A REMARK ON INVARIANT SUBSPACES OF POSITIVE OPERATORS

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Abstract. If S, T, R, and K are non-zero positive operators on a Banach lattice such that $S \leftrightarrow T \leftrightarrow R \leqslant K$, where "$\leftrightarrow$" stands for the commutation relation, T is non-scalar, and K is compact, then S has an invariant subspace.

Throughout this note, X is a (real or complex) Banach lattice. For two operators S and T on X, the notation $S \leftrightarrow T$ means that S and T commute. A (norm closed) subspace Y of X is said to be invariant under an operator T in $L(X)$ if $\{0\} \neq Y \neq X$ and $TY \subseteq Y$. We follow the notations and terminology of [AA02].

There have been many extensions of Lomonosov's theorem [Lom73] to positive operators; see Chapter 10 of [AA02] for a review of the subject. In particular, if $T \leftrightarrow R \geqslant K$ for some positive non-zero operators T, R, and K with T quasinilpotent and K compact, then T has an invariant subspace (even an invariant closed ideal). The condition $T \leftrightarrow R \geqslant K$ can be replaced with $T \leftrightarrow R \leqslant K$ or, even more generally, with $T \leftrightarrow R \geqslant C \leqslant K$ for some non-zero positive operator C; in the latter case, T is said to be *compact friendly*. There have been several more recent similar extensions of Lomonosov's theorem to positive quasinilpotent operators: [Drn01, IM04, AT05, ÇE07, FTT08, PT09, Ges09, FV09, DK11]. In this note we do not require that T be quasinilpotent. Our result was motivated by Theorem 3.5 of [ÇM11], where quasinilpotence is not required either.

Theorem 1. *Suppose that S, T, R, and K are non-zero positive operators on a Banach lattice such that $S \leftrightarrow T \leftrightarrow R \leqslant K$, T is non-scalar, and K is compact. Then S has an invariant subspace.*

Proof. Suppose that S has no invariant subspaces. Let $\widetilde{S} = \sum_{n=0}^{\infty} t^n S^n$ where t is a positive real such that series converges. Then $\widetilde{S} \geqslant I$, $\widetilde{S} \geqslant tS$, and $\widetilde{S}$ commutes with T.

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Claim: for every $x > 0$, the vector $\widetilde{S}x$ is quasi-interior, that is, the order ideal J generated by $\widetilde{S}x$ is dense in X. Indeed, $\widetilde{S}x \geqslant x > 0$, so that $J \neq \{0\}$. Note that J is invariant under S because for every $z \in J$ we have $|z| \leqslant \lambda\widetilde{S}x$ for some $\lambda > 0$, so that

$$|Sz| \leqslant S|z| \leqslant \lambda S\widetilde{S}x = \lambda \sum_{n=0}^{\infty} t^n S^{n+1} x = \tfrac{\lambda}{t} \sum_{n=0}^{\infty} t^{n+1} S^{n+1} x \leqslant \tfrac{\lambda}{t}\widetilde{S}x.$$

Since S has no invariant subspaces, J has to be dense in X. This proves the claim.

Since $R \neq 0$, there exists $x_0 > 0$ such that $Rx_0 > 0$. By the claim, $\widetilde{S}Rx_0$ is quasi-interior. Since R is positive and non-zero, it cannot vanish on a quasi-interior vector, hence $R\widetilde{S}Rx_0 > 0$. Iterating this step, we get $R\widetilde{S}R\widetilde{S}Rx_0 > 0$. It follows that $R\widetilde{S}R\widetilde{S}R \neq 0$. Since $\widetilde{S}R \leqslant \widetilde{S}K$ and the latter operator is compact, $R\widetilde{S}R\widetilde{S}R$ is compact by Aliprantis-Burkinshaw's Cube Theorem [AA02, Theorem 2.34]. Hence, T commutes with a non-zero compact operator. Therefore, T has a hyperinvariant subspace: in case of a complex Banach lattice this follows from Lomonosov's Theorem, while in the case of a real Banach lattice we use Corollary 2.4 of [Sir05]. □

Remark 2. We have, actually, proved more than stated: we proved that either S has an invariant closed ideal or T commutes with a non-zero compact operator and, therefore, has a hyperinvariant subspace. We would also like to point out that the assumption that T is positive is not really needed.

