ON SEMITRANSITIVE COLLECTIONS OF OPERATORS

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Abstract. A collection $\mathcal{F}$ of operators on a vector space V is said to be semitransitive if for every pair of nonzero vectors x and y in V there exists a member T of $\mathcal{F}$ such that either $Tx = y$ or $Ty = x$ (or both). We study semitransitive algebras and semigroups of operators. One of the main results is that if the underlying field is algebraically closed, then every semitransitive algebra of operators on a space of dimension n contains a nilpotent element of index n. Among other results on semitransitive semigroups, we show that if the rank of nonzero members of such a semigroup acting on an n-dimensional space is a constant k, then k divides n.

1. Introduction

As is well known, a collection $\mathcal{F}$ of linear transformations on a vector space V is said to be ***transitive*** if for every pair of nonzero vectors $x, y \in V$ there exists a transformation $T \in \mathcal{F}$ such that $Tx = y$.

We shall consider a weaker version of this property, which was recently introduced in [6] and proved not only interesting in its own right, but also fruitful in connection with the properties of lattices of invariant subspaces of algebras of operators. Although our primary interest in this paper lies in algebras and semigroups, we state the definition for an arbitrary collection. We say that $\mathcal{F}$ is ***semitransitive*** if for every pair of nonzero vectors $x, y \in V$ there exists $T \in \mathcal{F}$ such that either $Tx = y$ or $Ty = x$. Note, that for groups the notions of transitivity and semitransitivity coincide.

It should be noted that if V is a topological vector space, then these two properties are sometimes referred to as ***strict transitivity*** and ***strict semitransitivity*** respectively, while ***transitivity*** and ***semitransitivity*** are understood in the topological sense, that is, for every pair of nonzero vectors $x, y \in V$ and for every neighborhood U of 0 there exists a transformation $T \in \mathcal{F}$ such that $Tx \in y + U$ (or $Ty \in x + U$, for semitransitivity). We will refer to the latter properties as ***topological transitivity*** and ***topological semitransitivity***.

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Semitransitive algebras of bounded operators on a Banach space were introduced and investigated in [6]. There it is observed that a unital algebra of bounded operators on a Banach space is semitransitive if and only if its invariant linear subspaces are totally ordered by inclusion, and it is topologically semitransitive if and only if all its closed invariant subspaces are totally ordered by inclusion. This implies that in the finite-dimensional case the terms *topological semitransitivity* and *semitransitivity* are equivalent for unital algebras of real or complex matrices.

If $\mathcal{S}$ is a uniformly bounded semitransitive semigroup of operators on a Banach space $\mathcal{B}$, then $\mathcal{S}$ has no invariant subspaces. Indeed, suppose that $\|T\| < M$ for all $T \in \mathcal{S}$. Consider a subspace Z of $\mathcal{B}$ and choose a vector $x \in Z$ with $\|x\| > M$ and a vector $y \notin Z$ with $\|y\| = 1$. Since no operator in $\mathcal{S}$ can map y to x, there must exist an operator $T \in \mathcal{S}$ that maps x to y and hence Z is not invariant under T.

In this paper we study semitransitive algebras and semigroups of matrices, or equivalently, operators on a finite-dimensional space. It turns out that even the case of dimension one is not totally trivial.

2. Semitransitivity and preorders for abelian groups

We start with some algebraic considerations. For this section, let $\mathcal{G}$ be an abelian group, and $\mathcal{S}$ a subsemigroup of $\mathcal{G}$. We say that $\mathcal{G}$ is ***preordered*** (***ordered***) if it is equipped with a reflexive, transitive (and antisymmetric for *ordered*) binary relation "$\preceq$" such that $x \preceq y$ implies $xz \preceq yz$ for all $x, y, z \in \mathcal{G}$. We write $\mathcal{G}_+$ for the positive cone $\{x \in \mathcal{G} \mid x \succeq e\}$. Note that $\mathcal{G}_+$ is a subsemigroup of $\mathcal{G}$. Clearly, a preorder is total if and only if $\mathcal{G}_+ \cup \mathcal{G}_+^{-1} = \mathcal{G}$. A preorder is an order if and only if $\mathcal{G}_+ \cap \mathcal{G}_+^{-1} = \{e\}$.

We say that $\mathcal{S}$ is ***semitransitive***, or ***acts semitransitively*** on $\mathcal{G}$, if for any $x, y \in \mathcal{G}$ there exists $s \in \mathcal{S}$ such that $sx = y$ or $sy = x$. One can easily verify the following characterization.

Proposition 1. *The following are equivalent.*

(i) $\mathcal{S}$ *is semitransitive;*
(ii) $\mathcal{S}^{-1}$ *is semitransitive;*
(iii) $\mathcal{S} \cup \mathcal{S}^{-1} = \mathcal{G}$*;*
(iv) $\mathcal{S} = \mathcal{G}_+$ *for some total preorder on* $\mathcal{G}$*.*

Proposition 2. *Every semitransitive semigroup* $\mathcal{S}$ *in* $\mathcal{G}$ *contains the torsion group* $\mathrm{t}(\mathcal{G})$ *of* $\mathcal{G}$*.*

Proof. Let $a \in \mathrm{t}(\mathcal{G})$, then $a^m = e$ for some m. By Proposition 1(iii) either $a \in \mathcal{S}$ or $a \in \mathcal{S}^{-1}$. In the latter case, $a^{-1} \in \mathcal{S}$, so that $a = (a^{-1})^{m-1} \in \mathcal{S}$. □

We say that $\mathcal{S}$ is ***minimal semitransitive*** if it is semitransitive, and no proper subsemigroup of $\mathcal{S}$ acts semitransitively on $\mathcal{G}$.

Let $\widehat{\mathcal{G}} = \mathcal{G}/\,\mathrm{t}(\mathcal{G})$, then $\widehat{\mathcal{G}}$ is a torsion-free group. Let $\pi\colon \mathcal{G} \to \widehat{\mathcal{G}}$ be the canonical epimorphism.

Theorem 3. *The following are equivalent:*

(i) $\mathcal{S}$ *is minimal semitransitive;*

(ii) $\mathcal{S} \cup \mathcal{S}^{-1} = \mathcal{G}$ *and* $\mathcal{S} \cap \mathcal{S}^{-1} = \mathrm{t}(\mathcal{G})$*;*

(iii) $\mathcal{S} = \pi^{-1}(\widehat{\mathcal{G}}_+)$ *for some total order on* $\widehat{\mathcal{G}}$*.*

Furthermore, every semitransitive subsemigroup of $\mathcal{G}$ *contains a minimal semitransitive subsemigroup.*

Proof. The implication (ii)⇒(i) follows from Propositions 1 and 2. To show that (iii)⇒(ii), let $\mathcal{S} = \pi^{-1}(\widehat{\mathcal{G}}_+)$ for some total order on $\widehat{\mathcal{G}}$, then

$$\begin{aligned} \mathcal{S} \cup \mathcal{S}^{-1} &= \pi^{-1}\big(\widehat{\mathcal{G}}_+ \cup \widehat{\mathcal{G}}_+^{-1}\big) = \pi^{-1}(\widehat{\mathcal{G}}) = \mathcal{G}, \text{ and} \\ \mathcal{S} \cap \mathcal{S}^{-1} &= \pi^{-1}\big(\widehat{\mathcal{G}}_+ \cap \widehat{\mathcal{G}}_+^{-1}\big) = \pi^{-1}(\{e\}) = \mathrm{t}(\mathcal{G}). \end{aligned} \tag{1}$$

