we can consider $h = \varphi^{\otimes}$, then $0 \leqslant h \in (E \otimes_{|\pi|} F)^*$ and $\|h\| = \|\varphi\| = \|T\|$. It is easy to see that $T = \alpha(h)$. Hence, the restriction of $\alpha$ to the positive cones of $(E \otimes_{|\pi|} F)^*$ is a bijective isometry onto the positive cone of $L(E, F^*)$. It follows by [AB06, Theorem 2.15] that $\alpha$ is a latice isomorphism between $(E \otimes_{|\pi|} F)^*$ and $L^r(E, F^*)$. Moreover, if $T = \alpha(h)$ for some $h \in (E \otimes_{|\pi|} F)^*$ then $\alpha\bigl(|h|\bigr) = |T|$ yields $\|h\| = \bigl\||h|\bigr\| = \bigl\||T|\bigr\| = \|T\|_r$. It follows that $\alpha$ is a lattice isometry between $(E \otimes_{|\pi|} F)^*$ and $L^r(E, F^*)$. Therefore, $(E \otimes_{|\pi|} F)^{**}$ is lattice isometric to $L^r(E, F^*)^*$. Since $E \otimes_{|\pi|} F$ can be viewed as a sublattice of $(E \otimes_{|\pi|} F)^{**}$, it is lattice isometric to a closed sublattice of $L^r(E, F^*)^*$. ☐

1.2. **Functional calculus.** Given $x$ and $y$ in a Banach lattice $E$, one would like to define expressions like $(x^2 + y^2)^{\frac{1}{2}}$ and $x^{\frac{1}{2}} y^{\frac{1}{2}}$ to be elements of $E$. This can be done point-wise if $E$ can be represented as a function space. One could object, however, that the definition may then depend on the choice of a functional representation. Theorem 1.d.1 in [LT79] (see also [BdPvR91]) proves that there is a unique way to extend all continuous homogeneous[1] functions from $\mathbb{R}^n$ to $\mathbb{R}$ to functions from $E^n$ to $E$ which does not depend on a particular representation of $E$ as a function space. More precisely, for any $x_1, \dots, x_n \in E$ there exists a unique lattice homomorphism $\tau$ from the space of all continuous homogeneous functions on $\mathbb{R}^n$ to $E$ such that if $f(t_1, \dots, t_n) = t_i$ then $\tau(f) = x_i$ as $i = 1, \dots, n$. We denote $\tau(f)$ by $f(x_1, \dots, x_n)$. In

[1]Recall that a function $f\colon \mathbb{R}^n \to R$ is called ***homogeneous*** if $f(\lambda x_1, \dots, \lambda x_n) = \lambda f(x_1, \dots, x_n)$ for any $x_1, \dots, x_n$, and $\lambda \geqslant 0$.

particular, all identities and inequalities for homogeneous expressions that are valid in $\mathbb{R}$ remain valid in $E$. For example,

$$(x_1+x_2)^{\frac12}y^{\frac12}=\bigl((x_1^{\frac12}y^{\frac12})^2+(x_2^{\frac12}y^{\frac12})^2\bigr)^{\frac12} \tag{2}$$

for every $x_1$, $x_2$, and $y$ in every Banach lattice $E$. Note that, following convention from [LT79, p. 53], for $t\in\mathbb{R}$ and $p>0$, by $t^p$ we mean $|t|^p\operatorname{sign}t$. There is a certain inconsistency in notation: for example, $t^2$ equals $t|t|$, not $tt$, so that $(x^2)^{\frac12}=x$ while $(xx)^{\frac12}=|x|$. To avoid confusion, we will distinguish $xx$ from $x^2$ throughout the paper. Note also that

$$x^{\frac12}|x|^{\frac12}=x. \tag{3}$$

In the following lemma, we collect several standard facts that we will routinely use.

**Lemma 3.** *Given any $x,y\in E$ and $p>0$.*

(i) $\bigl|x^{\frac12}y^{\frac12}\bigr|=|x|^{\frac12}|y|^{\frac12}$;
(ii) $\bigl\|x^{\frac12}y^{\frac12}\bigr\|\leqslant\|x\|^{\frac12}\|y\|^{\frac12}$;
(iii) *If* $x\perp y$ *then* $x^{\frac12}y^{\frac12}=0$;
(iv) *If* $x,y\geqslant 0$ *then* $(x^p+y^p)^{\frac1p}\geqslant 0$;
(v) *If* $x\wedge y=0$ *then* $\bigl(x^p+y^p\bigr)^{\frac1p}=x+y$.

*Proof.* (i) follows from the fact that the identity holds for real numbers.

(ii) By Proposition 1.d.2(i) from [LT79], we have $\bigl\||x|^{\frac12}|y|^{\frac12}\bigr\|\leqslant\|x\|^{\frac12}\|y\|^{\frac12}$. Combining this with (i), we get the required inequality.

(iii) follows from the fact that $\bigl|x^{\frac12}y^{\frac12}\bigr|=\bigl(|x|\vee|y|\bigr)^{\frac12}\bigl(|x|\wedge|y|\bigr)^{\frac12}$.

(iv) Note that $\bigl(|x|^p+|y|^p\bigr)^{\frac1p}\geqslant 0$ for every $x,y\in E$ because this inequality is true for real numbers. It follows that if $x,y\geqslant0$ then $(x^p+y^p)^{\frac1p}=\bigl(|x|^p+|y|^p\bigr)^{\frac1p}\geqslant 0$.

(v) Again, for every $x,y\in E$ we have $|x|\vee|y|\leqslant\bigl(|x|^p+|y|^p\bigr)^{\frac1p}\leqslant|x|+|y|$ because this is true for real numbers. But if $x\wedge y=0$ then $x,y\geqslant0$ and $x\vee y=x+y$. □

A Banach lattice $E$ is said to be $p$-***convex*** for some $1\leqslant p<\infty$ if there is a constant $M>0$ such that $\bigl\|\bigl(\sum_{i=1}^n x_i^p\bigr)^{\frac1p}\bigr\|\leqslant M\bigl(\sum_{i=1}^n\|x_i\|^p\bigr)^{\frac1p}$ whenever $x_1,\dots,x_n\in E_+$. Similarly, $E$ is $p$-***concave*** if there is a constant $M>0$ such that $\bigl\|\bigl(\sum_{i=1}^n x_i^p\bigr)^{\frac1p}\bigr\|\geqslant\frac1M\bigl(\sum_{i=1}^n\|x_i\|^p\bigr)^{\frac1p}$ whenever $x_1,\dots,x_n\in E_+$.

A Banach lattice $E$ satisfies the ***upper*** $p$-***estimate*** with constant $M$ if $\bigl\|\sum_{k=1}^n x_k\bigr\|\leqslant M\bigl(\sum_{k=1}^n\|x_k\|^p\bigr)^{\frac1p}$ whenever $x_1,\dots,x_n$ are disjoint. Similarly, $E$ satisfies the ***lower*** $p$-***estimate*** with constant $M$ if $\bigl\|\sum_{k=1}^n x_k\bigr\|\geqslant\frac1M\bigl(\sum_{k=1}^n\|x_k\|^p\bigr)^{\frac1p}$ whenever $x_1,\dots,x_n$ are disjoint. It follows from Lemma 3(v) that $p$-convexity implies the upper $p$-estimate and $p$-concavity implies the lower $p$-estimate.

