

STRICTLY SEMI-TRANSITIVE OPERATOR ALGEBRAS

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ABSTRACT. An algebra \mathfrak{A} of operators on a Banach space X is called strictly semi-transitive if for all non-zero $x, y \in X$ there exists an operator $A \in \mathfrak{A}$ such that $Ax = y$ or $Ay = x$. We show that if \mathfrak{A} is norm-closed and strictly semi-transitive, then every \mathfrak{A} -invariant linear subspace is norm-closed. Moreover, $\text{Lat } \mathfrak{A}$ is totally and well ordered by reverse inclusion. If X is complex and \mathfrak{A} is transitive and strictly semi-transitive, then \mathfrak{A} is WOT-dense in $\mathcal{L}(X)$. It is also shown that if \mathfrak{A} is an operator algebra on a complex Banach space with no invariant operator ranges, then \mathfrak{A} is WOT-dense in $\mathcal{L}(X)$. This generalizes a similar result for Hilbert spaces proved by Foiaş.

1. INTRODUCTION

Throughout the paper X is a real or complex Banach space, B_X stands for the closed unit ball of X . Let \mathfrak{A} be a subalgebra of $\mathcal{L}(X)$, the algebra of all bounded linear operators on X . For $x \in X$ the orbit of x under \mathfrak{A} is defined by $\mathfrak{A}x = \{Ax \mid A \in \mathfrak{A}\}$. We say that $x_0 \in X$ is *cyclic* for \mathfrak{A} if $\mathfrak{A}x_0$ is dense in X ; x_0 is called *strictly cyclic* if $\mathfrak{A}x_0 = X$. The algebra \mathfrak{A} is called *transitive* (*strictly transitive*) if every non-zero element of X is cyclic (respectively, strictly cyclic). We write $\text{Lat } \mathfrak{A}$ for the lattice of all closed \mathfrak{A} -invariant subspaces of X . It is easy to see that \mathfrak{A} is transitive if and only if $\text{Lat } \mathfrak{A} = \{\{0\}, X\}$. Similarly, \mathfrak{A} is strictly transitive if and only if it has no invariant linear subspaces other than $\{0\}$ and X (by a linear subspace we mean a subspace which is not necessarily closed). A remarkable result due independently to Yood [Yood49] and Rickart [Ric50] yields that if \mathfrak{A} is a strictly transitive algebra of operators on a complex Banach space, then it is WOT-dense in $\mathcal{L}(X)$. We refer the reader to [RR73] for a detailed introduction to transitive algebras.

An algebra \mathfrak{A} is called *semi-transitive* if for all non-zero $x, y \in X$ and $\varepsilon > 0$ there exists an operator $A \in \mathfrak{A}$ such that $\|Ax - y\| < \varepsilon$ or $\|Ay - x\| < \varepsilon$. We say that \mathfrak{A}

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is **strictly semi-transitive** if for all non-zero $x, y \in X$ there is an operator $A \in \mathfrak{A}$ with $Ax = y$ or $Ay = x$. One can show that a unital algebra \mathfrak{A} is semi-transitive if and only if it is **unicellular**, that is, $\text{Lat } \mathfrak{A}$ is totally ordered by inclusion (see [RR73] for a treatment of unicellular algebras). Similarly, \mathfrak{A} is strictly semi-transitive if and only if all the \mathfrak{A} -invariant linear subspaces are totally ordered by inclusion. In Section 3 we investigate the structure of strictly semi-transitive algebras. We show that if \mathfrak{A} is norm closed and strictly semi-transitive, then every \mathfrak{A} -invariant linear subspace is norm-closed and $\text{Lat } \mathfrak{A}$ is well-ordered by reverse inclusion. We deduce that a transitive strictly semi-transitive operator algebra on a complex Banach space is WOT-dense in $\mathcal{L}(X)$. In the special case of CSL-algebras on a Hilbert space similar results were obtained in [Hop01, DHP01].

Foiaş proved in [Foi72] that if \mathfrak{A} is a WOT-closed algebra of operators on a Hilbert space H , and \mathfrak{A} has no invariant operator ranges, then $\mathfrak{A} = \mathcal{L}(H)$. In Section 4 we generalize the result of Foiaş to complex Banach spaces using a version of Arveson's Lemma of [Arv67]. For $Y \subseteq X$, we say that Y is an **injective operator range** if $Y = \text{Range } \vec{T}$ for an injective bounded operator $\vec{T} \in \mathcal{L}(\mathcal{M}, X)$, where \mathcal{M} is a closed subspace of X^n for some $n \geq 1$. We show that if \mathfrak{A} has no invariant injective operator ranges in X , then \mathfrak{A} is WOT-dense in $\mathcal{L}(X)$.

2. PRELIMINARIES

We first formulate a version of the standard lemma for the Open Mapping Theorem, and then use it to deduce a results concerning strictly cyclic vectors.

Lemma 2.1 (Open Mapping Lemma). *Suppose that Y is a normed space, and $T: X \rightarrow Y$ is a bounded operator. Assume that there exist $K > 0$ and $0 < \varepsilon < 1$ such that*

$$(2.1) \quad B_Y \subseteq KT(B_X) + \varepsilon B_Y.$$

Then T is a surjective open map, and Y is complete.

Remark 2.2. In particular, (2.1) is satisfied if $\overline{T(B_X)}$ has non-empty interior.

Proof. Fix $y \in B_Y$, choose $x_1 \in KB_X$ such that $\|y - Tx_1\| \leq \varepsilon$. Denote $y_1 = y - Tx_1$, then $y_1 \in \varepsilon B_Y$, so that there exists $x_2 \in \varepsilon KB_X$ such that $\|y_1 - Tx_2\| \leq \varepsilon^2$, let $y_2 = y_1 - Tx_2$. Continuing, we obtain sequences (x_n) in X and (y_n) in Y so that for all n we have

$$(2.2) \quad \|x_n\| \leq \varepsilon^{n-1} K, \text{ and}$$

$$(2.3) \quad \|y_n\| \leq \varepsilon^n, \text{ and } y_n = y - T \sum_{j=1}^n x_j.$$

Since X is complete, (2.2) yields that $\sum_{j=1}^{\infty} x_j$ converges to an element x of X . Since T is continuous, $Tx = y$ by (2.3). Thus T is surjective. Moreover, it follows from (2.2) that $\|x\| \leq M$ where $M = K \sum_{j=0}^{\infty} \varepsilon^j = \frac{K}{1-\varepsilon}$, so that $B_Y \subseteq MT(B_X)$, hence T is open. Now let $Z = X/\text{Null}T$ and let $\pi: X \rightarrow Z$ be the canonical quotient map. Then Z is complete and there is a one-to-one operator $\tilde{T}: Z \rightarrow Y$ such that $\tilde{T}\pi = T$. Since T is open, the set $\tilde{T}(U) = T(\pi^{-1}U)$ is open whenever $U \subseteq Z$ is open, so that \tilde{T}^{-1} is continuous. Hence, \tilde{T} is an isomorphism between Z and Y , so that Y is complete. \square

Proposition 2.3. *Let \mathfrak{A} be norm closed and $x_0 \in X$ with $x_0 \neq 0$. The following are equivalent.*

- (i) x_0 is strictly cyclic.
- (ii) There is a constant $C > 0$ so that for all $y \in B_X$ there is an operator $A \in \mathfrak{A}$ with $\|A\| \leq C$ such that $Ax_0 = y$.
- (iii) There are $C > 0$ and $0 < \varepsilon < 1$ so that for all $y \in B_X$ there is an operator $A \in \mathfrak{A}$ with $\|A\| \leq C$ and $\|y - Ax_0\| < \varepsilon$.

