SEMITRANSITIVE SPACES OF OPERATORS

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Abstract. A collection $\mathcal{S}$ of linear maps on a vector space X is strictly semitransitive if for every two vectors x, y there is $A \in \mathcal{S}$ such that $Ax = y$ or $Ay = x$. There is also a topological version of this property for bounded maps on a Banach space. In this paper we discuss semitransitive subspaces of $L(X)$. We also study k-semitransitivity, which is the multi-variable version of semitransitivity, the corresponding weakening of the well-known notion of k-transitivity. We establish, in particular, that every strictly k-semitransitive subspace is strictly $(k-1)$-transitive. We also show that if $2k > \dim X$, then every k-semitransitive subspace is k-transitive. Finally, we extend Jacobson's theorem to semitransitive rings.

1. Introduction and notation

Throughout this paper, X will be a real or complex Banach space, and by $L(X)$ we denote the space of all continuous linear operators on X. In the finite-dimensional case we will write M_n instead of $L(X)$, where $n = \dim X$. In fact, most of the results in the finite-dimensional case remain valid for $M_n(\mathbb{F})$ where $\mathbb{F}$ is an arbitrary field.

A subset $\mathcal{S} \subseteq L(X)$ is said to be ***strictly transitive*** if for every two non-zero vectors $x, y \in X$ there is $A \in \mathcal{S}$ such that $Ax = y$. We say that $\mathcal{S}$ is ***topologically transitive*** if for every two non-zero vectors $x, y \in X$ and every $\varepsilon > 0$ there is $A \in \mathcal{S}$ such that $\|Ax - y\| < \varepsilon$. Given a positive integer k, we say that $\mathcal{S}$ is ***strictly*** (or ***topologically***) ***k-transitive*** if for every linearly independent k-tuple $x_1, \dots, x_k$ in X and for every k-tuple $y_1, \dots, y_k$ in X (and every $\varepsilon > 0$) there exists $A \in \mathcal{S}$ such that for every $i = 1, \dots, k$ one has $Ax_i = y_i$ (respectively, $\|Ax_i - y_i\| < \varepsilon$). Clearly, $\mathcal{S}$ is strictly (or topologically) 1-transitive if and only if it is strictly (respectively, topologically) transitive.

We say that $\mathcal{S}$ is ***strictly semitransitive*** if for every two non-zero vectors $x, y \in X$ there is $A \in \mathcal{S}$ such that $Ax = y$ or $Ay = x$. We say that $\mathcal{S}$ is ***topologically semitransitive*** if for every two non-zero vectors $x, y \in X$ and every $\varepsilon > 0$ there

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is $A \in \mathcal{S}$ such that $\|Ax-y\|<\varepsilon$ or $\|Ay-x\|<\varepsilon$. Given a positive integer k, we say that $\mathcal{S}$ is ***strictly k-semitransitive*** if for every two linearly independent k-tuples $x_1,\dots,x_k$ and $y_1,\dots,y_k$ in X there exists $A \in \mathcal{S}$ such that $Ax_i=y_i$ for all $i=1,\dots,k$, or $Ay_i=x_i$ for all $i=1,\dots,k$. Topological k-semitransitivity is defined accordingly.

For $x \in X$, we will write $\mathcal{S}x$ for the orbit of x under $\mathcal{S}$, i.e., $\mathcal{S}x=\{Ax \mid A \in \mathcal{S}\}$. We say that x is ***strictly cyclic*** under $\mathcal{S}$ if $\mathcal{S}x=X$, we say that x is ***topologically cyclic*** under $\mathcal{S}$ if $\mathcal{S}x$ is dense in X.

For $A \in L(X)$, let $A^{(k)}$ be an element of $L(X^k)$ defined by $A^{(k)}(x_1,\dots,x_k)=(Ax_1,\dots,Ax_k)$. Let $\mathcal{S}^{(k)}=\{A^{(k)} \mid A \in \mathcal{S}\}$.

These definitions immediately yield the following characterization. A subset $\mathcal{S}$ in $L(X)$ is strictly (or topologically) k-transitive if and only if every linearly independent k-tuple in X^k is strictly (respectively, topologically) cyclic for $\mathcal{S}^{(k)}$. That is, if $x=(x_1,\dots,x_k)$ is a linearly independent k-tuple, then $\mathcal{S}^{(k)}x=X^k$ (respectively, $\overline{\mathcal{S}^{(k)}x}=X^k$). Similarly, $\mathcal{S}$ is strictly (or topologically) k-semitransitive if and only if for every two linearly independent k-tuples x and y in X^k we have $x \in \mathcal{S}^{(k)}y$ or $y \in \mathcal{S}^{(k)}x$ (respectively, $x \in \overline{\mathcal{S}^{(k)}y}$ or $y \in \overline{\mathcal{S}^{(k)}x}$).

One usually equips $\mathcal{S}$ with some additional structure. It is easy to see that if $\mathcal{S}$ is a group then strict semitransitivity coincides with strict transitivity. For bounded groups, topological semitransitivity coincides with topological transitivity. There is extensive literature on topologically transitive and n-transitive algebras, see [RR] for a survey. Strictly semitransitive algebras of operators on Banach spaces were investigated in [RT]. It is easy to see that a unital algebra of operators is topologically semitransitive if and only if it is unicellular; such algebras were studied in [RR]. We refer the reader to [BGMRT] for a study of strictly semitransitive semigroups and algebras in M_n, and to [DLMR] for a study of transitive subspaces of M_n. In this paper we will be primarily interested in semitransitive and k-semitransitive subspaces of M_n. Note that if $\mathcal{L}$ is a linear (i.e., not necessarily closed) subspace of $L(X)$ then $\mathcal{L}x$ is a linear subspace of X for every x. Therefore, it follows from the previous paragraph that a linear subspace of M_n is strictly k-semitransitive if and only if it is topologically k-semitransitive as every linear subspace is closed. Hence, when talking about subspaces of M_n we will be omitting the adverbs "strictly" or "topologically".