## 2. The concavification of a Banach lattice

The concavification procedure is motivated by the fact that if $(x_i) \in \ell_r$ and $1 < p < r$, then the sequence $(x_i^p)$ belongs to $\ell_{\frac{r}{p}}$.

This section is partially based on Section 1.d in [LT79]. Throughout this section, $E$ is a Banach lattice and $p \geqslant 1$.

We define new vector operations on $E$ via $x \oplus y = (x^p + y^p)^{\frac{1}{p}}$ and $\alpha \odot x = \alpha^{\frac{1}{p}} x$ whenever $x, y \in E$ and $\alpha \in \mathbb{R}$. (Here again, if $x$, $y$, or $\alpha$ are not positive then we use the convention described earlier.) Note that $E$ endowed with these new addition and multiplication operations and the original order is again a vector lattice by Lemma 3(iv).

Define

$$\|x\|_{(p)} = \inf\Big\{\sum_{i=1}^{n} \|v_i\|^p \ :\ |x| \leqslant v_1 \oplus \dots \oplus v_n, v_i \geqslant 0\Big\}. \tag{4}$$

**Remark 4.** Note that being a vector lattice, $(E, \oplus, \odot, \leqslant)$ satisfies the Riesz Decomposition Property (see, e.g., Theorem 1.13 in [AB06]), so that the inequality $|x| \leqslant v_1 \oplus \dots \oplus v_n$ in (4) can be replaced by equality.

It is easy to see that (4) defines a lattice semi-norm on $(E, \oplus, \odot, \leqslant)$. This semi-normed vector lattice will be denoted by $E_{(p)}$. It is called the ***$p$-concavification*** of $E$. As a partially ordered set, $E_{(p)}$ coincides with $E$. We will see in Examples 18 and 26 that $\|\cdot\|_{(p)}$ does not have to be a norm, and when it is a norm, it need not be complete.

The following fact is standard, we include the proof for completeness.

**Proposition 5.** *If $E$ is a $p$-convex Banach lattice then $E_{(p)}$ is a Banach lattice.*

*Proof.* Suppose that $E$ is $p$-convex with constant $M$. Given $x \in E$. Suppose that

$$|x| = v_1 \oplus \dots \oplus v_n = \big(v_1^p + \dots + v_n^p\big)^{\frac{1}{p}}$$

for some $v_i \geqslant 0$. Then $\|x\| \leqslant M\Big(\sum_{i=1}^n \|v_i\|^p\Big)^{\frac{1}{p}}$. It follows that $\frac{1}{M^p}\|x\|^p \leqslant \|x\|_{(p)} \leqslant \|x\|^p$. This yields that $\|\cdot\|_{(p)}$ is a complete norm on $E_{(p)}$. □

Recall that if $E$ is a Banach lattice and $x > 0$, then $x$ is an ***atom*** in $E$ if $0 \leqslant z \leqslant x$ implies that $z$ is a scalar multiple of $x$. We say that $E$ is ***atomic*** or ***discrete*** if for every $z > 0$ there exists an atom $x$ such that $0 < x \leqslant z$.

**Lemma 6.** *If $x$ is an atom in a Banach lattice $E$ then $\|x\|_{(p)} = \|x\|^p$*

*Proof.* Take $v_1, \dots, v_n \in E_+$ such that $x = v_1 \oplus \dots \oplus v_n$. It follows that $0 \leqslant v_k \leqslant x$ for each $k = 1, \dots, n$, hence $v_k = \alpha_k \odot x = \alpha_k^{1/p} x$ for some $\alpha_k \in \mathbb{R}_+$. Also,

$$x = v_1 \oplus \dots \oplus v_n = (\alpha_1 \odot x) \oplus \dots \oplus (\alpha_n \odot x) = (\alpha_1 + \dots + \alpha_n) \odot x,$$

so that $\sum_{k=1}^n \alpha_k = 1$. It follows that $\sum_{k=1}^n \|v_k\|^p = \sum_{k=1}^n \big\|\alpha_k^{\frac{1}{p}} x\big\|^p = \|x\|^p$, so that $\|x\|_{(p)} = \|x\|^p$. □

**Corollary 7.** *If $E$ is a discrete Banach lattice then $E_{(p)}$ is a normed lattice.*

*Proof.* Since we know that $\|\cdot\|_{(p)}$ is a lattice semi-norm on $E_{(p)}$, it suffices to prove that it has trivial kernel. Suppose that $y \in E$ with $y \neq 0$. There is an atom $x$ such that $0 < x \leqslant |y|$. Then $\|y\|_{(p)} \geqslant \|x\|_{(p)} = \|x\|^p > 0$. □

**Remark 8.** Thus, we know that $E_{(p)}$ is a normed lattice in two important special cases: when $E$ is discrete or $p$-convex. It would be interesting to find a general characterization of Banach lattices $E$ for which $\|\cdot\|_{(p)}$ is a norm. That is, characterize all Banach lattices $E$ such that

$$\inf\Big\{\sum_{i=1}^n \|v_i\|^p \ : \ x = \big(v_1^p + \dots + v_n^p\big)^{\frac{1}{p}}, v_i > 0\Big\} > 0$$

for every non-zero $x \in E_+$.

In general, we can only say that $\|\cdot\|_{(p)}$ is a lattice seminorm on $E_{(p)}$. It follows that its kernel is an ideal, so that the quotient space $E_{(p)}/\ker\|\cdot\|_{(p)}$ is a normed lattice. Denote its completion by $E_{[p]}$. Clearly, $E_{[p]}$ is a Banach lattice.

Let $E$ be a Banach lattice. It is a standard fact (c.f., the proof of [LT79, Lemma 1.b.13]) that if there exists $c > 0$ such that $\big\|\sum_{k=1}^n x_k\big\| \geqslant c \sum_{k=1}^n \|x_k\|$ whenever $x_1, \dots, x_n$ are disjoint (that is, if $E$ satisfies the lower 1-estimate), then $E$ is lattice isomorphic to an $AL$-space. Indeed, put

$$|||x||| = \sup\Big\{\sum_{i=1}^n \|x_i\| \ : \ x_1, \dots, x_n \text{ are positive and disjoint and } |x| = x_1 + \dots + x_n\Big\}.$$

It can be easily verified that this is an equivalent norm on $E$ which makes $E$ into an AL-space (with the same order).

The following lemma establishes that if $E$ satisfies the lower $p$-estimate then $E_{(p)}$ satisfies the lower 1-estimate.

**Lemma 9.** *Suppose that $E$ is a Banach lattice satisfying the lower $p$-estimate with constant $M$. Then $\left\|\sum_{k=1}^n x_k\right\|_{(p)} \geqslant \frac{1}{M^p}\sum_{k=1}^n \|x_k\|_{(p)}$ whenever $x_1,\dots,x_n$ are disjoint in $E$.*

*Proof.* Suppose $x_1,\dots,x_n$ are disjoint in $E$. Since $\left|\sum_{k=1}^n x_k\right| = \sum_{k=1}^n |x_k|$, we may assume without loss of generality that $x_k \geqslant 0$ for each $k$. Note that $\sum_{k=1}^n x_k = x_1 \oplus \cdots \oplus x_n$ by Lemma 3(v).