Suppose $\mathcal{S}$ is a semitransitive subsemigroup of $\mathcal{G}$. Consider the group $G = \mathcal{S} \cap \mathcal{S}^{-1}$. Propositions 1 and 2 yield $\mathrm{t}(\mathcal{G}) \subseteq G$. Let $B = G/\,\mathrm{t}(\mathcal{G})$, then B is a torsion-free subgroup of $\widehat{\mathcal{G}}$. It is well known (see [2] or [3, p.5]) that every torsion-free group admits a total order. Let $C = \pi^{-1}(B_+)$ for some total order on B. Note that $C \subseteq G$ is a semigroup. Also, as in (1), $C \cup C^{-1} = G$, and $C \cap C^{-1} = \mathrm{t}(\mathcal{G})$. Put $\mathcal{S}_0 = C \cup (\mathcal{S} \setminus G)$. Then $\mathcal{S}_0 \subseteq \mathcal{S}$. Show that $\mathcal{S}_0$ is a semigroup: if $x, y \in \mathcal{S}_0$ then $xy \in \mathcal{S}_0$. Suppose not, since $xy \in \mathcal{S}$ then $xy \in G \subseteq \mathcal{S}^{-1}$ so that $x \in y^{-1}\mathcal{S}^{-1} \subseteq \mathcal{S}^{-1}$. Hence, $x, y \in \mathcal{S}^{-1}$, so that $x, y \in G$ and, therefore, $x, y \in C$. It follows that $xy \in C \subseteq \mathcal{S}_0$. Thus, $\mathcal{S}_0$ is a semigroup. Furthermore,

$$\begin{aligned} \mathcal{S}_0 \cup \mathcal{S}_0^{-1} &= C \cup (\mathcal{S} \setminus G) \cup C^{-1} \cup (\mathcal{S} \setminus G)^{-1} \\ &= (C \cup C^{-1}) \cup \big((\mathcal{S} \cup \mathcal{S}^{-1}) \setminus G\big) = G \cup (\mathcal{G} \setminus G) = \mathcal{G}, \text{ and} \end{aligned}$$

$$\begin{aligned} \mathcal{S}_0 \cap \mathcal{S}_0^{-1} &= \big(C \cup (\mathcal{S} \setminus G)\big) \cap \big(C^{-1} \cup (\mathcal{S} \setminus G)^{-1}\big) \\ &= (C \cap C^{-1}) \cup \big(C \cap (\mathcal{S} \setminus G)^{-1}\big) \cup \big((\mathcal{S} \setminus G) \cap C^{-1}\big) \cup \big((\mathcal{S} \setminus G) \cap (\mathcal{S} \setminus G)^{-1}\big) \\ &= \mathrm{t}(\mathcal{G}) \cup \varnothing \cup \varnothing \cup \varnothing = \mathrm{t}(\mathcal{G}), \end{aligned}$$

so that $\mathcal{S}_0$ is minimal semitransitive. Hence, every semitransitive semigroup in $\mathcal{G}$ contains a minimal one.

Finally, show that (i)⇒(iii). Suppose that $\mathcal{S}$ is minimal semitransitive and construct $\mathcal{S}_0$ as in the previous paragraph. The minimality of $\mathcal{S}$ implies that $\mathcal{S}_0 = \mathcal{S}$, hence $G = C$, and, therefore,

$$G = G \cap G^{-1} = C \cap C^{-1} = \mathrm{t}(\mathcal{G}).$$

Let $K = \pi(\mathcal{S})$, then it is easy to verify that $\mathcal{S} = \pi^{-1}(K)$. Observe that K is a semigroup, $K \cup K^{-1} = \widehat{\mathcal{G}}$, and $K \cap K^{-1} = \{e\}$. Then K is the positive cone of a total order on $\widehat{\mathcal{G}}$ as claimed. □

Corollary 4. *If $\mathcal{S}$ is minimal semitransitive then $T = \mathcal{S} \setminus \mathrm{t}(\mathcal{G})$ is a semigroup ideal in $\mathcal{S}$.*

Proof. Suppose that $x \in \mathcal{S} \setminus \mathrm{t}(\mathcal{G})$, $y \in \mathcal{S}$. Theorem 3(ii) implies that $x \notin \mathcal{S}^{-1}$. But $xy \in \mathrm{t}(\mathcal{G})$ would imply $x = (xy)y^{-1} \in \mathcal{S}^{-1}$, a contradiction. □

3. Examples: one-dimensional case

Consider a semigroup of operators acting on a vector space V. In case $\dim V = 1$, the field $\mathbb{F}$ acts on itself by multiplication. Then $\mathbb{F}^\bullet = \mathbb{F} \setminus \{0\}$ is an abelian group, and it follows from Theorem 3 that every semitransitive semigroup contains a minimal such semigroup and that minimal semitransitive semigroups are the preimages of positive cones of total preorders on $\widehat{\mathbb{F}^\bullet}$.

We also study those semitransitive semigroups of $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}\}$ that are either bounded or compact.

3.1. **Real line.** The semigroups $[-1,1]$ and $[-1,1]\backslash\{0\}$ are the only bounded semitransitive semigroups in $\mathbb{R}$.

3.2. **Complex plane.** It is easy to see that the closed unit disk $D = \{|z| \leqslant 1\}$ is the only compact semitransitive semigroup in $\mathbb{C}$ and that any bounded semitransitive subsemigroup of $\mathbb{C}$ is contained in D.

Proposition 5. *A bounded semigroup $\mathcal{S}$ in $\mathbb{C}^\bullet$ is semitransitive if and only if it is the union of the punctured open disk $\{0 < |z| < 1\}$ and a semigroup $\mathcal{S}' \subseteq \partial D$, which acts semitransitively on the circle $\partial D = \{|z| = 1\}$.*

Proof. It is clear that any z outside D can be mapped to any nonzero element in D by multiplication with a nonzero number in the interior of D. It is also easy to see, that the entire punctured interior of D is needed for that purpose. Hence $\mathcal{S} \subseteq D$ is semitransitive if and only if it contains $\{0 < |z| < 1\}$ and $\mathcal{S} \cap \partial D$ acts semitransitively on ∂D. $\square$

Remark 6. It follows from Theorem 3 that a semigroup $\mathcal{S}' \subseteq \partial D$ acts semitransitively on ∂D if an only if it contains the preimage of the positive cone of some total order on $\widehat{\partial D}$.

Theorem 3 and Proposition 5 also yield the following.

Corollary 7. *There exists a bounded minimal semitransitive semigroup $\mathcal{S}$ in $\mathbb{C}$.*

4. Examples: n-dimensional case

In this section we are going to consider some examples of semitransitive algebras and semigroups of matrices. Let $M_n = M_n(\mathbb{F})$. For the following examples let $\mathcal{S}_{\mathbb{F}} = \mathcal{S}'_{\mathbb{F}} \cup \{0\}$, where $\mathcal{S}'_{\mathbb{F}}$ is a fixed minimal semitransitive semigroup in $\mathbb{F}^{\bullet}$.