Proof. Define $T: \mathfrak{A} \rightarrow X$ by $T(A) = Ax_0$, then T is a bounded linear operator from \mathfrak{A} to X . Since \mathfrak{A} is norm closed, it is a Banach space. Thus, (i) \Rightarrow (ii) follows immediately from the Open Mapping Theorem. The implication (ii) \Rightarrow (iii) is trivial. Finally, (iii) \Rightarrow (i) follows from the Open Mapping Lemma. \square

Corollary 2.4. *If \mathfrak{A} is norm-closed, then the set of strictly cyclic vectors for \mathfrak{A} is open.*

Proof. Let x_0 be a strictly cyclic vector for \mathfrak{A} , and choose C as in Proposition 2.3(ii). Let $0 < \delta < 1/C$, and suppose that $x \in X$ with $\|x - x_0\| < \delta$. Now if $y \in B_X$, we choose $A \in \mathfrak{A}$ with $\|A\| \leq C$ and $Ax_0 = y$. But then $\|Ax - y\| = \|Ax - Ax_0\| \leq C\delta < 1$. Hence x is strictly transitive by Proposition 2.3(iii). \square

This yields an alternate proof of a result due to A. Lambert.

Corollary 2.5 ([Lam71]). *If X is a complex Banach space and \mathfrak{A} is a transitive operator algebra on X with a strictly cyclic vector, then \mathfrak{A} is WOT-dense in $\mathcal{L}(X)$.*

Proof. We may assume that \mathfrak{A} is norm-closed by replacing \mathfrak{A} by $\overline{\mathfrak{A}}$. Let x_0 be a strictly cyclic vector for \mathfrak{A} . By Corollary 2.4 there exists $\delta > 0$ such that $\|x - x_0\| < \delta$ implies x is strictly cyclic. Let $y \in X$ with $y \neq 0$. Since \mathfrak{A} is transitive we may choose $A \in \mathfrak{A}$

with $\|Ay - x_0\| < \delta$. But then Ay is strictly cyclic, and so, of course, y is also strictly cyclic. Thus, \mathfrak{A} is strictly transitive and so is WOT-dense in $\mathcal{L}(X)$. \square

The next result shows that transitive algebras always have operators which are almost zero on prescribed vectors.

Proposition 2.6. *Let \mathfrak{A} be a transitive operator algebra on a complex Banach space X , then for all $x \in X$ and $\varepsilon > 0$ there is an $A \in \mathfrak{A}$ with $\|A\| = 1$ and $\|Ax\| < \varepsilon$.*

Proof. Without loss of generality, \mathfrak{A} is WOT-closed. Indeed, suppose that the conclusion holds for $\overline{\mathfrak{A}}^{\text{WOT}}$, show that it also holds for \mathfrak{A} . Fix $x \in X$ and $0 < \varepsilon < 1$, and let $\delta = \varepsilon/4$. There exists $A \in \overline{\mathfrak{A}}^{\text{WOT}}$ with $\|A\| = 1$ and $\|Ax\| < \delta$. Then $\|Ay\| \geq 1 - \delta$ for some $y \in X$ with $\|y\| = 1$. Since $\overline{\mathfrak{A}}^{\text{WOT}} = \overline{\mathfrak{A}}^{\text{SOT}}$, there exists $B \in \mathfrak{A}$ such that $\|(A - B)x\| < \delta$ and $\|(A - B)y\| < \delta$. Then $\|Bx\| < 2\delta$ and $\|By\| \geq 1 - 2\delta$, so that $\|B\| \geq 1 - 2\delta$. Let $C = \frac{B}{\|B\|}$, then $\|Cx\| < \frac{2\delta}{1-2\delta} < \varepsilon$.

If $\mathfrak{A} = \mathcal{L}(X)$ then the conclusion is trivially satisfied. Suppose that \mathfrak{A} is a WOT-closed proper subalgebra of $\mathcal{L}(X)$. Were the conclusion false, we could choose x of norm one and $\varepsilon > 0$ so that $\|Ax\| > \varepsilon$ whenever $A \in \mathfrak{A}$ and $\|A\| = 1$. But then the operator $T: \mathfrak{A} \rightarrow X$ defined by $T(A) = Ax$ for $A \in \mathfrak{A}$ is an isomorphism. In particular, $\mathfrak{A}x$ is norm-closed being isomorphic to \mathfrak{A} . Then $\mathfrak{A}x = X$ since \mathfrak{A} is transitive. This contradicts Corollary 2.5. \square

The following generalization of this result is due to Jiaosheng Jiang, and we are grateful to him for giving us permission to present his proof here.

Theorem 2.7 (J. Jiang). *Let \mathfrak{A} be a commutative transitive operator algebra on a complex Banach space. Then for all $x_0, x_1, \dots, x_n \in X$ and $\varepsilon > 0$ there exists an operator $A \in \mathfrak{A}$ with $\|A\| = 1$ and $\|Ax_i\| < \varepsilon$ for all $0 \leq i \leq n$.*

Proof. We may assume without loss of generality that $x_0 \neq 0$. By the transitivity of \mathfrak{A} , for each $1 \leq i \leq n$ we may choose $B_i \in \mathfrak{A}$ so that $\|B_i x_0 - x_i\| \leq \varepsilon/2$. Let $\delta = \min\{\varepsilon, \frac{\varepsilon}{2 \cdot \max_i \|B_i\|}\}$. By Proposition 2.6, we may choose $A \in \mathfrak{A}$ with $\|A\| = 1$ and $\|Ax_0\| < \delta$. Then $\|Ax_0\| < \varepsilon$ and for all $1 \leq i \leq n$,

$$\begin{aligned} \|Ax_i\| &\leq \|A(x_i - B_i x_0)\| + \|AB_i x_0\| \\ &\leq \|x_i - B_i x_0\| + \|B_i Ax_0\| \leq \varepsilon/2 + \|B_i\| \cdot \delta < \varepsilon. \end{aligned}$$

\square

We do not know if the conclusion of this proposition holds for general transitive algebras. The following result gives a partial answer. Recall that \mathfrak{A} is said to be *n-transitive* for some $n \geq 1$ if every linearly independent n -tuple in X^n is cyclic for the algebra $\mathfrak{A}^{(n)} = \{A \oplus \cdots \oplus A \mid A \in \mathfrak{A}\}$.