Starting with [BGMRT], several authors have studied naturally arising semitransitivity questions on finite-dimensional spaces, including reducibility and triangularizability of semitransitive subspace of M_n. We would like to mention the two recent papers [Bled] and [BDKKO] which contain many new results in this direction.

2. Cyclic vectors of semitransitive subspaces

Theorem 1. *Suppose that X is a separable Banach space and $\mathcal{L}$ is a linear subspace of $L(X)$. Suppose that $\mathcal{L}$ is topologically semitransitive. Then it has a topologically cyclic vector. Moreover, the set of topologically cyclic vectors for $\mathcal{L}$ contains a dense G_δ set.*

Proof. Let C be the set of all topologically cyclic vectors in X. For $x \in X$ write

$$\mathcal{L}^\circ x = \bigl\{ y \in X \mid x \in \overline{\mathcal{L}y} \bigr\}.$$

Clearly, topological semitransitivity of $\mathcal{L}$ is equivalent to $\overline{\mathcal{L}x} \cup \mathcal{L}^\circ x = X$ for every non-zero $x \in X$. In particular, if $x \in X \setminus C$, then $\overline{\mathcal{L}x}$ is a proper closed subspace, so that $\mathcal{L}^\circ x$ contains an open dense subset, namely, $X \setminus \overline{\mathcal{L}x}$.

If C contains a dense open subset, then we are done. Otherwise, the closure of $X \setminus C$ contains an open set. Since X is separable, there is a sequence (x_i) in $X \setminus C$ whose linear span is dense in X. Put $G = \bigcap_{i=1}^\infty \mathcal{L}^\circ x_i$; then, by the Baire Category Theorem, G contains a dense G_δ subset. We show that $G \subseteq C$. Indeed, if $y \in G$, then for every i we have $y \in \mathcal{L}^\circ x_i$, so that $x_i \in \overline{\mathcal{L}y}$. Since (x_i) spans a dense subspace of X, it follows that $\overline{\mathcal{L}y} = X$, hence y is topologically cyclic. □

Remark 2. We would like to mention here that Corollary 3.10 of [RT] asserts that if X is a Banach space and $\mathcal{A}$ is a strictly semitransitive norm-closed subalgebra of $L(X)$, then the set of strictly cyclic vectors for $\mathcal{A}$ is residual, i.e., its complement is of first category.

Corollary 3. *If $\mathcal{L}$ is a semitransitive subspace of M_n then* $\dim \mathcal{L} \geqslant n$.

Proof. By Theorem 1, $\mathcal{L}$ has a cyclic vector. Let x be a cyclic vector for $\mathcal{L}$. Then $\dim \mathcal{L} \geqslant \dim \mathcal{L}x = n$. □

3. k-semitransitive sets

We start with a simple observation that generally k-semitransitivity implies $\frac{k}{2}$-transitivity. We will see later that better estimates hold when $\mathcal{S}$ is a subspace or a subring.

Proposition 4. *Suppose that X is a Banach space and $\mathcal{S}$ is a topologically k-semitransitive subset of $L(X)$ for some even $k \leqslant \dim X$. Then $\mathcal{S}$ is topologically $\frac{k}{2}$-transitive.*

Proof. Put $m = \frac{k}{2}$. Assume that we have linearly independent vectors $x_1, \dots, x_m$ in X, arbitrary $y_1, \dots, y_m$ in X, and an arbitrary $\varepsilon > 0$. For every $i = 1, \dots, m$ one can find

$\tilde{y}_1, \dots, \tilde{y}_m$ such that $\|\tilde{y}_i - y_i\| < \frac{\varepsilon}{2}$ and so that $x_1, \dots, x_m, \tilde{y}_1, \dots, \tilde{y}_m$ are all linearly independent. Applying the definition of k-semitransitivity to the k-tuples

$$(x_1, \dots, x_m, \tilde{y}_1, \dots, \tilde{y}_m) \text{ and } (\tilde{y}_1, \dots, \tilde{y}_m, x_1, \dots, x_m)$$

we conclude that there is $A \in \mathcal{S}$ such that $\|Ax_i - \tilde{y}_i\| < \frac{\varepsilon}{2}$ and, therefore, $\|Ax_i - y_i\| < \varepsilon$ for all $i = 1, \dots, m$. □

If $\mathcal{S}$ is strictly k-semitransitive then, by the preceding proposition, $\mathcal{S}$ is topologically m-transitive for every $m \leqslant \frac{k}{2}$. Hence, $\mathcal{S}^{(m)}x$ is dense in X^m for every linearly independent $x \in X^m$. We claim that if, in addition, $\mathcal{S}$ is convex then $\mathcal{S}^{(m)}x = X$ for every such x, so that $\mathcal{S}$ is strictly m-transitive. Indeed, let $x \in X^m$ be linear independent and $y \in X^m$ be arbitrary. Choose $z \in X^m$ so that the $2m$-tuple (x, z) is linearly independent. Then $(x, \varepsilon y + z)$ and $(x, \varepsilon y - z)$ are still linear independent for some sufficiently small ε. Hence $(x, y + \varepsilon^{-1}z)$ and $(x, y - \varepsilon^{-1}z)$ are linearly independent. Applying strict $2m$-semitransitivity to the following pairs of $2m$-tuples: $(x, y + \varepsilon^{-1}z)$ and $(y + \varepsilon^{-1}z, x)$, and $(x, y - \varepsilon^{-1}z)$ and $(y - \varepsilon^{-1}z, x)$ we conclude that $y + \varepsilon^{-1}z$ and $y - \varepsilon^{-1}z$ are both in $\mathcal{S}^{(m)}x$. Since $\mathcal{S}^{(m)}$ is convex, it follows that $y \in \mathcal{S}^{(m)}x$.