4.1. **Upper-triangular Toeplitz algebra and semigroup.** Let $\mathcal{A}$ be the algebra of all upper-triangular Toeplitz operators. More precisely, $\mathcal{A}$ is the algebra generated by the shift S, that is, the algebra of all the operators of the form $a_1 I + a_2 S + a_3 S^2 + \dots + a_n S^{n-1}$ or, in matrix form,

$$\begin{pmatrix} a_1 & a_2 & a_3 & \dots & a_n \\ 0 & a_1 & a_2 & \dots & a_{n-1} \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & a_1 \end{pmatrix}$$

where $a_1, \dots, a_n$ are arbitrary scalars. Let $\mathcal{S}$ be the set of all matrices in $\mathcal{A}$ with $a_1 \in \mathcal{S}_{\mathbb{F}}$. We claim that $\mathcal{A}$ is a semitransitive algebra and $\mathcal{S}$ is a semitransitive semigroup.

The following shows that $\mathcal{S}$ is semitransitive. We claim, that given $\mathbf{x}, \mathbf{y} \in \mathbb{F}^n$, there is a $T \in \mathcal{S}$ such that either $T\mathbf{x} = \mathbf{y}$ or $T\mathbf{y} = \mathbf{x}$. Let x_k and y_m be the last non-zero components of $\mathbf{x}$ and $\mathbf{y}$ respectively. Without loss of generality we may assume that $k \geqslant m$ and that $\frac{y_k}{x_k} \in \mathcal{S}_{\mathbb{F}}$. Finding $T \in \mathcal{A}$ satisfying $T\mathbf{x} = \mathbf{y}$ is equivalent to solving the system

$$\left\{ \begin{array}{ccccccccccc} a_1 x_1 & + & a_2 x_2 & + & \dots & + & a_{k-1} x_{k-1} & + & a_k x_k & = & y_1 \\ a_1 x_2 & + & a_2 x_3 & + & \dots & + & a_{k-1} x_k & & & = & y_2 \\ & & & & & & & & & \vdots & \\ a_1 x_k & & & & & & & & & = & y_k \end{array} \right. \tag{2}$$

for $a_1, a_2, \dots, a_k$.

The system (2) is consistent, so that there exists $T \in \mathcal{A}$ such that $T\mathbf{x} = \mathbf{y}$. Since $\frac{y_k}{x_k} \in \mathcal{S}_{\mathbb{F}}$, it follows from the last equation of (2) that $a_1 \in \mathcal{S}_{\mathbb{F}}$, hence $T \in \mathcal{S}$.

Any semitransitive subset of $\mathcal{A}$ must contain the identity matrix (I is the only element of $\mathcal{A}$ that maps $\mathbf{e}_n$ to itself) and also any matrix of the form $a_2S + \ldots + a_nS^{n-1}$ (we denote the semigroup of all the matrices of this form by $\mathcal{R}$), since such a matrix is the only matrix in $\mathcal{A}$ mapping $\mathbf{e}_n$ to $\mathbf{x} = a_n\mathbf{e}_1 + \ldots + a_2\mathbf{e}_{n-1}$ and no matrix in $\mathcal{A}$ can map $\mathbf{x}$ to $\mathbf{e}_n$. Therefore $\mathcal{A}$ is a minimal semitransitive algebra and, as we shall see later, is unique up to conjugation if the field $\mathbb{F}$ is algebraically closed.

We will show that $\mathcal{S}$ contains a minimal semitransitive semigroup. To make notation less cumbersome we identify the field $\mathbb{F}$ with the scalars in M_n. As above, let $\mathcal{R}$ denote the semigroup of all the matrices of the form $a_2S + \ldots + a_nS^{n-1}$. Observe that $\mathcal{R}$ coincides with the set of matrices in $\mathcal{S}$ of determinant zero. Thus, $\mathcal{S}$ is a disjoint union of $\mathcal{R}$ and the semigroup of the form $\mathcal{S}'_{\mathbb{F}} + \mathcal{R}$ and $\mathcal{R}$ is a semigroup ideal in $\mathcal{S}$. As we noted, any minimal subsemigroup of $\mathcal{S}$ will contain $\mathcal{R}$. So we study the semigroup $\mathcal{S}'_{\mathbb{F}} + \mathcal{R}$. It is easy to see that any semitransitive subsemigroup of $\mathcal{S}$ must also contain all matrices of the form $a_1I + \mathcal{R}$, where $a_1 \in \mathcal{S}'_{\mathbb{F}}$ is not a root of unity. Now, $\mathcal{S}'_{\mathbb{F}} + \mathcal{R}$ is a disjoint union of $\mathrm{t}(\mathbb{F}^\bullet) + \mathcal{R}$, which is a group, and $(\mathcal{S}'_{\mathbb{F}} \setminus \mathrm{t}(\mathbb{F}^\bullet)) + \mathcal{R}$, which is a semigroup ideal in $\mathcal{S}'_{\mathbb{F}} + \mathcal{R}$ by Corollary 4. So, finally, we consider the abelian group $\mathrm{t}(\mathbb{F}^\bullet) + \mathcal{R}$. Observe that it acts transitively and freely on the set of vectors with $x_n \in \mathrm{t}(\mathbb{F}^\bullet)$. So by Theorem 3 there exists a minimal semitransitive semigroup for this action, which we denote by $\mathcal{P}$. It is easy to see that $\mathcal{P} \cup ((\mathcal{S}'_{\mathbb{F}} \setminus \mathrm{t}(\mathbb{F}^\bullet)) + \mathcal{R}) \cup \mathcal{R}$ is minimal semitransitive. Now, the structure of this semigroup depends on the characteristic of the field $\mathbb{F}$. If $char(\mathbb{F}) = p > 0$, then every element of $\mathrm{t}(\mathbb{F}^\bullet) + \mathcal{R}$ is torsion, so $\mathcal{P} = \mathrm{t}(\mathbb{F}^\bullet) + \mathcal{R}$ in this case. If, in addition, $\mathbb{F}$ is algebraic over the prime field, then $\mathbb{F}^\bullet = \mathrm{t}(\mathbb{F}^\bullet)$ and $\mathcal{S}$ was minimal to begin with.

If $char(\mathbb{F}) = 0$, then $\mathrm{t}(\mathbb{F}^\bullet) + \mathcal{R}$, as an abelian group, is a product of its torsion part $\mathrm{t}(\mathbb{F}^\bullet)$ and the group $I + \mathcal{R}$, where the latter is torsion free. So $\mathcal{P}$, and thus any minimal subsemigroup in $\mathcal{S}$, corresponds to a certain total order on $I + \mathcal{R}$ by Theorem 3.

4.2. **Upper triangular one-column matrices.** Consider the set

$$\mathcal{S} = \{\mathbf{x}\mathbf{e}_i^T \mid 1 \leqslant i \leqslant n,\, \mathbf{x} \in \mathbb{F}^n,\, x_i \in \mathcal{S}_{\mathbb{F}} \text{ and } x_j = 0 \text{ for } j > i\}$$

in $M_n(\mathbb{F})$. It can be easily verified that $\mathcal{S}$ is a semigroup. We claim that it is minimal semitransitive.