Proposition 2.8. *Let \mathfrak{A} be an operator algebra on a complex Banach space. If \mathfrak{A} is n -transitive for some $n \geq 1$, then for all $x_1, \dots, x_n \in X$ and $\varepsilon > 0$ there exists an operator $A \in \mathfrak{A}$ with $\|A\| = 1$ and $\|Ax_i\| < \varepsilon$ for all i .*

Proof. As in the proof of Proposition 2.6, we may assume that \mathfrak{A} is a WOT-closed proper subalgebra of $\mathcal{L}(X)$. Were the conclusion false, we could find $x_1, \dots, x_n \in X$ and $\varepsilon > 0$ such that $\max_{1 \leq i \leq n} \|Ax_i\| \geq \varepsilon$ for all $A \in \mathfrak{A}$ with $\|A\| = 1$. Without loss of generality, we may assume that x_1, \dots, x_k are linearly independent for some $k \leq n$, and $x_j \in \text{span}\{x_1, \dots, x_k\}$ whenever $k < j \leq n$. It follows that there is a $C > 1$ such that

$$\max_{1 \leq i \leq n} \|Ax_i\| \leq C \cdot \max_{1 \leq i \leq k} \|Ax_i\|$$

whenever $A \in \mathfrak{A}$ with $\|A\| = 1$, so that $\max_{1 \leq i \leq k} \|Ax_i\| \geq \varepsilon/C$. Then the map $T: \mathfrak{A} \rightarrow X^k$ defined by $T(A) = (Ax_1, \dots, Ax_k)$ for $A \in \mathfrak{A}$ is an isomorphism, and hence $T(\mathfrak{A})$ is closed. Since \mathfrak{A} is n -transitive, it is also k -transitive. Then $T(\mathfrak{A}) = X^k$ because x_1, \dots, x_k are linearly independent. This implies, in particular, that x_1 is strictly cyclic for \mathfrak{A} , and, since \mathfrak{A} is transitive, \mathfrak{A} is WOT-dense in $\mathcal{L}(X)$ by Corollary 2.5, a contradiction. \square

Recall that $T \in \mathcal{L}(X, Y)$ is called a *semi-embedding* if it is one-to-one and $T(B_X)$ is closed. The following result is certainly known, and we include a proof for the sake of completeness.

Proposition 2.9. *Let Y be a normed space, and $T: Y \rightarrow X$ a semi-embedding. Then Y is complete.*

Proof. Let $S: \text{Range}(T) \rightarrow Y$ be the inverse of T , and let \tilde{Y} be the completion of Y . Then T may be extended to a continuous operator $\tilde{T}: \tilde{Y} \rightarrow X$. Since $T(B_Y)$ is closed in X we have $\tilde{T}(B_{\tilde{Y}}) = T(B_Y)$, and hence $\text{Range } \tilde{T} = \text{Range } T$. Consider the operator $S\tilde{T}: \tilde{Y} \rightarrow Y$. If $y \in Y$ then $S\tilde{T}y = STy = y$. Moreover, $S\tilde{T}(B_{\tilde{Y}}) = S(T(B_Y)) = B_Y \subseteq B_{\tilde{Y}}$. It follows that $S\tilde{T}$ is a bounded projection of \tilde{Y} onto Y , and thus Y is complemented in \tilde{Y} , so that it is closed in \tilde{Y} , hence $\tilde{Y} = Y$. \square

We are now prepared for the final result of this section. For $W \subseteq X$ and $\varepsilon > 0$, let $W_\varepsilon = \{x \in X \mid \text{dist}(x, W) \leq \varepsilon\}$. The continuity of the map $x \mapsto \text{dist}(x, W)$ from X to \mathbb{R} implies that W_ε is a closed set.

Lemma 2.10. *Let X be a Banach space and W a bounded circled closed convex subset of X with void interior. Then for any $\varepsilon > 0$ the set W_ε contains no ball of radius larger than ε .*

Proof. Let $Y = \text{span } W = \bigcup_{n=1}^{\infty} nW$. Then Y equipped with the norm given by the Minkowski functional of W is a normed space. Let $T: Y \rightarrow X$ be the inclusion operator. Then T is bounded, one-to-one, and $T(B_Y) = W$. Hence, T is a semi-embedding, so by Proposition 2.9 Y is complete.

Clearly, W_ε is also circled and convex. Suppose that W_ε contains a ball of radius $r > \varepsilon$ centered at x_0 . Since W_ε is circled, it also contains the ball of radius r centered at $-x_0$, and since W_ε is convex, it contains the ball of radius r centered at the origin. Indeed, if $\|z\| \leq r$ then $x_0 + z$ and $-x_0 + z$ belong to W_ε , so that $z = \frac{(x_0+z)+(-x_0+z)}{2} \in W_\varepsilon$. Now let $\varepsilon < \varepsilon' < r$. Then $rB_X \subseteq W_\varepsilon \subseteq W + \varepsilon'B_X$, thus, $B_X \subseteq \frac{1}{r}W + \frac{\varepsilon'}{r}B_X$. Since $\frac{\varepsilon'}{r} < 1$, we conclude by Lemma 2.1 that T is surjective. This yields $X = Y = \bigcup_{n=1}^{\infty} nW$, which contradicts the Baire Category Theorem since W has void interior. \square

3. THE STRUCTURE OF STRICTLY SEMI-TRANSITIVE ALGEBRAS

A subalgebra \mathfrak{A} of $\mathcal{L}(X)$ is called *strictly semi-transitive* if for all non-zero $x, y \in X$ there exists an operator $A \in \mathfrak{A}$ so that $Ax = y$ or $Ay = x$.

Proposition 3.1. *Let \mathfrak{A} be a unital subalgebra of $\mathcal{L}(X)$. Then the following are equivalent.*

- (i) \mathfrak{A} is strictly semi-transitive;
- (ii) All the \mathfrak{A} -invariant linear subspaces of X are totally ordered by inclusion;
- (iii) All the orbits are totally ordered by inclusion.

Proof. (i) \Rightarrow (ii). Let Y and Z be two \mathfrak{A} -invariant linear subspaces of X , and suppose that $Z \not\subseteq Y$. Show that $Y \subseteq Z$. Choose $z \in Z \setminus Y$ and $y \in Y$. There is no $T \in \mathfrak{A}$ with $Ty = z$, for otherwise $z \in Y$ since Y is \mathfrak{A} -invariant. Hence there must be a $T \in \mathfrak{A}$ with $Tz = y$, and so $y \in Z$ since Z is \mathfrak{A} -invariant.

(ii) \Rightarrow (iii) is trivial.

(iii) \Rightarrow (i). Let $x, y \in X$. Then $\mathfrak{A}x \subseteq \mathfrak{A}y$ or $\mathfrak{A}y \subseteq \mathfrak{A}x$. Since \mathfrak{A} is unital, it follows that $x \in \mathfrak{A}y$ or $y \in \mathfrak{A}x$, hence \mathfrak{A} is strictly semi-transitive. \square

Remark 3.2. The implications (i) \Rightarrow (ii) \Rightarrow (iii) are still valid for non-unital algebras.