The following example shows that for arbitrary sets strict k-semitransitivity does not imply $\frac{k}{2}$-transitivity.

Example. Let $\mathcal{S}$ be the subset of M_2 consisting of all the 2×2 matrices except the matrices of the form $\left(\begin{smallmatrix} a & b \\ 0 & c \end{smallmatrix}\right)$ with $|a| > 1$. Clearly, if $A \in M_2$ is invertible then either A or A^{-1} belongs to $\mathcal{S}$. It follows that $\mathcal{S}$ is strictly 2-semitransitive. However, it is not strictly transitive as no matrix in $\mathcal{S}$ takes e_1 into $2e_1$.

4. k-SEMITRANSITIVE SUBSPACES

We show in this section that a much stronger result than Proposition 4 holds for subspaces of M_n. Namely, every k-semitransitive subspace of M_n is $(k-1)$-transitive. Here, again, we will assume that the scalar field is $\mathbb{R}$ or $\mathbb{C}$, though many of the proofs remain valid for arbitrary fields.

Let M_{nk} be the space of all $n \times k$ matrices. It is well known that M_{nk} becomes a Hilbert space if equipped with scalar product $\langle A, B\rangle = \operatorname{tr}(A^*B) = \sum_{i,j} a_{ij}\bar{b}_{ij}$, where $A = (a_{ij})$ and $B = (b_{ij})$ are two matrices in M_{nk}. It follows from $\operatorname{tr}(AB) = \operatorname{tr}(BA)$ for any $A, B \in M_n$ that $\langle \cdot, \cdot \rangle$ is stable under unitary equivalences. That is, if U and V are unitaries in M_n and M_k respectively, then $\langle UAV, UBV\rangle = \langle A, B\rangle$ for any $A, B \in M_{nk}$. If $\mathcal{L}$ is a linear subspace of M_{nk} then, clearly, $\mathcal{L}$ is proper if and only if $\mathcal{L} \perp T$ for some $T \in M_{nk}$.

The following lemma is well known. For completeness, we provide the proof.

Lemma 5. *Let $\mathcal{L}$ be a subspace of M_n and $k \leqslant n$. Then $\mathcal{L}$ is not k-transitive if and only if there is a nonzero $T \in M_n$ such that* $\operatorname{rank} T \leqslant k$ *and $\mathcal{L} \perp T$.*

Proof. For $A \in M_n$ and $k \leqslant n$ let $\widetilde{A}$ denote the matrix in M_{nk} composed of the first k columns of A. Furthermore, if $\mathcal{M}$ is a subspace of M_n, let $\widetilde{\mathcal{M}} = \{\widetilde{A} \mid A \in \mathcal{M}\}$. Clearly, $\mathcal{M}$ is a linear subspace of M_{nk}.

Suppose that $\mathcal{L}$ is not k-transitive. Then there exists a linearly independent k-tuple $(x_1, \dots, x_k)$ and a k-tuple $(y_1, \dots, y_k)$ such that no $A \in \mathcal{L}$ satisfies $Ax_i = y_i$ for all $i = 1, \dots, k$. Let S be an invertible operator in M_n such that $Sx_i = e_i$ for $i = 1, \dots, k$, and put $\mathcal{M} = S\mathcal{L}S^{-1}$. Let A be a matrix in M_n whose first k columns are $Sy_1, \dots, Sy_k$. Then $ASx_i = Ae_i = Sy_i$, so that $S^{-1}ASx_i = y_i$ for $i = 1, \dots, k$. It follows that $S^{-1}AS \notin \mathcal{L}$ so that $A \notin \mathcal{M}$. Since this is true for every such A, we have $\widetilde{A} \notin \widetilde{\mathcal{M}}$, hence $\widetilde{\mathcal{M}}$ is a proper subspace of M_{nk}. Then there exists $T_0 \in M_{nk}$ such that $\widetilde{\mathcal{M}} \perp T_0$ in M_{nk}. Extend T_0 to a matrix T_1 in M_n, that is $T_1 = (T_0\ 0)$. Clearly, $\operatorname{rank} T_1 \leqslant k$ and $\mathcal{M} \perp T_1$. Let $T = S^* T_1 S^{-1*}$, then $\operatorname{rank} T \leqslant k$ and $\mathcal{L} \perp T$.

Conversely, if a non-zero $T \in M_n$ satisfies $\operatorname{rank} T \leqslant k$ and $\mathcal{L} \perp T$, we can assume without loss of generality that $\operatorname{Range} T \subseteq \operatorname{span}\{e_1, \dots, e_k\}$, so that $T = (T_0, 0)$ for some non-zero $T_0 \in M_{nk}$. It follows that $\widetilde{\mathcal{L}} \perp T_0$, so that $\widetilde{\mathcal{L}}$ is a proper subspace of M_{nk}. Let $A_0 \in M_{nk} \setminus \widetilde{\mathcal{L}}$, and let $y_1, \dots, y_k$ be the columns of A_0, then no matrix in $\mathcal{L}$ sends $e_1, \dots, e_k$ into $y_1, \dots, y_k$. ☐

Recall that an operator T is an ***involution*** if $T^2 = I$.