First we show that $\mathcal{S}$ is semitransitive. Let $\mathbf{x}$ and $\mathbf{y}$ be arbitrary nonzero vectors in $\mathbb{F}^n$. Define i and j to be the smallest integers such that $x_k = 0 = y_l$ for all $k > i$

and $l > j$. Without any loss of generality we can assume that $i \geqslant j$ and that $\frac{y_i}{x_i} \in \mathcal{S}_{\mathbb{F}}$. Now note that $\frac{1}{x_i}\mathbf{y}\mathbf{e}_i^T \in \mathcal{S}$ maps $\mathbf{x}$ to $\mathbf{y}$.

Now suppose that $\mathcal{S}' \subseteq \mathcal{S}$ is a semitransitive semigroup. Choose $A = \mathbf{x}\mathbf{e}_i^T \in \mathcal{S}$. We will show that $A \in \mathcal{S}'$, thus proving minimality of $\mathcal{S}$. We consider two cases.

Case: x_i is not a root of unity. Then A is the only element of $\mathcal{S}$ that maps $\mathbf{e}_i$ to $\mathbf{x}$ and there is no element in $\mathcal{S}$ that would map $\mathbf{x}$ to $\mathbf{e}_i$. Hence $A \in \mathcal{S}'$.

Case: $x_i^k = 1$ for some positive integer k. Then A is the only element of $\mathcal{S}$ that maps $\mathbf{x}$ to $x_i\mathbf{x}$ and $\frac{1}{x_i^2}A$ is the only element of $\mathcal{S}$ that maps $x_i\mathbf{x}$ to $\mathbf{x}$. Hence $A \in \mathcal{S}'$, since $(\frac{1}{x_i^2}A)^{k-1} = A$.

4.3. **One-column contractions on ℓ_∞^n.** Consider the set

$$\mathcal{S} = \left\{\mathbf{x}\mathbf{e}_i^T \mid 1 \leqslant i \leqslant n,\ \|\mathbf{x}\|_\infty \leqslant 1,\ |x_j| < 1 \text{ for all } j > i, \text{ and } x_i \in \mathcal{S}_{\mathbb{F}}\right\}.$$

Here we assume that $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}\}$ and that $\mathcal{S}_{\mathbb{F}}$ is bounded. Clearly, $\mathcal{S}$ is a subsemigroup of contractions on ℓ_∞^n. We claim that $\mathcal{S}$ is a minimal semitransitive semigroup. First, show that $\mathcal{S}$ is semitransitive. Fix $\mathbf{x}, \mathbf{y} \in \mathbb{F}^n$.

Case $\|\mathbf{x}\|_\infty > \|\mathbf{y}\|_\infty$. If k is such that $\|\mathbf{x}\|_\infty = |x_k|$ then the operator $T = \frac{1}{x_k}\mathbf{y}\mathbf{e}_k^T \in \mathcal{S}$, since $\|\frac{1}{x_k}\mathbf{y}\|_\infty < 1$, and maps $\mathbf{x}$ to $\mathbf{y}$.

Case $\|\mathbf{x}\|_\infty = \|\mathbf{y}\|_\infty$. Let k and l be largest integers, such that $\|\mathbf{x}\|_\infty = |x_k|$ and $\|\mathbf{y}\|_\infty = |y_l|$. Without any loss of generality assume that $k \geqslant l$ and that $\frac{y_k}{x_k} \in \mathcal{S}_{\mathbb{F}}$ and note that the matrix $T = \frac{1}{x_k}\mathbf{y}\mathbf{e}_k^T \in \mathcal{S}$ maps $\mathbf{x}$ to $\mathbf{y}$.

Now assume that $\mathcal{S}'$ is a semitransitive subsemigroup of $\mathcal{S}$. We show that any operator $T = \mathbf{x}\mathbf{e}_j^T \in \mathcal{S}$, must also belong to $\mathcal{S}'$. Note that T is the only matrix in $\mathcal{S}$ mapping $\mathbf{e}_j$ to $\mathbf{x}$.

If x_j is not a root of unity, then we show that no matrix $(\mathbf{y}\mathbf{e}_i^T) \in \mathcal{S}$ maps $\mathbf{x}$ to $\mathbf{e}_j$ and therefore $T \in \mathcal{S}'$. Indeed, if $(\mathbf{y}\mathbf{e}_i^T)\mathbf{x} = \mathbf{e}_j$ for some $\mathbf{y}\mathbf{e}_i^T \in \mathcal{S}$, then $\mathbf{y} = \frac{1}{x_i}\mathbf{e}_j$, thus $|x_i| = |y_j| = 1$ (since $|x_i| \leqslant 1$ and $|y_j| = \frac{1}{|x_i|} \leqslant 1$) and hence also $i = j$. But then $y_j \notin \mathcal{S}_{\mathbb{F}}$ (the only elements of $\mathcal{S}_{\mathbb{F}}$ whose inverses are in $\mathcal{S}_{\mathbb{F}}$ as well, are roots of unity), contradicting $\mathbf{y}\mathbf{e}_j^T \in \mathcal{S}$.

If x_j is a root of unity, say $x_j^k = 1$, then the only element of $\mathcal{S}$ mapping $\mathbf{x}$ to $x_j\mathbf{x}$ is T. Indeed, if $(\mathbf{y}\mathbf{e}_i^T)\mathbf{x} = x_j\mathbf{x}$, then $x_i\mathbf{y} = x_j\mathbf{x}$, hence $|x_i| = 1 = |y_j|$, thus $i = j$ and therefore $\mathbf{y}\mathbf{e}_i^T = T$. Also, $\frac{1}{x_j^2}T$ is the unique matrix in $\mathcal{S}$ mapping $x_j\mathbf{x}$ to $\mathbf{x}$. Hence $T \in \mathcal{S}'$, since $(\frac{1}{x_j^2}T)^{k-1} = T$.

4.4. **Rank-one Euclidean contractions.** Let $\mathcal{S}$ be the set of all rank-one complex matrices with Euclidean norm at most one together with the zero matrix. This is clearly a closed semigroup. It is easy to see that it is semitransitive. Indeed, given

$\mathbf{x},\mathbf{y}\in\mathbb{C}^n$, without loss of generality $\|\mathbf{x}\|_2\geqslant\|\mathbf{y}\|_2$. Let $T=\frac{1}{\|\mathbf{x}\|_2^2}\mathbf{y}\mathbf{x}^*$. Then T is clearly a contraction, and $T\mathbf{x}=\mathbf{y}$. It will follow from Corollary 21 that $\mathcal{S}$ contains a minimal closed semitransitive subsemigroup.

We claim that $\mathcal{S}$ also contains a minimal semitransitive subsemigroup $\mathcal{T}$. Note that any semitransitive semigroup contained in $\mathcal{S}$ must necessarily contain all self-adjoint rank-one projections. Indeed, if $\|\mathbf{x}\|=1$ then $\mathbf{x}\mathbf{x}^*$ is the only rank-one matrix that takes $\mathbf{x}$ to $\mathbf{x}$. As a consequence, it must contain all rank-one proper contractions (operators of norm strictly less than one) since every rank-one proper contractions is a product of rank-one projections; see [4]. Thus, we only have to care about the norm-one rank-one elements. These are uniquely determined by pairs of norm-one vectors: if $\|\mathbf{x}\|=1=\|\mathbf{y}\|$, then $\mathbf{y}\mathbf{x}^*$ is the only element of $\mathcal{S}$ that maps $\mathbf{x}$ to $\mathbf{y}$.