We will use (4) and Remark 4 to estimate $\left\|x_1 \oplus \cdots \oplus x_n\right\|_{(p)}$. Take $u_1,\dots,u_m$ in $E_+$ such that $x_1 \oplus \cdots \oplus x_n = u_1 \oplus \cdots \oplus u_m$. Since $E_{(p)}$ is a vector lattice, by the Riesz Decomposition Property [AB06, Theorem 1.20], for each $k = 1,\dots,n$ we find $v_{k,1},\dots,v_{k,m}$ in $E_+$ such that $x_k = v_{k,1} \oplus \cdots \oplus v_{k,m}$ and $u_i = v_{1,i} \oplus \cdots \oplus v_{n,i}$ for each $i = 1,\dots,m$. For each $k$ and $i$ we have $0 \leqslant v_{k,i} \leqslant x_k$, so that $v_{1,i},\dots,v_{n,i}$ are disjoint for every $i$. It follows that $u_i = v_{1,i} + \cdots + v_{n,i}$. By the lower $p$-estimate, we get $\|u_i\| \geqslant \frac{1}{M}\left(\sum_{k=1}^n \|v_{k,i}\|^p\right)^{\frac{1}{p}}$, so that $M^p\|u_i\|^p \geqslant \sum_{k=1}^n \|v_{k,i}\|^p$. For every $k$, we have $\|x_k\|_{(p)} \leqslant \sum_{i=1}^m \|v_{k,i}\|^p$, so that

$$\sum_{k=1}^n \|x_k\|_{(p)} \leqslant \sum_{k=1}^n \sum_{i=1}^m \|v_{k,i}\|^p \leqslant M^p \sum_{i=1}^m \|u_i\|^p.$$

Taking the infimum over all $u_1,\dots,u_m$ in $E_+$ such that $x_1 \oplus \cdots \oplus x_n = u_1 \oplus \cdots \oplus u_m$, we get the required inequality. □

**Theorem 10.** *If a Banach lattice $E$ satisfies the lower $p$-estimate with constant $M$ then $E_{[p]}$ is lattice isomorphic to an AL-space. Furthermore, if $M = 1$ then $E_{[p]}$ is an AL-space.*

*Proof.* Suppose that $E$ satisfies a lower $p$-estimate with constant $M$. Applying Lemma 9, we have $M^p\left\|\sum_{k=1}^n x_k\right\|_{(p)} \geqslant \sum_{k=1}^n \|x_k\|_{(p)}$ whenever $x_1,\dots,x_n$ are disjoint in $E$. It is easy to see that this inequality remains valid in $E_{(p)}/\ker\|\cdot\|_{(p)}$ and, furthermore, in $E_{[p]}$. □

## 3. Main results

Let $E$ be a Banach lattice. Let $I_{\mathrm{oc}}$ be the norm closed ideal generated in $E \otimes_{|\pi|} E$ by the elements of the form $x \otimes y$ where $x \perp y$ (without loss of generality, we may also assume that $x$ and $y$ are positive). We can view $I_{\mathrm{oc}}$ as the set of all "off-diagonal" elements of $E \otimes_{|\pi|} E$. Therefore, following [BB12], we think of $(E \otimes_{|\pi|} E)/I_{\mathrm{oc}}$ as the diagonal of $E \otimes_{|\pi|} E$. We claim that this space is lattice isometric to $E_{[2]}$.

**Theorem 11.** *Suppose that $E$ is a Banach lattice. Then there exists a surjective lattice isometry $T\colon E_{[2]} \to (E \otimes_{|\pi|} E)/I_{\mathrm{oc}}$ such that $T\bigl(x + \ker\|\cdot\|_{(2)}\bigr) = x \otimes |x| + I_{\mathrm{oc}}$ for each $x \in E$.*

*Proof.* Define a map $\varphi\colon E \times E \to E_{(2)}$ by $\varphi(x,y) = x^{\frac{1}{2}}y^{\frac{1}{2}}$. By the nature of the vector operations in $E_{(2)}$, this map is bilinear. Indeed,

$$\varphi(\lambda x, y) = (\lambda x)^{\frac{1}{2}}y^{\frac{1}{2}} = \lambda^{\frac{1}{2}}x^{\frac{1}{2}}y^{\frac{1}{2}} = \lambda \odot (x^{\frac{1}{2}}y^{\frac{1}{2}}) = \lambda \odot \varphi(x,y).$$

Similarly, $\varphi(x, \lambda y) = \lambda \odot \varphi(x,y)$. Also, $\varphi(x_1 + x_2, y) = \varphi(x_1, y) \oplus \varphi(x_2, y)$ by (2); we obtain $\varphi(x, y_1 + y_2) = \varphi(x, y_1) \oplus \varphi(x, y_2)$ in a similar fashion. For any $x, y \in E$ we have by Lemma 3(ii)

$$\bigl\|\varphi(x,y)\bigr\|_{(2)} = \bigl\|x^{\frac{1}{2}}y^{\frac{1}{2}}\bigr\|_{(2)} \leqslant \bigl\|x^{\frac{1}{2}}y^{\frac{1}{2}}\bigr\|^2 \leqslant \Bigl(\|x\|^{\frac{1}{2}}\|y\|^{\frac{1}{2}}\Bigr)^2 = \|x\|\|y\|,$$

so that $\|\varphi\| \leqslant 1$. Clearly, $\varphi$ is a continuous lattice bimorphism; it is orthosymmetric by Lemma 3(iii).

Put $N = \ker\|\cdot\|_{(2)}$ and let $r\colon E_{(2)} \to E_{(2)}/N$ be the canonical quotient map. Also, let $i\colon E_{(2)}/N \to E_{[2]}$ be the natural inclusion map. Consider the map $(ir\varphi)^{\otimes}\colon E \otimes_{|\pi|} E \to E_{[2]}$ as in Remark 1 (see Figure 1); then $(ir\varphi)^{\otimes}$ is a lattice homomorphism and $\|(ir\varphi)^{\otimes}\| \leqslant 1$. Note that if $x \perp y$ then $(ir\varphi)^{\otimes}(x \otimes y) = ir\varphi(x,y) = 0$. Since $(ir\varphi)^{\otimes}$ is positive, it vanishes on $I_{\mathrm{oc}}$. Consider the quotient space $(E \otimes_{|\pi|} E)/I_{\mathrm{oc}}$; let $q\colon E \otimes_{|\pi|} E \to (E \otimes_{|\pi|} E)/I_{\mathrm{oc}}$ be the canonical quotient map. Since $I_{\mathrm{oc}} \subseteq \ker(ir\varphi)^{\otimes}$, we can consider the induced map $\widetilde{(ir\varphi)^{\otimes}}\colon (E \otimes_{|\pi|} E)/I_{\mathrm{oc}} \to E_{[2]}$ such that $\widetilde{(ir\varphi)^{\otimes}}q = (ir\varphi)^{\otimes}$.