The next result is fundamental for our development. Recall that a set is *residual* if its complement is of first category; residual subsets of X are dense by Baire Category Theorem.

Lemma 3.3. *Let (W_n) be a sequence of circled bounded closed convex sets, each with void interior, such that $W_n \subseteq W_{n+1}$ and $\bigcup_{n=1}^{\infty} W_n$ is dense in X , and let (ε_n) and (δ_n) be sequences of positive reals tending to zero. Let*

$$\begin{aligned} G_1 &= \{x \in X \mid \text{for all } \lambda > 0, \text{dist}(\lambda x, W_n) > \varepsilon_n \text{ for infinitely many } n\}, \\ G_2 &= \{x \in X \mid \text{for all } \lambda > 0, \text{dist}(\lambda x, W_n) < \delta_n \text{ for infinitely many } n\}. \end{aligned}$$

Then $G_1 \cap G_2$ is residual.

Proof. Note that $x \in G_1$ if and only if

$$(3.1) \quad \forall \lambda > 0 \quad \forall m \in \mathbb{N} \quad \exists n \geq m \text{ such that } \text{dist}(\lambda x, W_n) > \varepsilon_n.$$

In particular,

$$(3.2) \quad \forall k \in \mathbb{N} \quad \forall m \in \mathbb{N} \quad \exists n \geq m \text{ such that } \text{dist}\left(\frac{1}{k}x, W_n\right) > \varepsilon_n.$$

We claim that (3.1) and (3.2) are, in fact, equivalent. Indeed, suppose that (3.2) holds. Given $\lambda > 0$ and $m \in \mathbb{N}$, find a positive integer k such that $\frac{1}{k} < \lambda$, then there exists $n \geq m$ such that $\text{dist}\left(\frac{1}{k}x, W_n\right) > \varepsilon_n$. Since W_n is convex and circled, it follows that $\text{dist}(\lambda x, W_n) \geq \text{dist}\left(\frac{1}{k}x, W_n\right) > \varepsilon_n$.

It follows from (3.2) that $x \notin G_1$ if and only if there exist $k, m \in \mathbb{N}$ such that $\frac{1}{k}x \in (W_n)_{\varepsilon_n}$ for all $n \geq m$. Therefore,

$$\sim G_1 = \bigcup_{k,m=1}^{\infty} \bigcap_{n=m}^{\infty} k(W_n)_{\varepsilon_n} = \bigcup_{k,m=1}^{\infty} k \bigcap_{n=m}^{\infty} (W_n)_{\varepsilon_n}.$$

It follows from Lemma 2.10 that $(W_n)_{\varepsilon_n}$ contains no balls of radius larger than ε_n . Therefore, for every m the set $\bigcap_{n=m}^{\infty} (W_n)_{\varepsilon_n}$ has void interior, so that $\sim G_1$ is of first category.

Similarly, $x \in G_2$ if and only if

$$(3.3) \quad \forall k \in \mathbb{N} \quad \forall m \in \mathbb{N} \quad \exists n \geq m \text{ such that } \text{dist}(kx, W_n) < \delta_n.$$

Indeed, if $x \in G_2$ then (3.3) is satisfied trivially. Conversely, suppose that x satisfies (3.3), let $\lambda > 0$ and $m \in \mathbb{N}$. Let $k \in \mathbb{N}$ be such that $\lambda \leq k$. By (3.3) there exists $n \geq m$ such that $\text{dist}(kx, W_n) < \delta_n$. But since W_n is convex and circled we

have $\text{dist}(\lambda x, W_n) \leq \text{dist}(kx, W_n) < \delta_n$. Note that $\text{dist}(kx, W_n) < \delta_n$ is equivalent to $x \in \frac{1}{k}(W_n + \delta_n B_X^\circ)$, where B_X° stands for the open unit ball of X . Thus,

$$G_2 = \bigcap_{k,m=1}^{\infty} \frac{1}{k} \bigcup_{n=m}^{\infty} (W_n + \delta_n B_X^\circ).$$

Since $\bigcup_{n=m}^{\infty} (W_n + \delta_n B_X^\circ)$ is open and dense in X for every m , then by Baire Theorem G_2 is residual. Since G_1 and G_2 are both residual, so is their intersection. \square

The next result is our main lemma. It shows that if W_n 's are as in the previous lemma, then there are always u and v in X which “see” the W_n 's very differently.

Lemma 3.4. *Let (W_n) be a nested increasing sequence of closed convex bounded circled sets, each with void interior, so that $Y = \bigcup_{n=1}^{\infty} W_n$ is dense in X . Then given any $u \in X \setminus Y$, there is a vector $v \in X \setminus Y$ so that*

$$(3.4) \quad \limsup_{n \rightarrow \infty} \frac{\text{dist}(v, W_n)}{\text{dist}(u, W_n)} = \infty \quad \text{and} \quad \liminf_{n \rightarrow \infty} \frac{\text{dist}(v, W_n)}{\text{dist}(u, W_n)} = 0.$$

Proof. Note that Y is a linear subspace of X of first category, so $Y \neq X$ by the Baire Category Theorem. Let $\tau_n = \text{dist}(u, W_n)$, then the sequence (τ_n) is decreasing and $\lim_{n \rightarrow \infty} \tau_n = 0$. Now let G_1 and G_2 be as in Lemma 3.3 with $\varepsilon_n = \sqrt{\tau_n}$ and $\delta_n = \tau_n/n$, then $G_1 \cap G_2$ is residual, hence non-empty. Let $v \in G_1 \cap G_2$, then

$$\frac{\text{dist}(v, W_n)}{\text{dist}(u, W_n)} > \frac{\varepsilon_n}{\tau_n} = \frac{1}{\sqrt{\tau_n}}$$

for infinitely many n , proving the first equality in (3.4), and

$$\frac{\text{dist}(v, W_n)}{\text{dist}(u, W_n)} < \frac{\delta_n}{\tau_n} = \frac{1}{n}$$

for infinitely many n , proving the second equality. \square

Remark 3.5. Notice that every v in $G_1 \cap G_2$, which is a residual set in X , satisfies (3.4). Furthermore, by definition the sets G_1 and G_2 are positively homogeneous, that is, with every vector they contain the entire ray passing through that vector, so that $G_1 \cap G_2$ is also positively homogeneous.

We are now ready for our first main structural result for strictly semi-transitive algebras.