For $\|\mathbf{x}\|=\|\mathbf{y}\|=1$ we write $\mathbf{x}\sim\mathbf{y}$ if $\lambda\mathbf{x}=\mathbf{y}$ for some $|\lambda|=1$, and $[\mathbf{x}]=\big\{\mathbf{z}\mid\|\mathbf{z}\|=1\text{ and }\mathbf{z}\sim\mathbf{x}\big\}$. Fix a total order on the set of all the equivalence classes of norm-one vectors. Let L be a set of norm one vectors such that L contains exactly one vector from each equivalence class. As before, let $\mathcal{S}_{\mathbb{C}}$ denotes a fixed bounded minimal semitransitive subsemigroup of the complex plane. Put $\mathcal{T}=\mathcal{T}_1\cup\mathcal{T}_2\cup\mathcal{T}_3$, where

$$\begin{aligned}\mathcal{T}_1&=\text{ all rank-one proper contractions,}\\ \mathcal{T}_2&=\big\{\,\lambda\mathbf{x}\mathbf{x}^*\mid\lambda\in\mathcal{S}_{\mathbb{C}},\ \mathbf{x}\in L\,\big\},\text{ and}\\ \mathcal{T}_3&=\big\{\,\mathbf{y}\mathbf{x}^*\mid\|\mathbf{x}\|=\|\mathbf{y}\|=1\text{ and }[\mathbf{y}]<[\mathbf{x}]\,\big\}.\end{aligned}$$

We claim that $\mathcal{T}$ is a minimal semitransitive semigroup. First, show that $\mathcal{T}$ is a semigroup. Let $S,T\in\mathcal{T}$. Clearly, if either T or S belongs to $\mathcal{T}_1$, then the product is also in $\mathcal{T}_1$. Suppose that $S,T\in\mathcal{T}_2\cup\mathcal{T}_3$. Then $S=\mathbf{y}\mathbf{x}^*$ and $T=\mathbf{v}\mathbf{u}^*$ where $\|\mathbf{x}\|=\|\mathbf{y}\|=\|\mathbf{u}\|=\|\mathbf{v}\|=1$, $[\mathbf{y}]\leqslant[\mathbf{x}]$, and $[\mathbf{v}]\leqslant[\mathbf{u}]$. Note that T attains its norm only on the elements of $[\mathbf{u}]$, so that $\|ST\|=1$ only when $\|ST\mathbf{u}\|=1$, but $\|ST\mathbf{u}\|=\|\mathbf{y}\mathbf{x}^*\mathbf{v}\mathbf{u}^*\mathbf{u}\|\leqslant\|\mathbf{x}^*\mathbf{v}\|$. It follows that if $[\mathbf{x}]\neq[\mathbf{v}]$ then $ST\in\mathcal{T}_1$. On the other hand, if $[\mathbf{x}]=[\mathbf{v}]$, then $\mathbf{v}=\lambda\mathbf{x}$ for some $|\lambda|=1$, and then $ST=\mathbf{y}\mathbf{x}^*\cdot\lambda\mathbf{x}\mathbf{u}^*=\lambda\mathbf{y}\mathbf{u}^*\in\mathcal{T}_2\cup\mathcal{T}_3$ because $[\mathbf{y}]\leqslant[\mathbf{x}]=[\mathbf{v}]\leqslant[\mathbf{u}]$.

In order to show that $\mathcal{T}$ is semitransitive, fix nonzero $\mathbf{x}$ and $\mathbf{y}$, and observe that if $\|\mathbf{x}\|>\|\mathbf{y}\|$ then $T\mathbf{x}=\mathbf{y}$ where $T=\frac{1}{\|\mathbf{x}\|_2^2}\mathbf{y}\mathbf{x}^*\in\mathcal{T}_1$. Suppose that $\|\mathbf{x}\|=\|\mathbf{y}\|$, we can assume that $\|\mathbf{x}\|=\|\mathbf{y}\|=1$. If $\mathbf{x}\sim\mathbf{y}$ then there is $T\in\mathcal{T}_2$ such that $T\mathbf{x}=\mathbf{y}$ or $T\mathbf{y}=\mathbf{x}$. Finally, if $\mathbf{x}\nsim\mathbf{y}$ then there is $T\in\mathcal{T}_3$ such that $T\mathbf{x}=\mathbf{y}$ when $[\mathbf{y}]<[\mathbf{x}]$ or $T\mathbf{y}=\mathbf{x}$ when $[\mathbf{x}]<[\mathbf{y}]$.

Finally, show that $\mathcal{T}$ is minimal. Suppose $\mathcal{T}_0$ is a semitransitive subsemigroup of $\mathcal{T}$. Then $\mathcal{T}_0$ contains all self-adjoint rank-one projections, hence $\mathcal{T}_1\subset\mathcal{T}_0$. Since the action of $\mathcal{T}_0$ on $[\mathbf{x}]$ is semitransitive for every $\mathbf{x}$ with $\|\mathbf{x}\| = 1$, it follows that $\mathcal{T}_2 \subset \mathcal{T}_0$. Finally, if $[\mathbf{y}] < [\mathbf{x}]$ for some $\|\mathbf{x}\| = \|\mathbf{y}\| = 1$ then $\mathbf{y}\mathbf{x}^*$ is the only element of $\mathcal{T}$ that maps $\mathbf{x}$ to $\mathbf{y}$, while no element of $\mathcal{T}$ maps $\mathbf{y}$ to $\mathbf{x}$, so that $\mathbf{y}\mathbf{x}^* \in \mathcal{T}_0$. Therefore, $\mathcal{T}_3 \subset \mathcal{T}_0$. It follows that $\mathcal{T}_0 = \mathcal{T}$.

4.5. **Multiples of unitaries.** Let $\mathcal{S}$ be the set of all matrices of the form λU, where $\lambda \in [0, 1]$ and U is a unitary matrix. This is clearly a closed semitransitive semigroup. As usual, let SU_n be the group of all special unitary matrices A in $M_n(\mathbb{C})$, i.e., $A^*A = I$ and $\det A = 1$.

Proposition 8. *The group SU_2 is a minimal semigroup acting semitransitively on norm-one vectors in $\mathbb{C}^2$.*

Proof. Since the columns of every matrix in SU_2 form an orthonormal basis, it follows that SU_2 consists of all the matrices of the form $\left(\begin{smallmatrix} x & -\bar{y} \\ y & \bar{x} \end{smallmatrix}\right)$ with $|x|^2 + |y|^2 = 1$. A direct calculation shows that whenever $\|\mathbf{u}\| = \|\mathbf{v}\| = 1$ in $\mathbb{C}^2$ then there exists a unique matrix $A \in SU_2$ such that $A\mathbf{u} = \mathbf{v}$. It follows that SU_2 acts transitively on the norm one vectors.