Lemma 6. *The set of all involutions in M_n spans M_n.*

Proof. It suffices to find n^2 linearly independent involutions in M_n. Consider all the matrices of the following forms:

(i) Diagonal $\operatorname{diag}\{\underbrace{1, \dots, 1}_{i}, \underbrace{-1, \dots, -1}_{n-i}\}$, $i = 1, \dots, n$;

(ii) The identity matrix with i-th and j-th rows interchanged and multiplied respectively by 2 and $\frac{1}{2}$.

It can be easily seen that all these matrices are involutions, they are linearly independent, and there are n^2 of them. ☐

Lemma 7. *Suppose that $\mathcal{L}$ is a k-semitransitive subspace of M_n for some $k \leqslant n$, and P is an orthogonal projection of rank k. Then $\mathcal{L}P$ contains PM_nP.*

Proof. Without loss of generality, up to a unitary equivalence, we can assume that P is the orthogonal projection onto $\operatorname{span}\{e_1,\dots,e_k\}$. Pick an invertible matrix V in M_k, and let $y_1,\dots,y_k$ be the columns of V extended by zeros at the end to n-tuples. Since $\mathcal{L}$ is k-semitransitive, there exists $A\in\mathcal{L}$ such that either $Ae_i=y_i$ as $i=1,\dots,k$, or $Ay_i=e_i$ as $i=1,\dots,k$. It follows that either $A=\left(\begin{smallmatrix}V&R\\0&S\end{smallmatrix}\right)$ or $A=\left(\begin{smallmatrix}V^{-1}&R\\0&S\end{smallmatrix}\right)$ for some R and S. In particular, for every involution V in M_k there are matrices R and S such that $\left(\begin{smallmatrix}V&R\\0&S\end{smallmatrix}\right)$ is in $\mathcal{L}$, hence, $\left(\begin{smallmatrix}V&0\\0&0\end{smallmatrix}\right)$ is in $\mathcal{L}P$. Lemma 6 yields that $\left(\begin{smallmatrix}B&0\\0&0\end{smallmatrix}\right)$ is in $\mathcal{L}P$ for every $B\in M_k$, but the set of all the matrices of this form is exactly PM_nP. □

Remark 8. One can easily verify that the proofs of Lemmas 6 and 7 remain valid for $M_n(\mathbb{F})$ for any field $\mathbb{F}$ with $\operatorname{char}\mathbb{F}\neq 2$.

Suppose now that $\operatorname{char}\mathbb{F}=2$. Then (i) and (ii) in the proof of Lemma 6 are not valid. However, we claim that Lemma 7 remains true in this case. A glance at the original proof reveals that it is sufficient to show that if $\mathcal{L}$ is a subspace of $M_n(\mathbb{F})$ such that for every invertible matrix $A\in M_n(\mathbb{F})$ either $A\in\mathcal{L}$ or $A^{-1}\in\mathcal{L}$, then $\mathcal{L}=M_n(\mathbb{F})$. Therefore, $\mathcal{L}$ contains all the involutions. In particular, $I\in\mathcal{L}$. Note that V is an involution if and only if $(V+I)^2=0$, it follows that every square-zero matrix is in $\mathcal{L}$. Denote by E_{ij} the standard basis matrix $e_ie_j^T$. Let $\mathcal{S}_1=\{E_{ij}\mid i\neq j\}$ and $\mathcal{S}_2=\{E_{11}+E_{1i}+E_{i1}+E_{ii}\mid 1<i\leqslant n\}$. Then $\mathcal{S}_1$ and $\mathcal{S}_2$ consist of square-zero matrices, so that $\mathcal{S}_1\cup\mathcal{S}_2\subset\mathcal{L}$. Furthermore, $\mathcal{S}_1\cup\mathcal{S}_2$ is linearly independent and has n^2-1 elements. Note also, that all the elements of $\mathcal{S}_1\cup\mathcal{S}_2$ have zero trace. If n is odd, then $\operatorname{tr}I=1$ so that I is linearly independent of $\mathcal{S}_1\cup\mathcal{S}_2$. It follows that $\mathcal{S}_1\cup\mathcal{S}_2\cup\{I\}$ spans M_n, hence $\mathcal{L}=M_n$. Suppose that n is even. Let $A=I+E_{12}+E_{21}-E_{22}$, then $A^{-1}=I+E_{12}+E_{21}-E_{11}$. Then $\operatorname{tr}A=\operatorname{tr}A^{-1}=1$ yields that both A and A^{-1} are linearly independent of $\mathcal{S}_1\cup\mathcal{S}_2$. Since either A or A^{-1} is in $\mathcal{L}$ then $\dim\mathcal{L}=n^2$, hence $\mathcal{L}=M_n$.

In the case $k=n$ and $P=I$, Lemma 7 yields the following.