Theorem 3.6. *Let \mathfrak{A} be norm-closed and strictly semi-transitive. Then $\mathfrak{A}x$ is closed for every $x \in X$.*

Proof. Let $X' = \overline{\mathfrak{A}x}$. Define $T: \mathfrak{A} \rightarrow X'$ by $T(A) = Ax$ for $A \in \mathfrak{A}$, and let

$$W_n = n\overline{T(B_{\mathfrak{A}})} = \overline{\{Ax \mid A \in \mathfrak{A}, \|A\| \leq n\}}$$

for all n . Of course, $W_n = nW_1$ and $W_n \subset X'$ for every n . It suffices to prove that W_1 has non-void relative interior in X' because in this case Remark 2.2 would imply that $X' = T(\mathfrak{A}) = \mathfrak{A}x$. Suppose not. Then, of course, each W_n is a closed convex bounded circled set, with void relative interior in X' . Pick any u in $X' \setminus \bigcup_{n=1}^{\infty} W_n$. It follows from Lemma 3.4 that we may choose v in $X' \setminus \bigcup_{n=1}^{\infty} W_n$ satisfying (3.4). Since \mathfrak{A} is strictly semi-transitive, there is an operator $A \in \mathfrak{A}$ so that $Au = v$ or $Av = u$. It follows from Remark 3.5 that by scaling v and A we can also assume without loss of generality that $\|A\| = 1$. Then $A(W_n) \subseteq W_n$ for every n . If $Au = v$, then

$$\text{dist}(v, W_n) \leq \text{dist}(Au, AW_n) \leq \text{dist}(u, W_n),$$

for every n , but this contradicts the first equality in (3.4). Similarly, the second equality in (3.4) is violated if $Av = u$. This contradiction completes the proof. \square

Theorem 3.7. *Let \mathfrak{A} be a strictly semi-transitive operator algebra on a complex Banach space X . If \mathfrak{A} is transitive, then it is WOT-dense in $\mathcal{L}(X)$.*

Proof. We may assume without loss of generality that \mathfrak{A} is norm-closed. But then \mathfrak{A} is strictly transitive by Theorem 3.6, hence WOT-dense in $\mathcal{L}(X)$. \square

We now arrive at a second basic structural result.

Theorem 3.8. *Let \mathfrak{A} is strictly semi-transitive. Then $\text{Lat } \mathfrak{A}$ is well-ordered by reverse inclusion. That is, every non-empty subset of $\text{Lat } \mathfrak{A}$ has a maximal element.*

Proof. We may without loss of generality assume that \mathfrak{A} is norm closed. Suppose that \mathcal{W} is a non-empty subset of $\text{Lat } \mathfrak{A}$ with no maximal element. Then we can find an infinite sequence $X_1 \subsetneq X_2 \subsetneq X_3 \subsetneq \dots$ in \mathcal{W} . Let $Y = \bigcup_{n=1}^{\infty} X_n$. Then Y is not closed; indeed each X_n is nowhere dense in the induced topology of \overline{Y} , so that Y is of first category in \overline{Y} . Furthermore, \overline{Y} is \mathfrak{A} -invariant (i.e., $\overline{Y} \in \text{Lat } \mathfrak{A}$), and \mathfrak{A} reduced to \overline{Y} is again a strictly semi-transitive algebra. Set $W_n = nB_{X_n}$ for all n . Again by Lemma 3.4 we may choose $u, v \in \overline{Y} \setminus Y$ satisfying (3.4). Again, choose $A \in \mathfrak{A}$ of norm one so that $Au = v$ or $Av = u$. Then $A(W_n) \subseteq W_n$ because X_n is an invariant subspace for \mathfrak{A} . Now the rest of the argument for Theorem 3.6 yields a contradiction. \square

The next result yields the surprising fact that every \mathfrak{A} -invariant linear subspace is in $\text{Lat } \mathfrak{A}$.

Corollary 3.9. *Let \mathfrak{A} be unital, norm-closed, and strictly semi-transitive. Then every \mathfrak{A} -invariant linear subspace Y of X is closed. Furthermore, $Y = \mathfrak{A}x$ for some $x \in X$.*

Proof. Let \mathcal{W} be the set of all orbits contained in Y , i.e., $\mathcal{W} = \{\mathfrak{A}x \mid \mathfrak{A}x \subseteq Y\}$. Note that $\mathcal{W} \subseteq \text{Lat } \mathfrak{A}$ by Theorem 3.6. Then Theorem 3.8 yields that \mathcal{W} has a maximal element, say $\mathfrak{A}x$. We claim that $Y = \mathfrak{A}x$. Suppose not, then there exists $y \in Y \setminus \mathfrak{A}x$. Then $\mathfrak{A}x \subseteq \mathfrak{A}y \subseteq Y$ by Proposition 3.1, and $\mathfrak{A}x \neq \mathfrak{A}y$ because $y \in \mathfrak{A}y \setminus \mathfrak{A}x$. But this contradicts the maximality of $\mathfrak{A}x$. Hence $Y = \mathfrak{A}x$, so that Y is closed by Theorem 3.6. \square

The following result may alternatively be deduced from [RR73, Theorem 4.4] and Theorem 3.6. We present a simple direct proof of it.

Corollary 3.10. *Let \mathfrak{A} be unital, norm-closed, and strictly semi-transitive. Then \mathfrak{A} has (a residual set of) strictly cyclic vectors.*

Proof. If \mathfrak{A} is transitive, then every non-zero vector is strictly cyclic by Theorem 3.6. Otherwise, \mathfrak{A} has a maximal proper closed invariant subspace Y by Theorem 3.8. Then $X \setminus Y$ is residual. But if $x \in X \setminus Y$, then $\mathfrak{A}x = X$ by Theorem 3.6 and the maximality of Y . \square

We conclude this section with an explicit description of $\text{Lat } \mathfrak{A}$ when \mathfrak{A} a strictly semi-transitive algebra acting on a separable space, and a description of all “full” strictly semi-transitive algebras.

Corollary 3.11. *Let X be separable and \mathfrak{A} be strictly semi-transitive. Then there exists a countable ordinal η and a family of closed subspaces $(X_\alpha)_{\alpha \leq \eta}$ such that*

- (i) $X_0 = X$, $X_\eta = \{0\}$, and $X_{\alpha+1} \subsetneq X_\alpha$ for all $\alpha < \eta$;
- (ii) $\text{Lat } \mathfrak{A} = \{X_\alpha \mid \alpha \leq \eta\}$.

Proof. This holds for arbitrary X by Theorem 3.8, except that η need not be countable. However, if X is assumed to be separable, it follows that η must be countable, otherwise X would have an uncountable strictly decreasing chain of closed subspaces, which is impossible. \square

Remark 3.12. Assume that η and X_α 's are as in the above statement. It follows that

$$(3.5) \quad \text{if } \alpha_1 < \alpha_2 < \dots \text{ with } \alpha_n < \eta \text{ for all } n \\ \text{and } \alpha_n \rightarrow \alpha, \text{ then } X_\alpha = \bigcap_{n=1}^{\infty} X_{\alpha_n}.$$

Indeed, let $Y = \bigcap_{n=1}^{\infty} X_{\alpha_n}$, then $Y \in \text{Lat } \mathfrak{A}$, hence $Y = X_{\beta}$ for some β . But then $\beta \geq \alpha_n$ for all n , hence $\beta \geq \alpha$. This yields $X_{\beta} \subseteq X_{\alpha}$, but $X_{\alpha} \subseteq X_{\alpha_n}$ for all n , so that

$$\bigcap_{n=1}^{\infty} X_{\alpha_n} = X_{\beta} \subseteq X_{\alpha} \subseteq \bigcap_{n=1}^{\infty} X_{\alpha_n},$$

hence $X_{\beta} = X_{\alpha}$.