Consider the map $q\otimes$ from $E \times E$ to $(E \otimes_{|\pi|} E)/I_{\mathrm{oc}}$. This map is clearly bilinear and orthosymmetric. Therefore, by Theorem 9(ii) of [BvR01], there exists a lattice homomorphism $S\colon E_{(2)} \to (E \otimes_{|\pi|} E)/I_{\mathrm{oc}}$ such that $q\otimes = S\varphi$. Note that for each $x, y \in E$ we have

$$S(x^{\frac{1}{2}}y^{\frac{1}{2}}) = S\varphi(x,y) = q \otimes (x,y) = x \otimes y + I_{\mathrm{oc}}. \tag{5}$$

In particular, taking $y = |x|$, we get $Sx = x \otimes |x| + I_{\mathrm{oc}}$.

We claim that $\|Sx\| \leqslant \|x\|_{(2)}$ for each $x \in E$. Indeed, take $v_1, \dots, v_n \in E_+$ such that $|x| = v_1 \oplus \dots \oplus v_n$. Since $S$ is a lattice homomorphism, we have

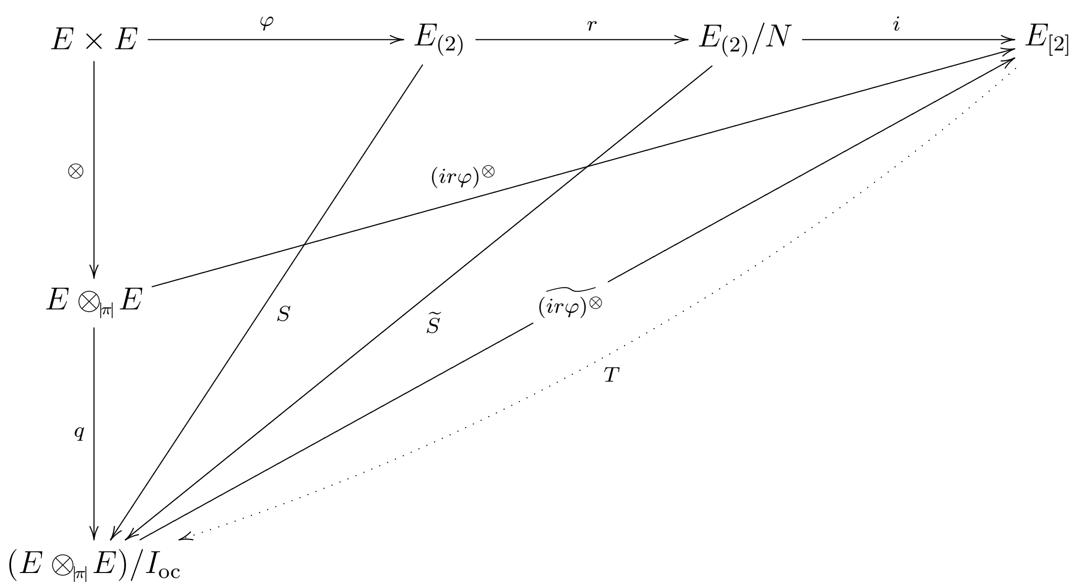
$$|Sx| = S|x| = Sv_1 + \dots + Sv_n = v_1 \otimes |v_1| + \dots + v_n \otimes |v_n| + I_{\mathrm{oc}}.$$

By the definition of a quotient norm,

$$\|Sx\| \leqslant \Bigl\|\sum_{i=1}^n v_i \otimes |v_i|\Bigr\|_{|\pi|} \leqslant \sum_{i=1}^n \bigl\|v_i \otimes |v_i|\bigr\|_{|\pi|} = \sum_{i=1}^n \|v_i\|^2$$

because $\|\cdot\|_{|\pi|}$ is a cross-norm. It follows now from (4) that $\|Sx\| \leqslant \|x\|_{(2)}$.

FIGURE 1




In particular, $N \subseteq \ker S$. It follows that $S$ induces a lattice homomorphism $\widetilde{S}\colon E_{(2)}/N \to (E \otimes_{|\pi|} E)/I_{\text{oc}}$ such that $S = \widetilde{S}r$. We now show that $\widetilde{S}$ is an isometry. For any $x \in E$ we have $\|\widetilde{S}(x+N)\| = \|Sx\| \leqslant \|x\|_{(2)} = \|x+N\|$, so that $\|\widetilde{S}\| \leqslant 1$. On the other hand, for every $v \in I_{\text{oc}}$ we have $(ir\varphi)^{\otimes}(v) = 0$, so that $(ir\varphi)^{\otimes}\big(x \otimes |x| + v\big) = r\varphi\big(x, |x|\big) = rx = x + N$ by (3). Since $\|(ir\varphi)^{\otimes}\| \leqslant 1$, we get $\|x+N\| \leqslant \big\|x \otimes |x| + v\big\|_{|\pi|}$. Taking infimum over all $v \in I_{\text{oc}}$, we get $\|x+N\| \leqslant \|Sx\| = \|\widetilde{S}(x+N)\|$. Therefore, $\widetilde{S}$ is an isometry. It follows that $\widetilde{S}$ extends to a lattice isometry $T\colon E_{[2]} \to (E \otimes_{|\pi|} E)/I_{\text{oc}}$. Note that $T(x+N) = Sx = x \otimes |x| + I_{\text{oc}}$ for each $x \in E$.

We claim that $T$ is the inverse of $\widetilde{(ir\varphi)}^{\otimes}$. Indeed, for every $x \in E$ we have

$$\widetilde{(ir\varphi)}^{\otimes} T(x+N) = \widetilde{(ir\varphi)}^{\otimes}\big(x \otimes |x| + I_{\text{oc}}\big) = (ir\varphi)^{\otimes}\big(x \otimes |x|\big) = ir\varphi\big(x, |x|\big) = irx = x + N$$

by (3). This means that $\widetilde{(ir\varphi)}^{\otimes} T$ is the identity on $E_{(2)}/N$ and, therefore, on $E_{[2]}$. On the other hand, for each $x, y \in E$ it follows from (5) that

$$\begin{aligned} T\widetilde{(ir\varphi)}^{\otimes}(x \otimes y + I_{\text{oc}}) &= T(ir\varphi)^{\otimes}(x \otimes y) = Tir\varphi(x, y) \\ &= Tr(x^{\frac{1}{2}}y^{\frac{1}{2}}) = \widetilde{S}r(x^{\frac{1}{2}}y^{\frac{1}{2}}) = S(x^{\frac{1}{2}}y^{\frac{1}{2}}) = x \otimes y + I_{\text{oc}}. \end{aligned}$$

Hence, $T\widetilde{(ir\varphi)}^{\otimes}$ is the identity on $q(E \otimes E)$. Since $E \otimes E$ is dense in $E \otimes_{|\pi|} E$ then $q(E \otimes E)$ is dense in $(E \otimes_{|\pi|} E)/I_{\text{oc}}$, so that $T\widetilde{(ir\varphi)}^{\otimes}$ is the identity on $(E \otimes_{|\pi|} E)/I_{\text{oc}}$. Therefore, $T$ is the inverse of $\widetilde{(ir\varphi)}^{\otimes}$. It follows that $T$ is onto. $\square$

Recall that if $E$ is discrete then $\|\cdot\|_{(2)}$ is a norm by Corollary 7, so that $E_{(2)}$ is a normed lattice and $E_{[2]}$ equals $\overline{E_{(2)}}$, the completion of $E_{(2)}$; if $E$ is 2-convex then $E_{(2)}$ is a Banach lattice by Proposition 5; in this case $E_{[2]} = E_{(2)}$.