Corollary 9. *M_n contains no proper n-semitransitive subspaces.*

Lemma 10. *Suppose that $\mathcal{L}$ is a k-semitransitive subspace of M_n for some $k\leqslant n$, and $T\in M_n$ such that $\operatorname{rank}T\leqslant k$ and $\mathcal{L}\perp T$. Then $T^2=0$.*

Proof. Without loss of generality (up to a unitary similarity) we can assume that T is of the form $\left(\begin{smallmatrix}R&0\\S&0\end{smallmatrix}\right)$, where R is $k\times k$. Let P be the projection on the first k coordinates. By Lemma 7, $\mathcal{L}P$ contains all the matrices of the form $\left(\begin{smallmatrix}A&0\\0&0\end{smallmatrix}\right)$ for all $A\in M_k$. Since T is orthogonal to $\mathcal{L}$, it follows that $R=0$, so that $T^2=0$. □

Theorem 11. *Suppose that $\mathcal{L}$ is a $(k+1)$-semitransitive subspace of M_n for some $k<n$. Then $\mathcal{L}$ is k-transitive.*

Proof. Suppose that $\mathcal{L}$ is not k-transitive. It follows from Lemma 5 that there is a non-zero $T \in M_n$ with $\mathcal{L} \perp T$ and $\operatorname{rank} T \leqslant k$. Since $\mathcal{L}$ is $(k+1)$-semitransitive and, therefore, k-semitransitive, Lemma 10 yields $T^2 = 0$.

Let $m = \operatorname{rank} T$. Since T is nilpotent, we may assume without loss of generality (up to a similarity) that T is in Jordan form, no matter what the underlying field may be. Since $T^2 = 0$, it follows that all the non-zero Jordan blocks of T are of the form $\left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right)$. Let (t_{ij}) be the matrix of T. Then $t_{2i-1,2i} = 1$ for all $i = 1, \dots, m$, and all the other entries of the matrix are zero.

It follows from $m \leqslant k$ that $\mathcal{L}$ is $(m+1)$-semitransitive. Apply the definition of $(m+1)$-semitransitivity to the following $(m+1)$-tuples:

$$(e_1, e_2, e_4, e_6, \dots, e_{2m}) \quad \text{and} \quad (e_2, e_1, e_4, e_6, \dots, e_{2m}).$$

Hence there exists $A \in \mathcal{L}$ such that $Ae_2 = e_1$ and $Ae_{2i} = e_{2i}$ for $i = 2, \dots, m$. Let (a_{ij}) be the matrix of A, then $a_{1,2} = 1$ and $a_{2i-1,2i} = 0$ for $i = 2, \dots, m$. It follows that $\langle A, T \rangle = 1$, which contradicts $\mathcal{L} \perp T$. $\square$

5. When a k-semitransitive subspace is k-transitive

Proposition 12. *Suppose that $\mathcal{L}$ is a k-semitransitive subspace of M_n for some $k \leqslant n$. If $\mathcal{L}$ is not k-transitive then there exists $T \in M_n$ such that $\mathcal{L} \perp T$, $\operatorname{rank} T = k$, and $T^2 = 0$.*

Proof. Suppose that $\mathcal{L}$ is a k-semitransitive subspace of M_n for some $k \leqslant n$, and $\mathcal{L}$ is not k-transitive. By Lemma 5 there exists a non-zero $T \in M_n$ such that $\mathcal{L} \perp T$ and $\operatorname{rank} T \leqslant k$. If $k > 1$ then Theorem 11 asserts that $\mathcal{L}$ is $(k-1)$-transitive, so that Lemma 5 yields $\operatorname{rank} T > k-1$, hence $\operatorname{rank} T = k$. If $k = 1$ then we still have $\operatorname{rank} T = k$ as $T \neq 0$. Finally, it follows from Lemma 10 that $T^2 = 0$. $\square$

Combining Proposition 12 with Lemma 5, we obtain the following characterization.

Corollary 13. *Suppose that $\mathcal{L}$ is a k-semitransitive subspace of M_n for some $k < n$. Then $\mathcal{L}$ is k-transitive if and only if $\mathcal{L}^\perp$ contains no operator of rank k with zero square.*

This also allows us to improve the result of Theorem 11 when $k > \frac{n}{2}$.

Corollary 14. *If $2k > n$ then every k-semitransitive subspace of M_n is k-transitive.*

Proof. Suppose that $2k > n$ and observe that no operator of rank k has zero square. Indeed, let $T \in M_n$ be such that $\operatorname{rank} T = k$. Then $\dim \operatorname{Range} T = k$ while $\dim \ker T = n - k > k$, so that $\operatorname{Range} T$ is not contained in $\ker T$, hence $T^2 \neq 0$. Therefore, the result follows from Proposition 12. ☐

The following result is, in a sense, a complement to Corollary 14. We show that if $2k \leqslant n$ then there exists a k-semitransitive subspace of M_n that is not k-transitive.

Proposition 15. *Let $T \in M_n$ such that $\operatorname{rank} T = k$ and $T^2 = 0$. Then $\{T\}^\perp$ is k-semitransitive, but not k-transitive.*

Proof. Let $\mathcal{L} = \{T\}^\perp$. Observe that $\mathcal{L}$ is not k-transitive by Lemma 5. On the other hand, since $\mathcal{L}^\perp$ consists of multiples of T only, no non-zero matrix of rank less than k is orthogonal to $\mathcal{L}$, so that Lemma 5 yields that $\mathcal{L}$ is $(k-1)$-transitive.