The above remark motivates our final result of this section, which is a partial converse to Theorem 3.8.

Proposition 3.13. *Let X be a separable Banach space, η a countable ordinal with $\eta > 1$, and $(X_{\alpha})_{\alpha \leq \eta}$ a family of closed subspaces of X satisfying (i) of Corollary 3.11 and (3.5). Let*

$$\mathfrak{A} = \{T \in \mathcal{L}(X) \mid TX_{\alpha} \subseteq X_{\alpha} \text{ for all } \alpha \leq \eta\}.$$

Then \mathfrak{A} is strictly semi-transitive and $\text{Lat } \mathfrak{A} = \{X_{\alpha} \mid \alpha \leq \eta\}$.

Proof. Let $x, y \in X \setminus \{0\}$. Put

$$\alpha = \sup\{\gamma \leq \eta \mid x \in X_{\gamma}\} \quad \text{and} \quad \beta = \sup\{\gamma \leq \eta \mid y \in X_{\gamma}\}$$

It follows from (3.5) that $x \in X_{\alpha}$ and $y \in X_{\beta}$. Without loss of generality, $\alpha \leq \beta$ and, of course, $\beta < \eta$. Then by the Hahn Banach Theorem, there is an $f \in X^*$ with $X_{\beta+1} \subseteq \ker f$ and $f(x) = 1$. Define $T \in \mathcal{L}(X)$ by $Tu = f(u)y$ for all $u \in X$. Evidently $Tx = y$. If $\gamma \geq \beta + 1$ then $X_{\gamma} \subseteq X_{\beta+1} \subseteq \ker f$, so $TX_{\gamma} = \{0\}$, and if $\gamma \leq \beta$, then $X_{\beta} \subseteq X_{\gamma}$ and, of course, $TX = [y] \subseteq X_{\gamma}$. In either case, $TX_{\gamma} \subseteq X_{\gamma}$. Thus, $T \in \mathfrak{A}$, and we have shown that \mathfrak{A} is strictly semi-transitive.

It follows by definition that $\{X_{\alpha} \mid \alpha \leq \eta\} \subseteq \text{Lat } \mathfrak{A}$, so we must prove that, conversely, if $Y \in \text{Lat } \mathfrak{A}$, then $Y = X_{\alpha}$ for some α . Assume $Y \neq \{0\}$ or X , and let α be the greatest such that $Y \subseteq X_{\alpha}$. We claim that $Y = X_{\alpha}$. Suppose this were false. Now since $Y \subsetneq X_{\alpha}$, it follows from Proposition 3.1 and the definition of α that $X_{\alpha+1} \subsetneq Y$. Now we may choose $z \in X_{\alpha} \setminus Y$ and $y \in Y \setminus X_{\alpha+1}$. But then by our initial discussion, since neither z nor y are in $X_{\alpha+1}$, yet both belong to X_{α} , there is a $T \in \mathfrak{A}$ with $Ty = z$. But then $TY \not\subseteq Y$, contradicting the assumption that $Y \in \text{Lat } \mathfrak{A}$. \square

Example 3.14. Let $X = \ell_p$ for $1 \leq p < +\infty$ and let (e_i) be the standard basis of ℓ_p . Let \mathfrak{A} be the set of all the bounded operators A on ℓ_p such that

- (i) the matrix of A is lower triangular, that is, $Ae_n \in [e_i]_{i=n}^{\infty}$ for every n , and
- (ii) the matrix of A contains only finitely many non-zero columns or, equivalently, there exists a positive integer n such that $Ae_i = 0$ for all $i \geq n$.

It is easy to see that \mathfrak{A} is an algebra. Show that \mathfrak{A} is strictly semi-transitive. Given $x = (x_i)$ and $y = (y_i)$ in ℓ_p , let x_k be the first non-zero component of x and y_m the first non-zero component of y . Without loss of generality, $k \leq m$. Define an operator A as follows: $Ae_k = \frac{1}{x_k}y$ and $Ae_i = 0$ for all $i \neq k$. Then $A \in \mathfrak{A}$ and $Ax = y$.

Notice that this algebra is not norm closed. However, it follows that the algebra of all lower-triangular compact operators and the algebra of all lower-triangular bounded operators are strictly semi-transitive.

On the other hand, consider the algebra of all upper-triangular bounded operators. The linear subspace of all sequences with finitely many non-zero entries is invariant under the algebra, but it is not closed in ℓ_p . It follows from Corollary 3.9 that the upper-triangular algebra is not strictly semi-transitive. However, it is easy to see that this algebra is unicellular, hence semi-transitive.

4. OPERATOR RANGES IN BANACH SPACES

A subspace Y of a Hilbert space H is termed an operator range by Foias [Foi72] if there exists an operator $T \in \mathcal{L}(H)$ with $Y = T(H)$. We generalize this to arbitrary Banach spaces X as follows.

Definition 4.1. Let X be a Banach space and Y a linear subspace of X . Then Y is called an **operator range** if there is a closed subspace $\mathcal{M} \subseteq X^n$ for some $n \geq 1$ and a bounded linear operator $\vec{T}: \mathcal{M} \rightarrow X$ with $Y = \text{Range } \vec{T}$. In this case, Y is called an **operator range of order n** . Finally, Y is called an **injective operator range** if \vec{T} may be chosen one-to-one.

Of course, every closed subspace of X is an operator range of order one. It is proved in [Foi72] (see also [RR73, Theorem 8.9]) that if \mathfrak{A} is a WOT-closed algebra of operators on a Hilbert space with no non-trivial invariant operator ranges, then $\mathfrak{A} = \mathcal{L}(H)$. The main result of this section generalizes this fact.

Theorem 4.2. *Let \mathfrak{A} be a subalgebra of $\mathcal{L}(X)$ for a complex Banach space X , and $n \geq 1$. Then either \mathfrak{A} has a non-trivial invariant injective operator range of order n , or \mathfrak{A} is n -transitive.*

Note that \mathfrak{A} is WOT-dense in $\mathcal{L}(X)$ if and only if \mathfrak{A} is n -transitive for all n , c.f. [RR73, Theorem 7.1]. This yields the following consequence.

Corollary 4.3. *Let \mathfrak{A} be a subalgebra of $\mathcal{L}(X)$ for a complex Banach space X such that \mathfrak{A} has no non-trivial invariant injective operator ranges. Then \mathfrak{A} is WOT-dense in $\mathcal{L}(X)$.*

Remark 4.4. It is easily seen that every operator range in a Hilbert space is injective of order one. It seems we need the concept of operator range of order n , however, to obtain a general result. Also, we don't know if every operator range in a Banach space is also an injective operator range. This question was posed by P. Rosenthal.