Let $\mathcal{S} \subseteq SU_2$ be a semigroup acting semitransitively on the unit sphere in $\mathbb{C}^2$. Then $\mathcal{S}$ contains all the elements of SU_2 of finite order. Indeed, suppose that $A \in SU_2$ and $A^n = I$. Fix any $\mathbf{u}$ with $\|\mathbf{u}\| = 1$, then A is the only matrix in SU_2 that takes $\mathbf{u}$ to $A\mathbf{u}$, while A^{-1} is the only matrix in SU_2 that takes $A\mathbf{u}$ to $\mathbf{u}$. Hence, either $A \in \mathcal{S}$ or $A^{-1} \in \mathcal{S}$. In the later case, $\mathcal{S} \ni (A^{-1})^{n-1} = A$.

Since $\left(\begin{smallmatrix} 0 & -\bar{\alpha} \\ \alpha & 0 \end{smallmatrix}\right)$ is of order 4 whenever $|\alpha| = 1$, it belongs to $\mathcal{S}$. In particular, $\left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right) \in \mathcal{S}$. Since

$$\begin{pmatrix} \alpha & 0 \\ 0 & \bar{\alpha} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \cdot \begin{pmatrix} 0 & -\bar{\alpha} \\ \alpha & 0 \end{pmatrix},$$

it follows that $\left(\begin{smallmatrix} \alpha & 0 \\ 0 & \bar{\alpha} \end{smallmatrix}\right) \in \mathcal{S}$ whenever $|\alpha| = 1$. Since every conjugate of $\left(\begin{smallmatrix} \alpha & 0 \\ 0 & \bar{\alpha} \end{smallmatrix}\right) \in \mathcal{S}$ can still be written as a product of matrices of order 4, it also has to be in $\mathcal{S}$.

It is left to show that every matrix in SU_2 is a conjugate of $\left(\begin{smallmatrix} \alpha & 0 \\ 0 & \bar{\alpha} \end{smallmatrix}\right)$ for some $|\alpha| = 1$. Indeed, every matrix A in SU_2 has an orthonormal basis of eigenvectors. Denote the first eigenvector by $\left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right)$, then, without loss of generality, we can assume that $\left(\begin{smallmatrix} -\bar{y} \\ \bar{x} \end{smallmatrix}\right)$ is the second eigenvector. Let α and β be the eigenvalues of A. Since A is unitary, then $|\alpha| = |\beta| = 1$. Furthermore, $\alpha\beta = \det A = 1$, so that $\beta = \bar{\alpha}$. It follows that

$$A = \begin{pmatrix} x & -\bar{y} \\ y & \bar{x} \end{pmatrix} \begin{pmatrix} \alpha & 0 \\ 0 & \bar{\alpha} \end{pmatrix} \begin{pmatrix} x & -\bar{y} \\ y & \bar{x} \end{pmatrix}^{-1}.$$

□

Corollary 9. *The semigroup* $(0,1]SU_2$ *is minimal semitransitive.*

5. Main results

Theorem 10. *If* $\mathbb{F}$ *is an algebraically closed field, then every semitransitive subalgebra of* $M_n(\mathbb{F})$ *contains a nilpotent element of index* n *(i.e.,* $T^n = 0$*, but* $T^{n-1} \neq 0$*).*

Proof. Let $\mathcal{A}$ be a semitransitive subalgebra of $M_n(\mathbb{F})$. It is easy to see that the invariant subspaces of $\mathcal{A}$ are totally ordered. First, we consider the case when $\mathcal{A}$ is unital. Furthermore, we can also assume that $\mathcal{A}$ has the block upper triangular form given in [5, p. 13]. Denoting the block form of $A \in \mathcal{A}$ by $(A_{ij})_{i,j=1}^k$, observe that the Jacobson radical $\mathcal{R}$ of $\mathcal{A}$ consists precisely of those R for which all diagonal blocks R_{ii} are zero. We shall show that for fixed i, $1 \leqslant i \leqslant k-1$, the set $\{R_{i,i+1} \mid R \in \mathcal{R}\}$ is the full space of matrices of the given size. First of all, if this set were just $\{0\}$, then the 2×2 algebra

$$\mathcal{L} = \left\{ \left. \begin{pmatrix} A_{ii} & A_{i,i+1} \\ 0 & A_{i+1,i+1} \end{pmatrix} \right| A \in \mathcal{A} \right\}$$

would be semisimple; we can assume after a similarity that $A_{i,i+1} = 0$ for all $A \in \mathcal{A}$. This is a contradiction to the total order of the invariant subspaces. Indeed, if the invariant subspaces of $\mathcal{A}$ given by the above triangularization are denoted by

$$\{0\} \subset \mathcal{M}_1 \subset \ldots \subset \mathcal{M}_k = \mathbb{F}^n,$$

then the identity $A_{i,i+1} = 0$ would give an invariant subspace

$$\mathcal{M}_{i-1} \oplus (\mathcal{M}_{i+1} \ominus \mathcal{M}_i),$$

not comparable to $\mathcal{M}_i$.

Now fix an R with $R_{i,i+1} \neq 0$, and note that the $(i, i+1)$-block of $AR + RB$, with $A, B \in \mathcal{A}$ is

$$A_{ii}R_{i,i+1} + R_{i,i+1}B_{i+1,i+1}.$$

Since A and B here can be chosen in $\mathcal{A}$ independently of each other, and since $\{A_{jj} \mid A \in \mathcal{A}\}$ is the full matrix algebra of its size for each j, we conclude that the $(i, i+1)$ block can be arbitrarily chosen in $\mathcal{R}$ as desired.

By construction, since $\mathbb{F}$ is algebraically closed, $\mathcal{A}$ has a member N whose diagonal is $N_1 \oplus \ldots \oplus N_k$, where each N_i is the $m_i \times m_i$ Jordan cell

$$\begin{pmatrix} 0 & 1 & & & \\ & 0 & 1 & & \\ & & \ddots & \ddots & \\ & & & 0 & 1 \\ & & & & 0 \end{pmatrix},$$

with m_i the size of the i^{th} block. If $k = 1$, then there is nothing to prove. Otherwise, we view each member of $\mathcal{A}$ as an operator expressed in the ordered basis

$$\{\mathbf{e}_{1,1}, \ldots, \mathbf{e}_{1,m_1}; \mathbf{e}_{2,1}, \ldots, \mathbf{e}_{2,m_2}; \ldots; \mathbf{e}_{k,1}, \ldots, \mathbf{e}_{k,m_k}\}$$

of the n-dimensional column space. By the preceeding paragraph, for each $j \leqslant k - 1$, $\mathcal{R}$ has a member R_j with

$$R_j \mathbf{e}_{j+1,1} = \mathbf{e}_{j,m_j}.$$

One can inductively choose the scalars $\alpha_1, \ldots, \alpha_{k-1}$ such that the $k - 1$ entries of $\alpha_1 R_1 + \ldots + \alpha_{k-1} R_{k-1}$ at $(m_1 + \ldots + m_j, m_1 + \ldots + m_j + 1)$ are all nonzero. Put

$$A = N + \alpha_1 R_1 + \ldots + \alpha_{k-1} R_{k-1},$$

then all the $(i, i + 1)$-entries of the nilpotent matrix A are nonzero, so that we have $A^{n-1} \neq 0$ and $A^n = 0$ as desired.