**Corollary 12.** *Suppose that $E$ is Banach lattice. If $E_{(2)}$ is a normed lattice then it is lattice isometric to a dense sublattice of $(E \otimes_{|\pi|} E)/I_{\mathrm{oc}}$ via $x \in E_{(2)} \mapsto x \otimes |x| + I_{\mathrm{oc}}$.*

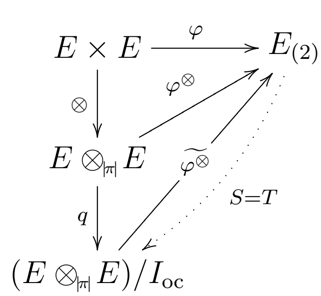
**Corollary 13.** *Suppose that $E$ is a Banach lattice such that $E_{(2)}$ is also a Banach lattice. Then the map $T\colon E_{(2)} \to (E \otimes_{|\pi|} E)/I_{\mathrm{oc}}$ given by $Tx = x \otimes |x| + I_{\mathrm{oc}}$ is a surjective linear lattice isometry.*

**Remark 14.** Theorem 11 provides a new characterization of the ideal $I_{\mathrm{oc}}$. It was observed in the proof of Theorem 11 that $(ir\varphi)^{\otimes}$ vanishes on $I_{\mathrm{oc}}$ and the induced map $\widetilde{(ir\varphi)}^{\otimes}$ on $(E \otimes_{|\pi|} E)/I_{\mathrm{oc}}$ is a bijection; hence $I_{\mathrm{oc}} = \ker(ir\varphi)^{\otimes}$. This can be used to easily verify whether certain elements belong to $I_{\mathrm{oc}}$. For example, it follows from $\varphi\bigl(x^{\frac{1}{2}}y^{\frac{1}{2}}, x^{\frac{1}{2}}y^{\frac{1}{2}}\bigr) = \varphi(x,y)$ that $x \otimes y - \bigl(x^{\frac{1}{2}}y^{\frac{1}{2}}\bigr) \otimes \bigl(x^{\frac{1}{2}}y^{\frac{1}{2}}\bigr)$ is in $\ker(ir\varphi)^{\otimes}$ and, hence, in $I_{\mathrm{oc}}$ for every $x, y \in E$.

Similarly, one can check that $x \otimes y - y \otimes x \in I_{\mathrm{oc}}$ for all $x, y \in E$. Let $Z$ be the closed sublattice of $E \otimes_{|\pi|} E$ generated by vectors of the form $x \otimes y - y \otimes x$. We refer to $Z$ as the ***antisymmetric part*** of $E \otimes_{|\pi|} E$. This yields $Z \subseteq I_{\mathrm{oc}}$ (this inclusion also follows from Proposition 4.33 of [L07], obtained there by very different means).

**Remark 15.** Suppose that $E$ is such that $E_{(2)}$ is a Banach lattice. Then we can identify $T^{-1}$ in Corollary 13. Indeed, in this case, the maps $i$ and $r$ in the proof of Theorem 11 are just the identity maps, so that $T^{-1} = \widetilde{\varphi^{\otimes}}$ where $\varphi(x,y) = x^{\frac{1}{2}}y^{\frac{1}{2}}$ (see Figure 2). Furthermore, in this case, we have $I_{\mathrm{oc}} = \ker \varphi^{\otimes}$.

FIGURE 2



**Remark 16.** Again, suppose that $E$ is such that $E_{(2)}$ is a Banach lattice. It follows from Corollary 13 that every equivalence class in $(E \otimes_{|\pi|} E)/I_{\rm oc}$ contains a representative of the form $x \otimes |x|$ for some $x \in E$. Therefore, $q(E \otimes E) = (E \otimes_{|\pi|} E)/I_{\rm oc}$, where $q\colon E \otimes_{|\pi|} E \to (E \otimes_{|\pi|} E)/I_{\rm oc}$ is the canonical quotient map. In other words, the elements of $E \otimes E$ (and even elementary tensor products) are sufficient to "capture all of the diagonal" in $E \otimes_{|\pi|} E$.

As usual, one can identify $q(E \otimes E)$ with the quotient of $E \otimes E$ over $I_{\rm oc}$ or, more precisely, with $(E \otimes E)/\big((E \otimes E) \cap I_{\rm oc}\big)$, where $E \otimes E$ is viewed as a (non-closed) subspace of $E \otimes_{|\pi|} E$. Therefore,

$$(E \otimes_{|\pi|} E)/I_{\rm oc} = (E \otimes E)/\big((E \otimes E) \cap I_{\rm oc}\big) \tag{6}$$

Combining Theorems 10 and 11, we immediately get the following.

**Corollary 17.** *Suppose that $E$ is a Banach lattice satisfying the lower 2-estimate with constant $M$. Then $(E \otimes_{|\pi|} E)/I_{\rm oc}$ is lattice isomorphic to an AL-space. If $M = 1$ then $(E \otimes_{|\pi|} E)/I_{\rm oc}$ is an AL-space.*

## 4. Function spaces

In this section, we consider the case when $E$ is a Köthe space on a $\sigma$-finite measure space $(\Omega, \Sigma, \mu)$ as in [LT79, Definition 1.b.17]. That is, $E$ is contained in the space $L_0(\Omega)$ of all measurable functions on $\Omega$ such that $E$ contains the characteristic functions of all sets of finite measure and if $f \in E$, $g \in L_0(\Omega)$ and $|g| \leqslant |f|$ then $g \in E$ and $\|g\| \leqslant \|f\|$.

It is easy to see that in a Köthe space, the functional calculus map $\tau$, described in Subsection 1.2, agrees with almost everywhere pointwise operations. Indeed, fix $x_1, \dots, x_n$ in $E$ and let $h\colon \mathbb{R}^n \to \mathbb{R}$ be a homogeneous continuous function. It is easy to see that

$$\big|h(t_1, \dots, t_n)\big| \leqslant M \max_{1 \leqslant i \leqslant n} |t_i|$$

for all $t_1, \dots, t_n \in \mathbb{R}$, where

$$M = \max\big\{|h(t_1, \dots, t_n)| \ : \ \max_{1 \leqslant i \leqslant n} |t_i| = 1\big\}.$$

It follows that

$$\Big|h\big(x_1(\omega), \dots, x_n(\omega)\big)\Big| \leqslant M \max_{1 \leqslant i \leqslant n} \big|x_i(\omega)\big|$$

for all $\omega \in \Omega$, so that the usual composition function $h(x_1, \dots, x_n)$ defined a.e. by

$$h(x_1, \dots, x_n)(\omega) = h\big(x_1(\omega), \dots, x_n(\omega)\big)$$

satisfies

$$\left|h(x_1,\dots,x_n)\right| \leqslant M \bigvee_{1\leqslant i\leqslant n} |x_i| \text{ a.e.;}$$

it follows that $h(x_1,\dots,x_n)\in E$. Thus, almost everywhere pointwise operations define a functional calculus on $E$. It follows from the uniqueness of functional calculus that this functional calculus agrees with $\tau$ [2].