Before proving the theorem, we need some preliminary results, which also hold in real Banach spaces. We start with the following simple consequences of the definition of an operator range, given in [Foi72] in a Hilbert space case.

Proposition 4.5. *Let Y_1 and Y_2 be operator ranges in X . Then $Y_1 + Y_2$ and $Y_1 \cap Y_2$ are operator ranges.*

Proof. For $i = 1, 2$ choose n_i , \mathcal{M}_i a closed subspace of X^{n_i} , and $T_i: \mathcal{M}_i \rightarrow X$ bounded linear with $Y_i = \text{Range } T_i$. Now if

$$\mathcal{M} = \mathcal{M}_1 \oplus \mathcal{M}_2 \subseteq X^{n_1} \oplus X^{n_2} = X^{n_1+n_2},$$

and $T = T_1 \oplus T_2$, then $T(\mathcal{M}) = Y_1 + Y_2$, hence $Y_1 + Y_2$ is an operator range. The argument for $Y_1 \cap Y_2$ is not quite obvious. Let

$$\mathcal{W} = \{(w_1, w_2) \in \mathcal{M} \mid T_1 w_1 = T_2 w_2\}.$$

Evidently, \mathcal{W} is a linear subspace of \mathcal{M} . If $(u_n, v_n) \rightarrow (u, v)$ with $(u_n, v_n) \in \mathcal{W}$ for all n , then $T_1 u = \lim_{n \rightarrow \infty} T_1 u_n = \lim_{n \rightarrow \infty} T_2 v_n = T_2 v$, so that $(u, v) \in \mathcal{W}$, hence \mathcal{W} is closed. Show that $T(\mathcal{W}) = Y_1 \cap Y_2$. Indeed, if $(w_1, w_2) \in \mathcal{W}$ then $T(w_1, w_2) = T_1 w_1 + T_2 w_2 = 2T_1 w_1 = 2T_2 w_2 \in Y_1 \cap Y_2$. Conversely, if $y \in Y_1 \cap Y_2$, then there exist $w_i \in \mathcal{M}_i$ with $T_i w_i = y$ for $i = 1, 2$. Then $(\frac{w_1}{2}, \frac{w_2}{2}) \in \mathcal{W}$ and $T(\frac{w_1}{2}, \frac{w_2}{2}) = y$. \square

The following theorems refine some of the results in [Arv67], we will use them in the proof of Theorem 4.2.

Theorem 4.6. *Let \mathfrak{A} be $(n-1)$ -transitive for some $n > 1$ and $\mathcal{M} = \overline{\mathfrak{A}^{(n)} \vec{x}}$ for some linearly independent n -tuple $\vec{x} = (x_1, \dots, x_n)$ in X^n . Then either $\mathcal{M} = X^n$ or \mathcal{M} consists only of linearly independent n -tuples and zero.*

Proof. First, we show that if \mathcal{N} is a closed $\mathfrak{A}^{(n)}$ -invariant subspace of X^n such that $\mathcal{M} \subseteq \mathcal{N}$, then \mathcal{N} satisfies the following two properties.

- (i) If \mathcal{N} contains an n -tuple of the form $(u_1, \dots, u_{n-1}, 0)$ where u_1, \dots, u_{n-1} are linearly independent, then $\mathcal{N} = X^n$.
- (ii) If \mathcal{N} contains both linearly independent and non-zero linearly dependent n -tuples, then $\mathcal{N} = X^n$.

Note that if \mathfrak{A} is unital, then $\vec{x} \in \mathcal{M}$; in this case applying (ii) with $\mathcal{N} = \mathcal{M}$ would immediately yield the conclusion of the theorem.

To prove (i), notice that since \mathfrak{A} is $(n-1)$ -transitive then $X^{n-1} \oplus \{0\} \subseteq \mathcal{N}$. Since \mathfrak{A} is transitive, then $Ax_n \neq 0$ for some $A \in \mathfrak{A}$. It follows that there exists (w_1, \dots, w_n) in \mathcal{M} with $w_n \neq 0$. But $X^{n-1} \oplus \{0\} \subseteq \mathcal{N}$ implies $(w_1, \dots, w_{n-1}, 0) \in \mathcal{N}$, so that $(0, \dots, 0, w_n) \in \mathcal{N}$, which yields $\{0\}^{n-1} \oplus X \subseteq \mathcal{N}$. Hence, $\mathcal{N} = X^n$.

To prove (ii) suppose that \mathcal{N} contains a linearly independent n -tuple $\vec{v} = (v_1, \dots, v_n)$ and a non-zero linearly dependent n -tuple $\vec{y} = (y_1, \dots, y_n)$. Without loss of generality, y_1, \dots, y_k are linearly independent for some $k < n$, and $y_i = \sum_{j=1}^k \alpha_{ij} y_j$ as $i = k+1, \dots, n$. Since \mathfrak{A} is k -transitive, there exists a sequence (A_m) in \mathfrak{A} such that $A_m y_1 \rightarrow v_1$ and $A_m y_i \rightarrow 0$ as $i = 2, \dots, k$. Then for $k+1 \leq i \leq n$ we have $A_m y_i \rightarrow \alpha_{i,1} v_1$. It follows that

$$A_m^{(n)} \vec{y} \rightarrow (v_1, 0, \dots, 0, \alpha_{k+1,1} v_1, \dots, \alpha_{n,1} v_1),$$

so that the latter n -tuple belongs to \mathcal{N} . Subtracting it from \vec{v} we see that \mathcal{N} contains an element of the form $(0, z_2, \dots, z_n)$, where z_2, \dots, z_n are linearly independent. Now $\mathcal{N} = X^n$ by (i).

To complete the proof, assume that \mathcal{M} contains a linearly dependent non-zero n -tuple and $\mathcal{M} \neq X^n$. Notice that $\mathcal{M} + [\vec{x}]$ is \mathfrak{A} -invariant, so that $\mathcal{M} + [\vec{x}] = X^n$ by (ii). In particular, there exists $\vec{w} \in \mathcal{M}$ and a scalar λ such that $\vec{w} + \lambda \vec{x} = (x_1, 0, \dots, 0)$. Then

$$\vec{w} = ((1-\lambda)x_1, -\lambda x_2, \dots, -\lambda x_n).$$

If $\lambda = 1$ then $\vec{w} = (0, -x_2, \dots, -x_n)$, so that $\mathcal{M} = X^n$ by (i), contradiction. If $\lambda \neq 1$ and $\lambda \neq 0$ then all the components of \vec{w} are linearly independent, hence $\mathcal{M} = X^n$ by (ii) which, again, would contradict our assumptions. It follows that $\lambda = 0$, and so $(x_1, 0, \dots, 0) \in \mathcal{M}$. Similarly, $(0, \dots, 0, x_i, 0, \dots, 0) \in \mathcal{M}$ for every $i = 1, \dots, n$. It follows that $\vec{x} \in \mathcal{M}$, so that $\mathcal{M} = \mathcal{M} + [\vec{x}] = X^n$, contradiction. \square

Remark 4.7. Assuming that \mathfrak{A} is unital and $(n-1)$ -transitive, it can be shown that any $\mathcal{M} \in \text{Lat } \mathfrak{A}^{(n)}$ is either in $\text{Lat}(\mathcal{L}(X))^{(n)}$ (that is, consists of n -tuples satisfying a fixed set of linear dependence relations), or consists only of linearly independent n -tuples and zero.