In the case when $\mathcal{A}$ is not unital, by applying the previous argument to the unitalization $\mathcal{A}'$ of $\mathcal{A}$ we know that there is a nilpotent A of index n in $\mathcal{A}'$. If $A \in \mathcal{A}$, we are done. Otherwise, we may assume, after multiplying A with a scalar, that there exists $T \in \mathcal{A}$ such that $A = I + T$. Then $T^2 + T$ is a nilpotent element of index n in $\mathcal{A}$. $\square$

Every semitransitive algebra of matrices contains a minimal semitransitive subalgebra because it is finite-dimensional.

Corollary 11. *The algebra of upper triangular Toepliz matrices, described in Section 4.1 is the unique (up to simultaneous similarity) minimal semitransitive algebra of matrices over an algebraically closed field. In particular, every minimal semitransitive algebra of matrices over an algebraically closed field is commutative.*

Proof. Let $\mathcal{A}$ be a semitransitive algebra of $n \times n$ matrices over an algebraically closed field. By Theorem 10, $\mathcal{A}$ must contain an matrix S such that $S^n = 0$, but $S^{n-1} \neq 0$ (i.e. S is the backward shift in some basis). The subalgebra of $\mathcal{A}$ generated by S is similar to the algebra of upper triangular Toepliz matrices and is semitransitive (see Section 4.1).

On the other hand, if $\mathcal{A}$ is minimal semitransitive, then it must be equal to the subalgebra generated by S. $\square$

We now turn our attention to semitransitive semigroups of matrices. The lemma below is needed to show that semitransitive semigroups in M_n of constant rank k exist only for those k that divide n.

Lemma 12. *Let $\mathcal{S}$ be a semigroup of linear transformations of constant rank k (or 0) acting on $\mathbb{F}^n$ and assume that U is a subspace of $\mathbb{F}^n$ of dimension $m > k$ containing the range of $\mathcal{S}$ (i.e. $T\mathbf{x} \in U$ for all $\mathbf{x} \in \mathbb{F}^n$ and $T \in \mathcal{S}$). If $\mathcal{S}$ acts semitransitively on U, then there exists a subspace W of U of dimension $m - k$ and a subsemigroup $\mathcal{S}'$ of $\mathcal{S}$ whose range is contained in W and acts semitransitively on W.*

Proof. It is easy to see that $\mathcal{S}$ must contain an element T for which $\operatorname{Range} T^2 = \operatorname{Range} T$ (any nonnilpotent operator will suffice). We will show that $\mathcal{S}' = \{S \in \mathcal{S} \mid TS = 0\}$ and $W = (\ker T) \cap U = \ker T|_U$ satisfy the required criteria. It is clear that $\operatorname{Range} \mathcal{S}' \subseteq W$ and that $\dim(W) = m - k$ ($\operatorname{rank} T|_U = k$, since $\operatorname{rank} T = k$ and $\operatorname{Range} T^2 = \operatorname{Range} T$). We complete the proof by showing that $\mathcal{S}'$ acts semitransitively on W. Let $\mathbf{x}$ and $\mathbf{y}$ be nonzero vectors in W. Since $\mathcal{S}$ acts semitransitively on U (hence also on W) there is $S \in \mathcal{S}$ that maps one of the vectors into the other; suppose $S\mathbf{x} = \mathbf{y}$. Note that $TS\mathbf{x} = T\mathbf{y} = 0$. If $TS \neq 0$, then $\operatorname{rank}(TS) = k$ and therefore S has the same kernel as TS (clearly $\ker TS \subseteq \ker S$, we must have equality since $\dim \ker TS = n - k = \dim \ker S$); but this would mean that $S\mathbf{x} = 0$. Thus $TS = 0$ and hence $S \in \mathcal{S}'$. ☐

The following result extends a known theorem for transitivity of semigroups [1, Theorem 3.13].

Theorem 13. *If $\mathcal{S}$ is a semitransitive semigroup of matrices in $M_n(\mathbb{F})$ of rank k or 0, then k divides n.*

Proof. Suppose, if possible, that k does not divide n. Then we can write $n = sk + r$, where $0 < r < k$. Using Lemma 12 (s times) we obtain a subspace W of $\mathbb{F}^n$ of dimension r and a subsemigroup $\mathcal{S}'$ of $\mathcal{S}$ whose image is contained in W and acts semitransitively on W. But then $\mathcal{S}'$ would be a nonzero semigroup of rank at most r; a contradiction to the assumption that all ranks in $\mathcal{S}$ and hence in $\mathcal{S}'$ are either 0 or k. ☐

Example 14. If $n = kl$ and $V = V_1 \oplus V_2 \oplus \ldots \oplus V_l$, is a decomposition of an n-dimensional vector space V into k-dimensional subspaces, then all linear transformations $A\colon V \to V$ of the form $A = A_{1,i} + \ldots + A_{l,i}$, where each $A_{j,i}\colon V_i \to V_j$ is either 0 or invertible, form a semitransitive semigroup of transformations of rank k or 0.

The lemma below will be used to construct examples of semitransitive semigroups that contain no minimal semitransitive semigroups.

Lemma 15. *If a semigroup $\mathcal{S}$ is either transitive or semitransitive, then so is $\{AB \mid A, B \in \mathcal{S}\}$, the subsemigroup of $\mathcal{S}$ consisting of products of two elements of $\mathcal{S}$.*

Proof. If $\mathbf{x}, \mathbf{y}$ is a pair of nonzero vectors, then note that $AB\mathbf{x} = \mathbf{y}$, provided that $B\mathbf{x} = \mathbf{y}$ and $A\mathbf{y} = \mathbf{y}$. $\square$

Corollary 16. *If $\mathbb{F}$ is a field of characteristic zero and $n \geqslant 2$, then*

$$\mathcal{S} = \{A \in M_n(\mathbb{F}) \mid \det(A) = 2^k \textit{ for some positive integer } k\}$$

is a transitive semigroup that contains no minimal transitive (semitransitive) subsemigroups. In particular, if $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}\}$, then $\mathcal{S}$ is closed and contains no minimal closed transitive (semitransitive) subsemigroups.

Proof. Note that if $\mathcal{S}'$ is a subsemigroup of $\mathcal{S}$, then $\mathcal{S}' \cdot \mathcal{S}'$ is a proper subsemigroup $\mathcal{S}'$. The exercise of showing that $\mathcal{S}$ is transitive is left to the reader. $\square$

Corollary 17. *Let $\mathcal{S}_\alpha$ be the subsemigroup of $M_n(\mathbb{C})$, generated by unitary matrices of determinant α with $|\alpha| = 1$. If $n \geqslant 2$ and α is not a root of unity, then*

$$\mathcal{S}'_\alpha = (0, 1] \cdot \mathcal{S}_\alpha$$

is a bounded transitive semigroup that contains no minimal transitive (semitransitive) subsemigroups.