We claim that $\mathcal{L}$ is k-semitransitive. Suppose not. Let $(x_1, \dots, x_k)$ and $(y_1, \dots, y_k)$ be two k-tuples, each linearly independent, such that no matrix in $\mathcal{L}$ takes all x_i's into the corresponding y_i's or vice versa. Let $H = \operatorname{span}\{x_1, \dots, x_k\}$ and put $Z = H^\perp$. Let $A\colon H \mapsto X$ be such that $Ax_i = y_i$ as $i = 1, \dots, k$. Choose an orthonormal basis $e_1, \dots, e_k$ of H and an orthonormal basis $e_{k+1}, \dots, e_n$ of Z, so that $e_1, \dots, e_n$ is an orthonormal basis of X. In these bases we can view A as an $n \times k$ matrix. Let $(t_{ij})_{i,j=1}^n$ be the matrix of T relative to the basis $e_1, \dots, e_n$. Let T_H and T_Z be the matrices consisting of the first k and of the last $(n-k)$ columns of $(t_{ij})_{i,j=1}$ respectively, so that $T = (T_H\, T_Z)$. For every $F \in M_{n,n-k}$ we have $(A\, F) \in M_n$ and $(A\, F)x_i = Ax_i = y_i$ for $i = 1, \dots, k$, so that $(A\, F) \notin \mathcal{L}$. It follows that $0 \neq \langle (A\, F), T \rangle = \langle A, T_H \rangle + \langle F, T_Z \rangle$. Since F was chosen arbitrarily, it follows that $T_Z = 0$, so that $Z \subseteq \ker T$. Since $\dim \ker T = n - k = \dim Z$, we have $Z = \ker T$. Therefore, $\operatorname{span}\{x_1, \dots, x_k\} = \big(\ker T\big)^\perp$. Since $(x_1, \dots, x_k)$ and $(y_i, \dots, x_i)$ could be interchanged in the construction, it follows that $\operatorname{span}\{y_1, \dots, y_k\} = \big(\ker T\big)^\perp = \operatorname{span}\{x_1, \dots, x_k\} = H$. It follows that $\operatorname{Range} A \subseteq H$, so that $A = \left(\begin{smallmatrix} B \\ 0 \end{smallmatrix}\right)$ for some $B \in M_k$. Let $C = \left(\begin{smallmatrix} B & 0 \\ 0 & 0 \end{smallmatrix}\right)$, then $Cx_i = y_i$ as $i = 1, \dots, k$, so that $C \notin \mathcal{L}$.

We know that $T = (T_H\, 0) = \left(\begin{smallmatrix} R & 0 \\ S & 0 \end{smallmatrix}\right)$ for some $R \in M_{k,k}$ and $S \in M_{n-k,k}$. Since $T^2 = 0$, it follows that $\operatorname{Range} T \subseteq \ker T = Z$. In particular, $T(H) \subseteq Z$, so that $R = 0$. Thus,

$$\langle C, T \rangle = \left\langle \begin{pmatrix} B & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ S & 0 \end{pmatrix} \right\rangle = 0,$$

contradiction. ☐

Corollary 16. *For every $k \leqslant \frac{n}{2}$ there exists a k-semitransitive subspace of M_n which fails to be k-transitive.*

Proof. Let $T \in M_n$ be as follows: let $t_{2i-1,2i} = 1$ as $i = 1, \ldots, k$, and let all other entries of the matrix of T be zeros. Then $\operatorname{rank} T = k$ and $T^2 = 0$. Now the conclusion follows from Proposition 15. □

Next, we show that k-transitivity does not imply $(k+1)$-semitransitivity.

Proposition 17. *Suppose that $\mathcal{L}$ is a subspace of M_n and $1 < k \leqslant n$ such that $\mathcal{L}$ is $(k-1)$-transitive but not k-transitive. Then there exist unitaries $U, V \in M_n$ such that $U\mathcal{L}$ and $\mathcal{L}V$ are $(k-1)$-transitive but not k-semitransitive.*

Proof. If $\mathcal{L}$ is not k-semitransitive then we are done. Suppose that $\mathcal{L}$ is k-semitransitive. Then by Proposition 12 there exists $T \in \mathcal{L}^\perp$ with $\operatorname{rank} T = k$ and $T^2 = 0$. Choose a unitary $U \in M_n$ so that $(UT)^2 \neq 0$. Observe that $\operatorname{rank} UT = k$ and $UT \in (U\mathcal{L})^\perp$. It follows from Lemma 5 that $U\mathcal{L}$ is not k-transitive. Since $(UT)^2 \neq 0$, Lemma 10 yields that $U\mathcal{L}$ is not k-semitransitive. The existence of V is proved in a similar fashion. □

Corollary 18. *If $1 < k \leqslant n$ then there exists a subspace of M_n that is $(k-1)$-transitive but not k-semitransitive.*

Proof. Let $T \in M_n$ with $\operatorname{rank} T = k$, and let $\mathcal{L} = \{T\}^\perp$. Lemma 5 yields that $\mathcal{L}$ is $(k-1)$-transitive but not k-transitive. Proposition 17 completes the proof. □

We conclude this section with a few examples.