We proceed with a functional representation of $E_{(2)}$ (see, e.g., [BvR01] or [JL01, p. 30]). The square of $E$ is defined via $E^2=\{x^2 : x\in E\}$, where, again, by $x^2$ we really mean $x|x|$ and the product is defined a.e.. Note that the map $S\colon x\in E_{(2)}\mapsto x^2\in E^2$ is a bijection. In view of this, we may transfer the Banach lattice structure from $E_{(2)}$ to $E^2$. In particular, with this identification, $E^2$ is a vector space. The main advantage of this approach is that addition and scalar multiplication in $E^2$ are defined a.e. pointwise (the vector operations on $E_{(2)}$ were defined exactly this way):

$$S(x\oplus y)=(x\oplus y)^2=\bigl((x^2+y^2)^{\frac12}\bigr)^2=x^2+y^2=S(x)+S(y)$$

and

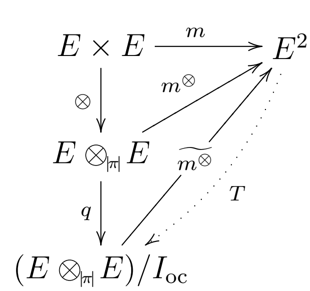
$$S(\lambda\odot x)=(\lambda\odot x)^2=\bigl(\lambda^{\frac12}x\bigr)^2=\lambda x^2=\lambda S(x).$$

Observe, also, that if $x,y\in E$ then the function $xy$ is in $E^2$. Indeed, $x^{\frac12}y^{\frac12}\in E$, so that $E^2\ni S\bigl(x^{\frac12}y^{\frac12}\bigr)=\bigl(x^{\frac12}y^{\frac12}\bigr)^2=xy$.

In view of this construction, we can replace $E_{(2)}$ with $E^2$ in the preceding section. In particular, instead of the map $\varphi\colon E\times E\to E_{(2)}$ defined by $\varphi(x,y)=x^{\frac12}y^{\frac12}$ in Remark 15, we can consider the corresponding map $m\colon E\times E\to E^2$ defined by $m(x,y)=xy$. This map is obviously a continuous orthosymmetric lattice bimorphism.

Suppose now that $E^2$ is a Banach lattice (for example, $E$ is 2-convex). Then the diagram in Figure 2 in Corollary 13 and Remark 15 becomes the diagram in Figure 3.

FIGURE 3




[2]The same argument shows that on $C(K)$-spaces, $\tau$ agrees with the pointwise operations.

For $x, y \in E$, their elementary tensor product $x \otimes y$ can be viewed as a function on $\Omega^2$ via $(x \otimes y)(s,t) = x(s)y(t)$ for $s, t \in \Omega$. This way, $E \otimes E$ is a subset of $L_0(\Omega^2)$. We do not know whether $E \otimes_{|\pi|} E$ can still be viewed as a sublattice of $L_0(\Omega^2)$, but this is definitely the case in many important special cases.

Let $D$ be the diagonal of $\Omega^2$, that is, $D = \{(s,s) \ : \ s \in \Omega\}$. Of course, the map $s \to (s,s)$ is a bijection between $\Omega$ and $D$, so that we can view $D$ as a copy of $\Omega$. For an arbitrary function $u$ in $L_0(\Omega^2)$, one cannot really consider the restriction of $u$ to $D$ because $D$ may have measure zero in $\Omega^2$. However, such a restriction may be defined for elementary tensors via $(x \otimes y)(s,s) = x(s)y(s)$, which is defined a.e. on $\Omega$. That is, the restriction of $x \otimes y$ to $D$ is exactly $xy = m(x,y) = m^{\otimes}(x \otimes y)$ (as we identify $D$ with $\Omega$). Extending this by linearity to $E \otimes E$, we can view $m^{\otimes}$ on $E \otimes E$ (or even on $E \otimes_{|\pi|} E$) as the *restriction to the diagonal* map. Note that, in view of Remark 16 and, in particular, (6), the space $E \otimes E$ is sufficient to capture the diagonal part of $E \otimes_{|\pi|} E$. Furthermore, for $u \in E \otimes E$ we have $u \in I_{\mathrm{oc}}$ iff $m^{\otimes}(u) = 0$ iff $u$ vanishes a.e. on the diagonal. It follows that the both quotient spaces in (6) can be viewed as the space of the restrictions of the functions in $E \otimes E$ to $D$. Therefore, in the case of Köthe spaces, Corollary 13 says that the restrictions of the elements of $E \otimes E$ (or $E \otimes_{|\pi|} E$) to the diagonal are exactly the functions in $E^2$ (again, we identify the diagonal with $\Omega$). Moreover, the norm of the restriction (that is, the quotient norm from (6)) is the same as its $E^2$ norm.

**Example 18.** If $E = L_p$ for $1 \leqslant p < \infty$ then $E^2$ as a vector lattice coincides with $L_{\frac{p}{2}}$. In the case $p \geqslant 2$, $E$ is 2-convex and hence $(E \otimes_{|\pi|} E)/I_{\mathrm{oc}} = E_{[2]} = E_{(2)} = L_{\frac{p}{2}}$. In the case $1 \leqslant p < 2$, the vector lattice $L_{\frac{p}{2}}$ (and, therefore, $E_{(2)}$) admits no non-trivial positive functionals by, e.g., [AB03, Theorem 5.24]. Note that every positive functional $f$ on $E_{[2]}$ gives rise to a positive functional $f \circ q$ on $E_{(2)}$, where $q\colon E_{(2)} \to E_{(2)}/\ker\|\cdot\|_{(2)}$ is the canonical quotient map. It follows that $E_{[2]}^*$ is trivial, and so is $E_{[2]}$. Hence $(E \otimes_{|\pi|} E)/I_{\mathrm{oc}} = E_{[2]} = \{0\}$, which is a trivial AL-space.

**Example 19.** Let $E = C[0,1]$. In this case, $E^2 = E$. Also, $E \otimes_{|\pi|} E = C[0,1]^2$ by Corollary 3F of [F74]. As before, we put $m(x,y) = xy$ for $x, y \in E$. In this case, the map $m^{\otimes}$ on $E \otimes E$ and, therefore, on $E \otimes_{|\pi|} E$, is the restriction to the diagonal, so that $I_{\mathrm{oc}}$ consists of those functions that vanish on the diagonal, while $(E \otimes_{|\pi|} E)/I_{\mathrm{oc}}$ is the space of the restrictions of the functions in $C[0,1]^2$ to the diagonal, which, naturally, can again be identified with $C[0,1]$.

## 5. Banach lattices with a basis

By a ***Banach lattice with a basis*** we mean a Banach lattice where the order is defined by a basis. That is, $E$ has a (Schauder) basis $(e_i)$ such that a vector $x = \sum_{i=1}^{\infty} x_i e_i$ is positive iff $x_i \geqslant 0$ for all $i$. It follows that the basis $(e_i)$ is 1-unconditional. The converse is also true: every Banach space with a 1-unconditional basis is a Banach lattice in the induced order. It is clear that every Banach lattice with a basis is discrete.

