REAL PARTITIONS OF MEASURE SPACES

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The notion of real partition introduced in the article presents a convenient tool for transferring many properties of the Lebesgue measure to a broad class of measure spaces. In particular, with the help of the notion introduced, existence is proved for measure-preserving mappings of several probability spaces onto the unit segment with the Lebesgue measure.

We recall that a measure space is called *nonatomic* if any measurable set in it of positive measure includes a measurable subset with a lesser but nonzero measure. A measure space is said to be *pointwise nonatomic* if all singletons in it are measurable and have zero measure. A space $(X, \mathcal{A}, \mu)$, with X a complete separable metric space and $\mathcal{A}$ its Borel σ-algebra, is referred to as *Polish space*.

We denote the Borel and Lebesgue algebras on the unit segment by Bo and Lb respectively. The symbol m will stand for the Lebesgue measure.

Theorem 1 (E. Marczewski (Szpilrajn) [1, 2]). *Every pointwise nonatomic probability Polish space $(X, \mathcal{A}, \mu)$ is isomorphic to the space $B = ([0,1], \mathrm{Bo}, m)$; i.e., there exists a mapping $T : X \to [0,1]$ which is one-to-one to within sets of zero measure and such that T and T^{-1} are measure-preserving.*

We introduce the notion of real partition.

DEFINITION 1. A *real partition* of a probability space $(X, \mathcal{A}, \mu)$ is defined to be a partition $(D_t)_{t\in[0,1]}$ of the space, such that $\mu(A_t^-) = \mu(A_t^+) = t$ for all $t \in [0,1]$, where $A_t^- = \bigcup\limits_{s<t} D_s$ and $A_t^+ = \bigcup\limits_{s\leq t} D_s$. A real partition is called *nondegenerate* if the set D_t is nonempty for every $t \in [0,1]$.

Theorem 2. *Every complete nonatomic probability space admits a real partition.*

PROOF. Let $X = (X, \mathcal{A}, \mu)$ be a complete nonatomic probability space. Let $\mathcal{S}$ be the set of chains in $\mathcal{A}$ of the form $(E_t)_{t\in T\subset[0,1]}$ such that $\mu(E_t) = t$ and $E_s \subset E_t$ for $s < t$. Introduce in $\mathcal{S}$ some natural order by putting $(E_t^1)_{t\in T_1} < (E_t^2)_{t\in T_2}$ when $T_1 \subset T_2$ and $E_t^1 = E_t^2$ for all t in T_1. By Zorn's lemma, there exists a maximal chain $(E_t)_{t\in T_0} \subset [0,1]$ in $\mathcal{S}$.

Demonstrate that $T_0 = [0,1]$. Let $t_0 \in [0,1] \setminus T_0$, $a = \sup\{t \in T_0 \mid t < t_0\}$, and $b = \inf\{t \in T_0 \mid t > t_0\}$. Prove that $a, b \in T_0$. Take an increasing sequence $(t_n)_{n\in\mathbb{N}} \subset T_0$ converging to a, and consider the set $A = \bigcup\limits_{n\in\mathbb{N}} E_{t_n}$. It is obvious that $A \in \mathcal{A}$ and $\mu(A) = a$. If $t \in T_0$ and $t < a$, then there exists an n such that $t < t_n$. Consequently, $E_t \subset A$. But if $t \in T_0$ and $t > a$, then all E_{t_n} lie in E_t; thus, $A \subset E_t$. Since $(E_t)_{t\in T_0}$ is maximal, we obtain $a \in T_0$. Analogously, $b \in T_0$. Assume that $a < b$; then $\mu(E_b \setminus E_a) = b - a > 0$. In this case, owing to the fact that X is nonatomic, there exists a subset $F \subset E_b \setminus E_a$ such that $0 < \mu(F) < b - a$. Hence $E_a \subset E_a \cup F \subset E_b$ and $\mu(E_a \cup F) = a + \mu(F) < b$, which contradicts the fact that $(E_t)_{t\in T_0}$ is maximal. Thus, $T_0 = [0,1]$.

Put $A_t^- = \bigcup\limits_{s<t} E_s$ and $A_t^+ = \bigcap\limits_{s>t} E_s$. Since the space is complete and the inner and outer measures of the sets A_t^- and A_t^+ coincide and equal t, it follows that A_t^- and A_t^+ are measurable and have measure t.

It is obvious that $A_{t_1}^- \subset E_{t_1} \subset A_{t_1}^+ \subset A_{t_2}^- \subset E_{t_2} \subset A_{t_2}^+$ for $t_1 < t_2$. Consider $D_t := A_t^+ \setminus A_t^-$. Then $D_{t_1} \cap D_{t_2} = \varnothing$. Demonstrate that $A_t^- = \bigcup\limits_{s<t} D_s$. Given an $x_0 \in A_t^-$, put $t_0 = \inf\{t \mid x_0 \in E_t\}$ and prove that $x_0 \in D_{t_0}$. If $s \in]t_0, t[$, then x_0 belongs to E_s and $A_{t_0}^+$. Similarly, $x_0 \notin A_{t_0}^-$, for there would otherwise exist an $s < t_0$ such that $x_0 \in E_s$.

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Consequently, $x_0 \in A^+_{t_0} \setminus A^-_{t_0} = D_{t_0}$. Thus, $A^-_t = \bigcup_{s<t} D_s$ and $A^+_t = \bigcup_{s\le t} D_s$.