Example. Recall that a matrix $A = (a_{i,j})$ in M_n is ***Toeplitz*** if $a_{i,j} = a_{i+1,j+1}$ for all $i, j < n$. Let $\mathcal{L}$ be the subspace of all Toeplitz matrices in M_n. It is known and easy to prove (see, e.g., [Az]) that $\mathcal{L}$ is a transitive subspace. We claim that it is not 2-semitransitive. Consider the following two pairs: (e_1, e_2) and (e_1+e_2, e_1-e_2). Suppose first that there is $A \in \mathcal{L}$ such that $Ae_1 = e_1 + e_2$, and $Ae_2 = e_1 - e_2$. But since A is Toeplitz, then $Ae_1 = e_1 + e_2$ implies $Ae_2 = e_2 + e_3$, contradiction. On the other hand, suppose that there is $A \in \mathcal{L}$ such that $A(e_1 + e_2) = e_1$, and $A(e_1 - e_2) = e_2$. Then

$$Ae_1 = A\Big(\frac{e_1+e_2}{2} + \frac{e_1-e_2}{2}\Big) = \tfrac{1}{2}(e_1+e_2). \tag{1}$$

Again, since A is Toeplitz, it follows that $Ae_2 = \frac{1}{2}(e_2 + e_3)$. However, as in (1), we have $Ae_2 = \frac{1}{2}(e_1 - e_2)$, contradiction. Therefore, $\mathcal{L}$ is not 2-semitransitive.

Example. Let $\mathcal{L} = \{A \in M_3 \mid \operatorname{tr}(A) = 0\}$. It is easy to see that $\mathcal{L}$ is 2-transitive but not 3-transitive. Observe that $\mathcal{L} = \{I\}^\perp$. Lemma 10 implies that $\mathcal{L}$ is not 3-semitransitive.

Example. Fix $t \neq 0$ and let $\mathcal{L}$ be the set of all the matrices in M_2 of the form $\left(\begin{smallmatrix} \alpha & \beta \\ 0 & t\alpha \end{smallmatrix}\right)$. Then $\mathcal{L}$ is a two-dimensional semitransitive subspace of M_2.

6. More on the infinite-dimensional case

In this section we show that some of the results of Section 4 remain valid in the infinite-dimensional setting. Namely, we present infinite-dimensional analogues of Lemmas 5 and 7, and of Theorem 11. Note that these results still hold if X is just a vector space, and bounded maps are replaced with linear maps.

The following generalization of Lemma 5 can be easily deduced from the definition of strict k-transitivity.

Lemma 19. *Suppose that $\mathcal{L}$ is a linear subspace of $L(X)$. Then $\mathcal{L}$ is strictly k-transitive if and only if $\mathcal{L}P = L(X)P$ for every projection $P \in L(X)$ with $\operatorname{rank} P \leqslant k$.*

Lemma 20. *Suppose that $\mathcal{L}$ is a strictly k-semitransitive subspace of $L(X)$, and $P \in L(X)$ is a projection with $\operatorname{rank} P \leqslant k$. Then $PL(X)P \subseteq \mathcal{L}P$*

Proof. Let $Y = \operatorname{Range} P$. Let $e_1, \dots, e_m$ be a basis of Y. Note that $m \leqslant k$. Relative to this basis, any $m \times m$ matrix A can be viewed as a bounded operator from Y to Y or from Y to X; then $AP = PAP \in L(X)$. Also, $PL(X)P$ can be identified with M_m. Let V be an $m \times m$ involution. Put $y_i = Ve_i$ for $i = 1, \dots, m$; they are linearly independent since V is invertible. Note that $\mathcal{L}$ is strictly m-semitransitive, hence there exists $A \in \mathcal{L}$ which either takes all e_i's into y_i's, or vice versa. Suppose that for each $i = 1, \dots, m$ we have $Ae_i = y_i$. Then $APe_i = y_i$. It follows that $AP = V$, so that $V \in \mathcal{L}P$. On the other hand, suppose that for each $i = 1, \dots, m$ we have $Ay_i = e_i$. Then $APy_i = e_i$, so that $AP = V$, so again $V \in \mathcal{L}P$. Lemma 6 now yields that $M_m \subseteq \mathcal{L}P$. ☐

Theorem 21. *If $\mathcal{L}$ is a strictly $(k+1)$-semitransitive subspace of $L(X)$ for some finite k, then $\mathcal{L}$ is strictly k-transitive.*

Proof. Suppose that $\mathcal{L}$ is not strictly k-transitive. Lemma 19 yields that there is a projection $P \in L(X)$ with $m := \operatorname{rank} P \leqslant k$ such that $\mathcal{L}P$ is contained in $L(X)P$. On the other hand, since $\mathcal{L}$ is strictly k-semitransitive, Lemma 20 yields $PL(X)P \subseteq \mathcal{L}P$. It follows that there exists $D \in L(X)$ such that $DP \notin \mathcal{L}P$ while $PDP \in \mathcal{L}P$, hence $(I-P)DP \notin \mathcal{L}P$.

Let $Y = \operatorname{Range} P$. Let $e_1, \dots, e_m$ be a basis of Y. Let $z_i = (I-P)DPe_i$. Then $z_i \in \operatorname{Range}(I-P)$.

Using strict k-semitransitivity of $\mathcal{L}$ on the k-tuples $(e_1, \dots, e_m)$ and $(e_1, \dots, e_m)$ we conclude that there exists $B \in \mathcal{L}$ such that $Be_i = e_i$ for $i = 1, \dots, m$.

Applying strict $(k+1)$-semitransitivity of $\mathcal{L}$ to the $(k+1)$-tuples

$$(z_1, e_1, e_2, e_3, \ldots, e_m) \text{ and } (e_1, z_1, e_2, e_3, \ldots, e_m),$$

we conclude that there exists $C_1 \in \mathcal{L}$ such that $C_1 e_i = e_i$ for $i = 2, \ldots, m$ and $C_1 e_1 = z_1$. Similarly, for each $j = 1, \ldots, m$ we find $C_j \in \mathcal{L}$ such that $C_j e_i = e_i$ if $i \neq j$ and $C_j e_j = z_j$. Let $A = C_1 + \cdots + C_m - (m-1)B$. Observe that $A \in \mathcal{L}$, hence $AP \in \mathcal{L}P$. On the other hand, $Ae_i = z_i$ for all $i = 1, \ldots, m$, so that $AP = (I-P)DP$, contradiction. □

7. More on 2-semitransitivity

In this section the vector spaces are finite or infinite dimensional. The following two results concern rings of linear transformations on a vector space over an arbitrary underlying field.