5.1. **Concavification of a Banach lattice with a basis.** Since $E$ is a Köthe space, its continuous homogeneous functional calculus in $E$ is coordinate-wise. For example, if $x = \sum_{i=1}^{\infty} x_i e_i$ and $y = \sum_{i=1}^{\infty} y_i e_i$ then

$$x^{\frac{1}{2}} y^{\frac{1}{2}} = \sum_{i=1}^{\infty} x_i^{\frac{1}{2}} y_i^{\frac{1}{2}} e_i \qquad \text{and} \qquad (x^p + y^p)^{\frac{1}{p}} = \sum_{i=1}^{\infty} (x_i^p + y_i^p)^{\frac{1}{p}} e_i.$$

As before, we use the conventions $t^p = |t|^p \operatorname{sign} t$ here for $t, p \in \mathbb{R}$.

Next, we fix $p \geqslant 1$ and consider $E_{(p)}$. Since $E$ is discrete, $E_{(p)}$ is a normed lattice by Corollary 7. Hence, in this case, $E_{[p]}$ equals $\overline{E_{(p)}}$, the completion of $E_{(p)}$. Since $(e_i)$ is disjoint in $E$, it follows from Lemma 3(v) that $x_1 e_1 + \dots + x_n e_n = x_1^p \odot e_1 \oplus \dots \oplus x_n^p \odot e_n$.

**Lemma 20.** *Suppose that $E$ is a Banach lattice with a basis $(e_i)$. Then*

(i) $\lVert e_i \rVert_{(p)} = \lVert e_i \rVert^p$ *for each $i$;*
(ii) *if $x = \sum_{i=1}^{\infty} x_i e_i$ in $E$ then $x = \oplus\!\!-\!\sum_{i=1}^{\infty} x_i^p \odot e_i$ in $E_{(p)}$; in particular, the series converges in $E_{(p)}$;*
(iii) *$(e_i)$ is a 1-unconditional basis of $\overline{E_{(p)}}$.*

*Proof.* (i) follows immediately from Lemma 6. To prove (ii), suppose that $x = \sum_{i=1}^{\infty} x_i e_i$ in $E$. For each $n$, we can write $x = u_n + v_n = u_n \oplus v_n$ where $u_n = \sum_{i=1}^{n} x_i e_i$ and $v_n = \sum_{i=n+1}^{\infty} x_i e_i$. Note that $\lVert v_n \rVert \to 0$ and $u_n = \oplus\!\!-\!\sum_{i=1}^{n} x_i^p \odot e_i$. Therefore,

$$\Big\lVert x \ominus \big(\oplus\!\!-\!\sum_{i=1}^{n} x_i^p \odot e_i\big) \Big\rVert_{(p)} = \lVert x \ominus u_n \rVert_{(p)} = \lVert v_n \rVert_{(p)} \leqslant \lVert v_n \rVert^p \to 0.$$

This proves (ii). It follows from (ii) that the closed linear span of $(e_i)$ is dense in $E_{(p)}$ and, therefore, in $\overline{E_{(p)}}$. Since the sequence $(e_i)$ remains disjoint in $\overline{E_{(p)}}$, this yields (iii). □

**Proposition 21.** *Suppose that $E$ is a Banach lattice with a normalized basis. If $E$ satisfies the lower $p$-estimate with constant $M$ then $\overline{E_{(p)}}$ is lattice isomorphic (isometric if $M = 1$) to $\ell_1$ via $(x_i) \in \ell_1 \mapsto \sum_{i=1}^{\infty} x_i \odot e_i \in E_{(p)}$.*

*Proof.* Let $x \in E$ such that $x = \sum_{i=1}^n x_i \odot e_i = \sum_{i=1}^n x_i^{1/p} e_i$. It follows from Lemma 9 that

$$\|x\|_{(p)} \geqslant \tfrac{1}{M^p} \sum_{i=1}^n \|x_i \odot e_i\|_{(p)} = \tfrac{1}{M^p} \sum_{i=1}^n |x_i|.$$

On the other hand, by the triangle inequality, we have $\|x\|_{(p)} \leqslant \sum_{i=1}^n \|x_i \odot e_i\|_{(p)} = \sum_{i=1}^n |x_i|$. □

5.2. **Fremlin tensor product of Banach lattices with bases.** Given Banach spaces $E$ and $F$ with bases $(e_i)$ and $(f_i)$, respectively, then the double sequence $(e_i \otimes f_j)$ is a basis for the Banach space projective tensor product $E \otimes_\pi F$, see [GGdL61]. However, even if these respective bases are unconditional then $(e_i \otimes f_j)$ is not necessarily an unconditional basis for $E \otimes_\pi F$. Indeed, it was shown in [KP70] that the Banach space projective tensor product $\ell_p \otimes_\pi \ell_q$ with $1/p + 1/q \leqslant 1$ does not have an unconditional basis.

Recall that if $E$ is a Banach lattice with a basis then the basis is automatically 1-unconditional.

**Lemma 22.** *Suppose that $E$ and $F$ are Banach lattices with bases, $(e_i)$ and $(f_j)$, respectively. Then the double sequence $(e_i \otimes f_j)_{i,j}$ is disjoint in $E \otimes_{|\pi|} F$. Moreover, this sequence is a 1-unconditional basis of $E \otimes_{|\pi|} F$ (under any enumeration).*

*Proof.* First, we will show that $(e_i \otimes f_j) \perp (e_k \otimes f_l)$ provided $(i, j) \neq (k, l)$. Using Proposition 2, we consider $E \otimes_{|\pi|} F$ as a sublattice of $L^r(E, F^*)^*$. It suffices to show that

$$\big\langle (e_i \otimes f_j) \wedge (e_k \otimes f_l), T \big\rangle = 0$$

for every positive $T \colon E \to F^*$. By [AB06, Theorem 3.49],

$$\big\langle (e_i \otimes f_j) \wedge (e_k \otimes f_l), T \big\rangle = \inf_{0 \leqslant S \leqslant T} \big\{ (e_i \otimes f_j)(S) + (e_k \otimes f_l)(T - S) \big\}. \tag{7}$$

Put $c = \langle Te_k, f_l \rangle$ and define $S \colon E \to F^*$ via $S = c e_k^* \otimes f_l^*$, where $e_k^*$ and $f_l^*$ are the appropriate bi-orthogonal functionals. That is, for $x \in E$ we have $Sx = c e_k^*(x) f_l^*$. Clearly, $S \geqslant 0$. We will show that $S \leqslant T$. It suffices to show that $Se_m \leqslant Te_m$ for every $m$. But if $m \neq k$ then $Se_m = 0 \leqslant Te_m$. It is left to prove that $Se_k \leqslant Te_k$. Note that $Se_k = c f_l^*$. It suffices to show that $\langle Se_k, f_n \rangle \leqslant \langle Te_k, f_n \rangle$ for all $n$. But this is true because $\langle Se_k, f_n \rangle = c f_l^*(f_n) = 0$, when $n \neq l$, and $\langle Se_k, f_l \rangle = c f_l^*(f_l) = c = \langle Te_k, f_l \rangle$ Now substituting this $S$ into (7), we get

$$(e_i \otimes f_j)(S) + (e_k \otimes f_l)(T - S) = c e_k^*(e_i) f_l^*(f_j) + \langle Te_k, f_l \rangle - \langle Se_k, f_l \rangle = 0 + c - c = 0$$

because $(i,j)\neq(k,l)$.