Corollary 4.8. *Let \mathfrak{A} be transitive and assume that for any linearly independent x_1 and x_2 in X there exists an operator $A \in \mathfrak{A}$ with (Ax_1, Ax_2) non-zero and linearly dependent. Then \mathfrak{A} is WOT-dense in $\mathcal{L}(X)$.*

Proof. We will show by induction that \mathfrak{A} is n -transitive for all $n \geq 1$, then $\overline{\mathfrak{A}}^{\text{wot}} = \mathcal{L}(X)$ by [RR73, Theorem 7.1]. It is given that \mathfrak{A} is 1-transitive. Suppose that \mathfrak{A} is $(n-1)$ -transitive for some $n > 1$, and let $\vec{x} = (x_1, \dots, x_n)$ be a linearly independent n -tuple. Choose $A \in \mathfrak{A}$ so that (Ax_1, Ax_2) is non-zero and linearly dependent. Then \vec{x} is cyclic for $\mathfrak{A}^{(n)}$ by Theorem 4.6. Hence, \mathfrak{A} is n -transitive. \square

Our next result is a refinement of Corollary 2.5 in [Arv67] (see also Lemma 8.8 in [RR73]).

Theorem 4.9 (Graph Theorem). *Suppose that \mathfrak{A} is n -transitive. Then \mathfrak{A} is not $(n+1)$ -transitive if and only if there exists a closed operator $\vec{T}: \mathcal{D} \subseteq X \rightarrow X^n$ where \mathcal{D} is a dense \mathfrak{A} -invariant linear subspace of X and $\vec{T} = T_1 \oplus \dots \oplus T_n$, each T_i commutes with \mathfrak{A} , and (x, T_1x, \dots, T_nx) is linearly independent for each non-zero $x \in \mathcal{D}$. In particular, each T_i is one-to-one and non-scalar.*

Proof. Suppose that \mathfrak{A} is n -transitive. If \mathfrak{A} is not $(n+1)$ -transitive then by Theorem 4.6 there exists a closed $\mathfrak{A}^{(n+1)}$ -invariant subspace \mathcal{M} of X^{n+1} such that every non-zero element of \mathcal{M} is a linearly independent $(n+1)$ -tuple. We claim that \mathcal{M} is the graph of a closed operator $\vec{T}: \mathcal{D} \subseteq X \rightarrow X^n$ satisfying the required conditions. Indeed, let $\mathcal{D} = \{x_0 \mid (x_0, x_1, \dots, x_n) \in \mathcal{M}\}$. For $(x_0, x_1, \dots, x_n) \in \mathcal{M}$ define $T_i x_0 = x_i$ for $i = 1, \dots, n$, and put $\vec{T} = T_1 \oplus \dots \oplus T_n$. Notice that \vec{T} is well-defined: suppose that (x_0, x_1, \dots, x_n) and (x_0, x'_1, \dots, x'_n) are both in \mathcal{M} , then $(0, x_1 - x'_1, \dots, x_n - x'_n) \in \mathcal{M}$, but this $(n+1)$ -tuple is linearly dependent, hence equals zero, so that $x_i = x'_i$ for all $1 \leq i \leq n$. Clearly, \vec{T} is closed because \mathcal{M} is closed.

Now let $x \in \mathcal{D}$, then $(x, T_1x, \dots, T_nx) \in \mathcal{M}$. Since \mathcal{M} is $\mathfrak{A}^{(n+1)}$ -invariant, if $A \in \mathfrak{A}$ then $(Ax, AT_1x, \dots, AT_nx) \in \mathcal{M}$. It follows that $Ax \in \mathcal{D}$, so that \mathcal{D} is \mathfrak{A} -invariant and, therefore, dense in X . Furthermore, it also follows that $AT_ix = T_i(Ax)$ for all $i = 1, \dots, n$, so that T_i commutes with \mathfrak{A} . Finally, (x, T_1x, \dots, T_nx) is in \mathcal{M} for each non-zero $x \in \mathcal{D}$, so that this $(n+1)$ -tuple is linearly independent. It follows that each T_i is non-scalar. It also follows that $T_ix = 0$ implies $x = 0$, so that T_i is one-to-one.

To prove the converse, suppose that \vec{T} is such an operator, and let \mathcal{M} be the graph of \vec{T} . Then \mathcal{M} is closed and every non-zero element of \mathcal{M} is linearly independent. It follows that $\mathcal{M} \neq X^{n+1}$. It is easy to see that \mathcal{M} is $\mathfrak{A}^{(n+1)}$ -invariant, so that every linearly independent $(n+1)$ -tuple in \mathcal{M} is not cyclic for $\mathfrak{A}^{(n+1)}$, hence \mathfrak{A} is not $(n+1)$ -transitive. \square

Now we are ready to prove Theorem 4.2.

Proof of Theorem 4.2. Suppose that \mathfrak{A} has no non-trivial invariant injective operator ranges of order n for some n . It follows that \mathfrak{A} has no non-trivial invariant injective operator ranges of order k whenever $1 \leq k \leq n$. In particular, \mathfrak{A} is transitive.

Note first, that $\mathfrak{A}' = [I]$. Indeed, let $T \in \mathfrak{A}'$. Pick $\lambda \in \sigma(T)$, and show that $T = \lambda I$. Suppose not, then $\text{Range}(T - \lambda I) \neq 0$. Since $\text{Null}(T - \lambda I)$ is an \mathfrak{A} -invariant closed subspace then $T - \lambda I$ is one-to-one. Then $\text{Range}(T - \lambda I)$ is an \mathfrak{A} -invariant injective operator range of order 1, so that $\text{Range}(T - \lambda I) = X$. But this would mean that $T - \lambda I$ is invertible, contradiction.

Suppose that \mathfrak{A} is not n -transitive, and show that this leads to a contradiction. Let k be the greatest such that \mathfrak{A} is k -transitive but not $(k+1)$ -transitive, then $1 \leq k < n$. Let $\vec{T}: \mathcal{D} \rightarrow X^k$ where $T = T_1 \oplus \cdots \oplus T_k$ is as in Theorem 4.9. Let \mathcal{M} be the graph of \vec{T} . Also, let $P: \mathcal{M} \rightarrow X$ be the projection on the first component. Then $\mathcal{D} = \text{Range } P$ is an injective operator range of order $(k+1)$. It follows that $\mathcal{D} = X$, so that $T_i \in \mathcal{L}(X)$ for each $i = 1, \dots, k$ and, therefore, $T_i \in \mathfrak{A}'$. But then T_i have to be scalar because $\mathfrak{A}' = [I]$, contradiction. \square

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