Proposition 22. *Let $\mathcal{R}$ be a ring of linear transformations on a vector space. Then $\mathcal{R}$ is strictly 2-semitransitive if and only if it is strictly 2-transitive.*

Proof. Obviously, if $\mathcal{R}$ is strictly 2-transitive then it is strictly 2-semitransitive. Suppose that $\mathcal{R}$ is strictly 2-semitransitive. Take two linearly independent vectors x and y, and two vectors u and v. We show that there is $R \in \mathcal{R}$ such that $Rx = u$ and $Ry = v$.

If $u = v = 0$ then $R = 0$ will do the job. Thus, we can assume that either $u \neq 0$ or $v \neq 0$. Note that given any two linearly independent vectors a and b, applying the definition of strict 2-semitransitivity to the pairs (a, b) and (b, a) one can find an operator $D_{(a,b)} \in \mathcal{R}$ such that $D_{(a,b)}a = b$ and $D_{(a,b)}b = a$.

Suppose first that the underlying field has characteristic different from 2. Applying the definition of strict 2-semitransitivity to the following pairs of pairs: (x, y) and (x, y), and to (x, y) and $(x, -y)$, we obtain operators J and A in $\mathcal{R}$ such that $Jx = x$, $Jy = y$, $Ax = x$, and $Ay = -y$. Put $B = J + A$ and $C = J - A$, then

$$Bx = 2x, \quad By = 0, \quad Cx = 0, \quad \text{and} \quad Cy = 2y.$$

Suppose that $u \neq 0$. We find $S \in \mathcal{R}$ such that $Sx = u$ and $Sy = 0$ as follows. If x and u are linearly independent, we take $S = D_{(2x,u)}B$. Otherwise, y and u have to be linearly independent, in which case we take $S = D_{(2y,u)}CD_{(x,y)}$. Similarly, if $v \neq 0$ then there exists $T \in \mathcal{R}$ such that $Tx = 0$ and $Ty = v$. Finally, if both u and v are non-zero, then we find S and T as before and put $R = S + T$. Clearly, $Rx = u$ and $Ry = v$.

Now suppose that the underlying field is of characteristic 2. As before, we can find $J \in \mathcal{R}$ such that $Jx = x$ and $Jy = y$. Observe that

$$D_{(x,x+y)}y = D_{(x,x+y)}\big((x+y)+x\big) = x + (x+y) = y.$$

Let $B = D_{(x,y)}\big(J + D_{(x,x+y)}\big)$, then $Bx = x$ and $By = 0$. Clearly, $B \in \mathcal{R}$. Similarly, one can find $C \in \mathcal{R}$ such that $Cx = 0$ and $Cy = y$. The rest of the proof is similar to the first case. □

It follows, in particular, under the hypotheses of Proposition 22, that if $\mathcal{R}$ is strictly 2-semitransitive then it is strictly transitive. Jacobson's Theorem [Jac] asserts that if $\mathcal{R}$ is strictly 2-transitive, then it is ***strictly dense***, i.e., strictly n-transitive for every n. Together with Proposition 22 it yields the following extension.

Corollary 23. *Let $\mathcal{R}$ be a unital ring of linear transformations on a vector space. If $\mathcal{R}$ is strictly 2-semitransitive, then it is strictly dense.*

Let X be a Banach space, $\mathcal{S}$ a subset of $L(X)$, and T a closed operator defined on a linear subspace of X. We say that T commutes with $\mathcal{S}$ if $\operatorname{dom} T$ is invariant under every operator $A \in \mathcal{S}$ and $ATx = TAx$ for every $x \in \operatorname{dom} T$.

Proposition 24. *Suppose that X is a Banach space, $\mathcal{S}$ is a topologically 2-semitransitive subset of $L(X)$, and T is a closed operator defined on a linear subspace of X. If $\mathcal{S}$ commutes with T then T is a multiple of the identity operator.*

Proof. Suppose not. Then there exists $x \in \operatorname{dom} T$ such that x and Tx are linearly independent. Apply the definition of topological 2-transitivity of $\mathcal{S}$ to the pairs (x, Tx) and $(x, 2Tx)$. Suppose first that there is a sequence of operators (A_n) in $\mathcal{S}$ such that $\|A_n x - x\| \to 0$ and $\|A_n(Tx) - 2Tx\| \to 0$. Since T is closed, this implies $Tx = 2Tx$, contradiction. On the other hand, suppose that there is (A_n) in $\mathcal{S}$ such that $\|A_n x - x\| \to 0$ and $\|A_n(2Tx) - Tx\| \to 0$, so that $Tx = \frac{1}{2}Tx$, contradiction. □

Corollary 25. *If X is Banach space, then no commutative subset of $L(X)$ is topologically 2-semitransitive.*

Suppose that T is an operator on a Banach space X such that T has no invariant subspaces. Let $\mathcal{A}$ be the subalgebra of $L(X)$ generated by T. Then, clearly, $\mathcal{A}$ is topologically transitive. On the other hand, Corollary 25 implies that $\mathcal{A}$ is not topologically 2-semitransitive.

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