Being a disjoint sequence in a Banach lattice, $(e_i\otimes f_j)_{i,j}$ is a 1-unconditional basic sequence. It is left to show that its closed span is all of $E\otimes_{|\pi|}F$. Take $x\in E$ and $y\in F$ with $\|x\|,\|y\|\leqslant 1$. Given any $\varepsilon\in(0,1)$, we can find basis projections $P$ and $Q$ on $E$ and $F$, respectively, such that $x_0=Px$ and $y_0=Qy$ satisfy $\|x-x_0\|<\varepsilon$ and $\|y-y_0\|<\varepsilon$. It follows that

$$\begin{aligned}\|x\otimes y-x_0\otimes y_0\|_{|\pi|}&=\|x_0\otimes(y-y_0)+(x-x_0)\otimes y_0+(x-x_0)\otimes(y-y_0)\|_{|\pi|}\\&\leqslant\|x_0\|\|y-y_0\|+\|x-x_0\|\|y_0\|+\|x-x_0\|\|y-y_0\|\leqslant 3\varepsilon.\end{aligned}$$

Since $x_0\otimes y_0$ is in $\operatorname{span}\{e_i\otimes f_j\,:\,i,j\in\mathbb{N}\}$, it follows that $x\otimes y$ can be approximated by elements of the span. It follows that the span is dense in $E\otimes F$ and, therefore, in $E\otimes_{|\pi|}F$. ◻

5.3. **The diagonal of $E\otimes_{|\pi|}E$.** Suppose that $E$ is a Banach lattice with a basis $(e_i)$. As we just observed, $(e_i\otimes e_j)_{i,j}$ is a 1-unconditional basis in $E\otimes_{|\pi|}E$. It is easy to see that

$$\begin{aligned}I_{\mathrm{oc}}&=\overline{\operatorname{span}}\big\{(e_i\otimes e_j)\,:\,i\neq j\big\}\\(E\otimes_{|\pi|}E)/I_{\mathrm{oc}}&=\overline{\operatorname{span}}\big\{(e_i\otimes e_i)\,:\,i\in\mathbb{N}\big\}\end{aligned}\tag{8}$$

In particular, we can view $I_{\mathrm{oc}}$ and $(E\otimes_{|\pi|}E)/I_{\mathrm{oc}}$ as two mutually complementary bands in $E\otimes_{|\pi|}E$. In view of this, our interpretation of $(E\otimes_{|\pi|}E)/I_{\mathrm{oc}}$ as the diagonal of $E\otimes_{|\pi|}E$ is consistent with, e.g., Examples 2.10 and 2.23 in [R02].

It follows immediately from Corollary 12 that $(E\otimes_{|\pi|}E)/I_{\mathrm{oc}}$ is lattice isometric to $\overline{E_{(2)}}$. Moreover, in view of (8), the map $T$ in Corollary 12 has a particularly simple form: $T\colon e_i\to e_i\otimes e_i$. Thus, Corollary 12 for Banach lattices with a basis can be stated as follows.

**Theorem 23.** *Suppose that $E$ is a Banach lattice with a basis $(e_i)$. Then the map that sends $\sum_{i=1}^\infty u_ie_i\otimes e_i$ in $E\otimes_{|\pi|}E$ into $\sum_{i=1}^\infty u_i\odot e_i$ in $\overline{E_{(2)}}$ is a surjective lattice isometry between $(E\otimes_{|\pi|}E)/I_{\mathrm{oc}}$ and $\overline{E_{(2)}}$.*

Combining this with Propositions 5 and 21, we get the following corollaries.

**Corollary 24.** *Suppose that $E$ is a Banach lattice with a basis. If $E$ is 2-convex then $(E\otimes_{|\pi|}E)/I_{\mathrm{oc}}$ is lattice isometric to $E_{(2)}$.*

**Corollary 25.** *Suppose that $E$ is a Banach lattice with a normalized basis $(e_i)$, satisfying a lower 2-estimate with constant $M$. Then $(E\otimes_{|\pi|}E)/I_{\mathrm{oc}}$ is lattice isomorphic (isometric if $M=1$) to $\ell_1$ via $(x_i)\in\ell_1\mapsto\sum_{i=1}^\infty x_ie_i\otimes e_i$.*

**Example 26.** If $E = \ell_p$ for $1 \leqslant p < \infty$ then $E^2$ (and, therefore, $E_{(2)}$) can be identified as a vector space with $\ell_{\frac{p}{2}}$. In the case $p \geqslant 2$, $E$ is 2-convex and hence $(E \otimes_{|\pi|} E)/I_{\rm oc} = E_{[2]} = E_{(2)} = \ell_{\frac{p}{2}}$. In the case $1 \leqslant p < 2$, $E$ satisfies the lower 2-estimate and hence $(E \otimes_{|\pi|} E)/I_{\rm oc} = E_{[2]} = \ell_1$. On the other hand, in the latter case, $\|\cdot\|_{(2)}$ is the $\ell_1$-norm on $E_{(2)} = \ell_{\frac{p}{2}}$ and we have $E_{[2]} = \overline{(E_{(2)}, \|\cdot\|_{(2)})} = \overline{(\ell_{\frac{p}{2}}, \|\cdot\|_{\ell_1})} = \ell_1$.

## References


[AB03] C.D. Aliprantis, O. Burkinshaw, *Locally solid Riesz spaces with applications to economics,* 2nd edition. Math. Surveys and Monographs, 105. AMS, Providence, RI, 2003.

[AB06] ______, *Positive operators*, Springer, 2006.

[BB12] Q. Bu, G. Buskes, Polynomials on Banach lattices and positive tensor products, *J. Math. Anal. Appl.* **388** (2012), 845-862.

[BdPvR91] G. Buskes, B. de Pagter, A. van Rooij, Functional calculus on Riesz spaces. *Indag. Math. (N.S.)* **2** (1991), no. 4, 423–436.

[BvR01] G. Buskes, A. van Rooij, Squares of Riesz spaces. *Rocky Mountain J. Math.* **31** (2001), no. 1, 45–56.

[F72] D.H. Fremlin, Tensor products of Archimedean vector lattices, *Amer. J. Math.* **94** (1972), 777–798.

[F74] ______, Tensor products of Banach lattices, *Math. Ann.* **211** (1974), 87–106.

[GGdL61] B.R. Gelbaum, J. Gil de Lamadrid, Bases of tensor products of Banach spaces, *Pacific J. Math.* **11** (1961), 1281–1286.

[JL01] W.B. Johnson, J. Lindenstrauss, Basic concepts in the geometry of Banach spaces. *Handbook of the geometry of Banach spaces*, Vol. I, 1-84, North-Holland, Amsterdam, 2001.

[KP70] S. Kwapień, A. Pełczyński, The main triangle projection in matrix spaces and its applications, *Studia Math.* **34** (1970), 43–68.

[LT79] J. Lindenstrauss, L. Tzafriri, *Classical Banach spaces. II*, Springer-Verlag, Berlin, 1979, Function spaces.

[L07] J. Loane, Polynomials on Riesz Spaces, *PhD Thesis*, Dept. of Math., National University of Ireland, Galway, 2007.

[R02] R.A. Ryan, *Introduction to tensor products of Banach spaces.* Springer-Verlag, London, 2002.

[S80] H.H. Schaefer, Aspects of Banach lattices. *Studies in functional analysis*, pp. 158–221, MAA Stud. Math., 21, Math. Assoc. America, Washington, D.C., 1980.



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