To put Theorem 1 in perspective, note that, under the assumptions of the theorem, the following facts are well known.

- If both X and X^* have order continuous norm, then R is compact by Dodds-Fremlin Theorem [AA02, Theorem 2.38], so that T has a hyperinvariant subspace by Lomonosov's Theorem.
- Note that R^3 is always compact by the Cube Theorem, and $T \leftrightarrow R^3$. Thus, if $R^3 \neq 0$ then it follows immediately from Lomonosov's Theorem that T has a hyperinvariant subspace. On the other hand, if $R^3 = 0$ then $\ker R$ is a non-trivial subspace invariant under T. Hence, in any case, T has an invariant subspace.
- Note that T is compact-friendly. Therefore, if T is quasinilpotent at a positive vector then Theorem 10.55 of [AA02] guarantees that S has an invariant closed ideal. The following result is an analogue of Theorem 10.55 in our setting.

Theorem 3. *Suppose that T, R, and K are non-zero positive operators on a Banach lattice X such that $T \leftrightarrow R \leqslant K$, T is non-scalar, and K is compact. If (S_n) is a sequence of positive operators commuting with T then there is a subspace invariant under T, R, and all S_n's.*

Proof. Let $S = T + R + \sum_{n=1}^{\infty} a_n S_n$, where (a_n) is a sequence of positive reals such that the series converges. Observe that S is a positive operator commuting with T. If S has an invariant closed ideal then this ideal remains invariant under T, R, and each S_n because these operators are dominated by S. However, if S has no invariant closed subspaces, then T has a hyperinvariant subspace by Remark 2. □

Example 4. *$0 \leqslant R \leqslant K$, K is compact, R is not compact, and $R^2 = 0$.*

This is the case in Example 5.19 of [AB06]; it is one of the few classical examples showing that Dodds-Fremlin Theorem may fail when X^* is not order continuous. Here is the example. Put $X = \ell_1 \oplus L_2$. Let $(e_i)_{i=1}^{\infty}$ stand for the unit vector basis of ℓ_1, $(r_i)_{i=1}^{\infty}$ stand for the sequence of the Rademacher functions in L_2, and $r_0 = \mathbb{1}$ stand for the constant one function in L_2. Recall that the sequence $(r_i)_{i=0}^{\infty}$ is an orthonormal sequence in L_2. Note also that $r_i^+ = \frac{1}{2}(r_i + \mathbb{1})$ for all i. We define $R_0, K_0 \colon \ell_1 \to L_2$ via $K_0 e_i = \mathbb{1}$ and $R_0 e_i = r_i^+$ for all $i \geqslant 1$. It is easy to see that the both operators are bounded, K_0 is compact, R_0 is not compact, and $0 \leqslant R_0 \leqslant K_0$. Now put $R = \left[\begin{smallmatrix} 0 & 0 \\ R_0 & 0 \end{smallmatrix}\right]$ and $K = \left[\begin{smallmatrix} 0 & 0 \\ K_0 & 0 \end{smallmatrix}\right]$. Then R and K are two operators on X with $0 \leqslant R \leqslant K$, K is compact, R is not compact, and $R^2 = 0$.