Proof. Note that $\mathcal{T} \cdot \mathcal{T} \subsetneq \mathcal{T}$,whenever $\mathcal{T}$ is a subsemigroup of $\mathcal{S}'_\alpha$. It remains to show that the semigroup $\mathcal{S}'_\alpha$ is transitive. For that, it is sufficient to prove that unitary matrices of determinant α act transitively on the unit sphere of vectors in $\mathbb{C}^n$. If $\mathbf{x}, \mathbf{y}$ is a pair of unit vectors in $\mathbb{C}^n$, then choose an orthonormal basis $\mathcal{V} = \{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ for $\mathbb{C}^n$, so that $\mathbf{x} = \mathbf{v}_1$ and $\mathbf{y} = y_1\mathbf{v}_1 + y_2\mathbf{v}_2$. Now note that (in basis $\mathcal{V}$),

$$A = \begin{pmatrix} y_1 & -\alpha\bar{y_2} \\ y_2 & \alpha\bar{y_1} \end{pmatrix} \oplus I_{n-2}$$

is a unitary transformation of determinant α that maps $\mathbf{x}$ to $\mathbf{y}$. $\square$

The examples above show that in general semitransitive semigroups need not contain any minimal semitransitive semigroups. In the remainder of our paper, we give some conditions which guarantee the existence of such a subsemigroup.

Theorem 18. *If $\mathcal{S}_0$ is a semitransitive semigroup in $M_n(\mathbb{F})$ consisting of matrices with at most one non-zero column then it contains a minimal semitransitive subsemigroup.*

Proof. Let $\mathcal{G}$ be the set of all semitransitive subsemigroups of $\mathcal{S}_0$, ordered by the reverse inclusion. We use Zorn's lemma to prove that $\mathcal{G}$ contains a maximal element. Suppose that $(\mathcal{S}_\alpha)_{\alpha \in \Lambda}$ is a chain in $\mathcal{G}$. We claim that the semigroup $\mathcal{S} = \bigcap_{\alpha \in \Lambda} \mathcal{S}_\alpha$ is semitransitive. We can assume that Λ is infinite. Pick a pair of nonzero vectors $\mathbf{x},\mathbf{y}$ and define

$$\begin{aligned}\Lambda' &= \big\{\alpha\in\Lambda \mid A\mathbf{x}=\mathbf{y} \text{ for some } A\in\mathcal{S}_\alpha\big\}\\ \Lambda'' &= \big\{\alpha\in\Lambda \mid A\mathbf{y}=\mathbf{x} \text{ for some } A\in\mathcal{S}_\alpha\big\}.\end{aligned}$$

Since $\Lambda'\cup\Lambda''=\Lambda$ we can, without any loss of generality, assume that $\bigcap_{\Lambda'}\mathcal{S}_\alpha=\mathcal{S}$. For each $\alpha\in\Lambda'$ choose a matrix $A_\alpha\in\mathcal{S}_\alpha$ so that $A_\alpha\mathbf{x}=\mathbf{y}$ and note that $A_\alpha=\mathbf{z}_\alpha\mathbf{e}_{j_\alpha}^T$ for some integer j_α and nonzero vector $\mathbf{z}_\alpha$. If $\Lambda_j=\{\alpha\in\Lambda' \mid j_\alpha=j\}$ then clearly $\Lambda'=\bigcup_{j=1}^n\Lambda_j$ and hence $\bigcap_{\Lambda_{j_0}}\mathcal{S}_\alpha=\mathcal{S}$ for some $1\leqslant j_0\leqslant n$, since at least one of the Λ_j has no upper bound.

For every $\alpha\in\Lambda_{j_0}$ we have $\mathbf{y}=A_\alpha\mathbf{x}=\mathbf{z}_\alpha\mathbf{e}_{j_0}^T\mathbf{x}=x_{j_0}\mathbf{z}_\alpha$, thus $\mathbf{z}_\alpha=\frac{1}{x_{j_0}}\mathbf{y}$, hence $A_\alpha=\frac{1}{x_{j_0}}\mathbf{y}\mathbf{e}_{j_0}$ for all $\alpha\in\Lambda_{j_0}$ and therefore $\frac{1}{x_{j_0}}\mathbf{y}\mathbf{e}_{j_0}\in\mathcal{S}$. □

The proof of the theorem above can be modified to show the following.

Theorem 19. *Suppose that X is a Banach space and $\mathcal{S}$ is a semitransitive semigroup of bounded operators on X. If $\mathcal{S}$ is compact in operator norm, then it contains a minimal compact semitransitive semigroup.*

Proof. Let $\mathcal{G}$ be the set of all compact semitransitive subsemigroups of $\mathcal{S}_0$, ordered by the reverse inclusion. We use Zorn's lemma to show that $\mathcal{G}$ has a maximal element. If $(\mathcal{S}_\alpha)_{\alpha\in\Lambda}$ is a chain in $\mathcal{G}$ then define $\mathcal{S}=\bigcap_\Lambda\mathcal{S}_\alpha$. We claim, that $\mathcal{S}$ is semitransitive (it is clearly a compact semigroup). Pick a pair of nonzero vectors $\mathbf{x},\mathbf{y}$ and define

$$\begin{aligned}\Lambda' &= \big\{\alpha\in\Lambda \mid A\mathbf{x}=\mathbf{y} \text{ for some } A\in\mathcal{S}_\alpha\big\}\\ \Lambda'' &= \big\{\alpha\in\Lambda \mid A\mathbf{y}=\mathbf{x} \text{ for some } A\in\mathcal{S}_\alpha\big\}.\end{aligned}$$

Since $\Lambda'\cup\Lambda''=\Lambda$ we can, without any loss of generality, assume that $\bigcap_{\Lambda'}\mathcal{S}_\alpha=\mathcal{S}$. If $\alpha\in\Lambda'$ then choose an operator $A_\alpha\in\mathcal{S}_\alpha$, so that $A_\alpha\mathbf{x}=\mathbf{y}$. Since $(A_\alpha)_{\alpha\in\Lambda'}$ is a chain in $\mathcal{S}_0$, it contains a convergent subnet $(A_\alpha)_{\alpha\in\Lambda_0'}\to A$. Note that $A\in\mathcal{S}_\alpha$ for all α, since every $\mathcal{S}_\alpha$ is closed and contains a tail of the net $(A_\alpha)_{\alpha\in\Lambda_0'}$. Hence $A\in\mathcal{S}$ and $A\mathbf{x}=\lim_{\alpha\in\Lambda_0'}A_\alpha\mathbf{x}=\mathbf{y}$. □

Remark 20. Observe that operator norm compactness in Theorem 19 can be replaced with compactness in any Hausdorff operator topology τ that satisfies the following condition: *if $A_\alpha\xrightarrow{\tau}A$ and $A_\alpha\mathbf{x}=\mathbf{y}$ for some net (A_α) and some nonzero vectors $\mathbf{x}$ and $\mathbf{y}$, then $A\mathbf{x}=\mathbf{y}$.* In particular, Theorem 19 remains valid for the weak operator topology and, in case when X is a dual space, for the weak* operator topology.

Corollary 21. *A closed bounded semitransitive semigroup in $M_n(\mathbb{F})$, where $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}\}$, contains a minimal closed semitransitive subsemigroup.*

Remark 22. The only semigroup property that we used in the proof of Theorem 19 is that the intersection of a chain of semigroups is a semigroup. Thus Theorem 19, Remark 20 and Corollary 21 apply also to semitransitive sets or groups of operators.

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