To complete the proof, it remains to observe that $A^-_0 = \varnothing$ and $A^+_1 = X$. ■

DEFINITION 2. We say that a measure space *includes a Cantor set* if in this space there is a subset of zero measure that has the cardinality of the continuum.

The following observation was made by S. A. Mal′ugin.

Theorem 3. *Let $X = (X, \mathcal{A}, \mu)$ be a complete nonatomic measure space. If the cardinality of the space X is greater than the cardinality of the continuum, then X contains a Cantor set.*

PROOF. Without loss of generality, we assume that X is a probability space; otherwise we can take a subset of finite measure. In this case, by Theorem 2, there exists a real partition $(D_t)_{t\in[0,1]}$; moreover, $\mu(D_t) = 0$ for all $t \in [0,1]$. If the cardinality of each of the D_t's were less than the cardinality of the continuum, then the cardinality of the space X would be less than the cardinality of the continuum, which fact would contradict the hypothesis. Thus, there exists t_0 such that the cardinality of the continual subset D_{t_0} is not less than the cardinality of the continuum. Since the measure is complete, every continual subset D_{t_0} is a Cantor set in X. ■

By using Theorem 2, we can essentially widen the class of spaces in Theorem 1 by replacing isomorphy with a weaker condition.

Theorem 4. *For every complete nonatomic probability space that includes a Cantor set, there exists a measure-preserving epimorphism of the space onto the unit segment with the Lebesgue measure.*

PROOF. Let $X = (X, \mathcal{A}, \mu)$ be a complete nonatomic probability space and let a set C of zero measure have the cardinality of the continuum. Our aim is to construct an epimorphism $\varphi : X \to I$, where $I = ([0,1], \mathrm{Lb}, m)$.

Let $C = \{C_t\}_{t\in[0,1]}$. Consider the space $(\tilde{X}, \widetilde{\mathcal{A}}, \tilde{\mu})$, where $\tilde{X} = X \setminus C$, $\widetilde{\mathcal{A}} = \{A \setminus C \mid A \in \mathcal{A}\}$, and $\tilde{\mu} = \mu|_{\widetilde{\mathcal{A}}}$. It is a complete nonatomic probability space; therefore, by Theorem 2, it admits a real partition, i.e., a partition $(\widetilde{D}_t)_{t\in[0,1]}$ such that $\tilde{\mu}(\widetilde{A}^-_t) = \tilde{\mu}(\widetilde{A}^+_t) = t$, where $\widetilde{A}^-_t = \bigcup_{s<t} \widetilde{D}_s$ and $\widetilde{A}^+_t = \bigcup_{s\le t} \widetilde{D}_s$. Put $D_t = \widetilde{D}_t \cup \{C_t\}$, $A^-_t := \bigcup_{s<t} D_s$, and $A^+_t := \bigcup_{s\le t} D_s$. The completeness of X and the equality $\mu(C) = 0$ yield measurability of A^-_t and A^+_t; moreover, $\mu(A^-_t) = \mu(A^+_t) = t$. Thereby, $(D_t)_{t\in[0,1]}$ presents a nondegenerate real partition of X.

We define the epimorphism $\varphi : X \to [0,1]$ by the rule $\varphi|_{D_t} = t$. In this event, $\varphi(A^-_t) = [0,t)$ and $\varphi(A^+_t) = [0,t]$. Since the segments of the forms $[0,t)$ and $[0,t]$, with the measure equal to t by definition, generate the Borel σ-algebra with the Lebesgue measure, the mapping φ^{-1} preserves the Lebesgue measure of Borel subsets in $[0,1]$. From the completeness of the space X, it follows that φ also preserves the measure of Lebesgue subsets. ■

As an illustration of Theorem 4, we present a simple proof for the following assertion:

Theorem 5. *In every nonatomic probability space there is a nonmeasurable set.*

This fact occurs implicitly in the article [3].

PROOF. Suppose that there exists a nonatomic probability space $X = (X, \mathcal{P}(X), \mu)$. The fact that the space X is nonatomic implies that X is pointwise nonatomic. Since a pointwise nonatomic measure cannot be defined on the algebra of all subsets of a set whose cardinality is less or equal to the cardinality of the continuum (see [3, 4]), we conclude that the cardinality of the space X is greater than the cardinality of the continuum. Now Theorems 3 and 4 guarantee existence for an epimorphism $\varphi : X \to I = ([0,1], \mathrm{Lb}, m)$; moreover, the sets $D_t = \varphi^{-1}(t)$ form a nondegenerate real partition of the space X. Denote this partition by ξ. Consider the σ-algebras

$$\mathcal{S} = \varphi^{-1}(\mathrm{Lb}) = \{\varphi^{-1}(A) \mid A \in \mathrm{Lb}\}, \quad \mathcal{P} = \varphi^{-1}(\mathcal{P}[0,1]) = \{\varphi^{-1}(A) \mid A \subset [0,1]\}.$$

Take the quotient algebras by the partition ξ (the notion of the quotient algebra by a partition can be found, for instance, in [5]). It is obvious that the space $(X_\xi, \mathcal{S}_\xi, \mu_\xi)$ is isomorphic to I. Observe that $\mathcal{S}_\xi \subset \mathcal{P}_\xi = \mathcal{P}(X_\xi) = (\mathcal{P}(X))_\xi$. Consequently, the space $(X_\xi, \mathcal{P}(X_\xi), \mu_\xi)$ is pointwise nonatomic (since $\mathcal{S}_\xi \subset \mathcal{P}(X_\xi)$) and has the cardinality of the continuum. The contradiction obtained proves the theorem. ■

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