Example 5. With R and K as in Example 4, we will construct T such that T commutes with R but not with K. Put $T = \left[\begin{smallmatrix} P & 0 \\ 0 & Q \end{smallmatrix}\right]$ where $P \colon \ell_1 \to \ell_1$ is the left shift: $Pe_i = e_{i-1}$ if $i > 1$ and $Pe_1 = 0$; and $Q \colon L_2 \to L_2$ is defined as follows. Put $Q\mathbb{1} = \mathbb{1}$, $Qr_1 = -\mathbb{1}$, $Qr_i = r_{i-1}$ for $i > 1$ and define Q arbitrarily on the orthogonal complement of the closed span of $(r_i)_{i=0}^{\infty}$ in L_2. Using the fact that $r_i^+ = \frac{1}{2}(r_i + \mathbb{1})$ we see that Q acts as a left shift on the sequence $(r_i^+)_{i=1}^{\infty}$. It is easy to see that T commutes with R because for every $\sum_{i=1}^{\infty} \alpha_i e_i$ in ℓ_1 and every $f \in L_2$ we have $TR\big(\sum_{i=1}^{\infty} \alpha_i e_i, f\big) = \big(0, \sum_{i=1}^{\infty} \alpha_{i+1} r_i^+\big) = RT\big(\sum_{i=1}^{\infty} \alpha_i e_i, f\big)$. However, T does not commute with K because $TK(e_1, 0) = (0, \mathbb{1})$ while $KT(e_1, 0) = (0, 0)$. Note that T is not positive.

Example 6. We construct three non-zero *positive* operators T, R, and K such that $0 \leqslant R \leqslant T$, K is compact, R is not compact, and T commutes with R but not with

K. In particular, the operators K, R, and T, together with any positive operator S which commutes with T satisfy the assumptions of Theorem 1.

We construct R and K similarly to Example 4. We again put $X = \ell_1 \oplus L_2$, but this time we consider ℓ_1 indexed by $\mathbb{N} \cup \{0\}$, so that the unit basis now starts with e_0. Again, we define $R = \left[\begin{smallmatrix} 0 & 0 \\ R_0 & 0 \end{smallmatrix}\right]$ and $K = \left[\begin{smallmatrix} 0 & 0 \\ K_0 & 0 \end{smallmatrix}\right]$ where $R_0 e_i = r_i^+$ and $K_0 e_i = \mathbb{1}$ for all $i = 0, 1, 2, \dots$ (recall that $r_0 = \mathbb{1}$). We still have $0 \leqslant R \leqslant K$, K is compact, R is not compact, and $R^2 = 0$. Put $T = \left[\begin{smallmatrix} P & 0 \\ 0 & Q \end{smallmatrix}\right]$ where $P\colon \ell_1 \to \ell_1$ and $Q\colon L_2 \to L_2$ are defined as follows. Fix a positive real parameter α. For $f \in L_2$, put

$$(Qf)(t) = f\left(\tfrac{t}{2}\right) + 2\alpha \int_{\frac{1}{2}}^{1} f, \qquad t \in [0,1].$$

It is easy to see that $Q\mathbb{1} = (1+\alpha)\mathbb{1}$, $Qr_1 = (1-\alpha)\mathbb{1}$, and $Qr_i = r_{i-1}$ for $i > 1$. It follows from $r_i^+ = \frac{1}{2}(r_i + \mathbb{1})$ that $Qr_1^+ = \mathbb{1}$ and $Qr_i^+ = r_{i-1}^+ + \frac{\alpha}{2}\mathbb{1}$ whenever $i > 1$. Now we define P so that the action of P on $(e_i)_{i=0}^\infty$ matches the action of Q on $(r_i^+)_{i=0}^\infty$, namely,

$$Pe_i = \begin{cases} (1+\alpha)e_0 & i = 0, \\ e_0 & i = 1, \\ e_{i-1} + \frac{\alpha}{2} e_0 & i > 1. \end{cases}$$

Clearly, Q and P are positive, hence so is T. It is easy to verify that T commutes with R. However, T does not commute with K as $TK(e_1, 0) = T(0, \mathbb{1}) = \bigl(0, (1+\alpha)\mathbb{1}\bigr)$, while $KT(e_1, 0) = K(e_0, 0) = (0, \mathbb{1})$.

Note that $(0, \mathbb{1})$ is an eigenvector of T; it follows that T has a hyperinvariant subspace. Also, if $\alpha = 1$ then T commutes with the compact positive operator C defined by $C(x, f) = \bigl(0, (\int_0^1 f)\mathbb{1}\bigr)$. We do not know whether T commutes with a compact operator when $\alpha \neq 1$.

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