MINIMAL VECTORS OF POSITIVE OPERATORS

RAZVAN ANISCA AND VLADIMIR G. TROITSKY

ABSTRACT. We use the method of minimal vectors to prove that certain classes of positive quasinilpotent operators on Banach lattices have invariant subspaces. We say that a collection of operators $\mathcal{F}$ on a Banach lattice X satisfies condition $(*)$ if there exists a closed ball $B(x_0,r)$ in X such that $x_0 \geqslant 0$ and $\|x_0\| > r$, and for every sequence (x_n) in $B(x_0,r) \cap [0,x_0]$ there exists a subsequence (x_{n_i}) and a sequence $K_i \in \mathcal{F}$ such that $K_i x_{n_i}$ converges to a non-zero vector. Let Q be a positive quasinilpotent operator on X, one-to-one, with dense range. Denote $\langle Q] = \{T \geqslant 0 : TQ \leqslant QT\}$. If either the set of all operators dominated by Q or the set of all contractions in $\langle Q]$ satisfies $(*)$, then $\langle Q]$ has a common invariant subspace. We also show that if Q is a one-to-one quasinilpotent interval preserving operator on $C_0(\Omega)$, then $\langle Q]$ has a common invariant subspace.

Lomonosov proved in [Lom73] that if T is not a multiple of the identity and commutes with a non-zero compact operator K, then T has a hyperinvariant subspace, that is, a proper closed nontrivial subspace invariant under every operator S in the commutant $\{T\}' = \{S \in \mathcal{L}(X) : ST = TS\}$. There has been numerous extensions and generalizations of the result of Lomonosov. In particular, Abramovich, Aliprantis, and Burkinshaw produced several generalizations of Lomonosov's theorem for Banach lattice setting [AAB93, AAB94, AAB98], see also [AA02]. In these generalizations commutation relations are substituted by a super-commutation relation $ST \leqslant TS$ or $ST \geqslant TS$ and domination $0 \leqslant K \leqslant T$. They proved a series of results of the following type: if S is related to a compact operator via a certain rather loose chain of super-commutations and dominations, then S has an invariant subspace.

Ansari and Enflo [AE98] have recently introduced the so-called technique of *minimal vectors* in order to prove the existence of invariant subspaces for certain classes of operators on a Hilbert space. The method was later modified so that it could be used in arbitrary Banach spaces in [JKP03, And03, CPS04, Tr04]. In particular, the method of minimal vectors allows to prove Lomonosov-type results where a compact operator is replaced with a family of operators that "mimic" a compact operator.

Date: August 13, 2005.

2000 *Mathematics Subject Classification.* Primary: 47A15; Secondary: 46B42, 47B65.

Key words and phrases. Invariant subspace, minimal vector, Banach lattice, positive operator.

Indiana University Math. J., **54**(3), 2005, 861–872.

Theorem 1 ([Tr04]). *Suppose that Q is a quasinilpotent operator on a Banach space, and there exists a closed ball $B \not\ni 0$ such that for every sequence (x_i) in B there is a subsequence (x_{n_i}) and a uniformly bounded sequence K_i in $\{Q\}'$ such that $K_i x_{n_i}$ converges to a non-zero vector. Then Q has a hyperinvariant subspace.*

In the present paper we adapt the technique of minimal vectors to positive operators on Banach lattices in the spirit of [AAB93, AAB94, AAB98].

In the following, X is a Banach lattice with positive cone X_+. For simplicity we assume that X is a real Banach lattice, however the arguments remain valid in the complex case after straightforward adjustments. By an ***operator*** we always mean a continuous linear operator from X to X. The symbol $B(x,r)$ stands for the closed ball of radius r centered at x. Let Q be a positive operator on X. We will be interested in the existence of (non-trivial proper) subspaces invariant under Q and operators commuting with Q. Therefore, we will usually assume that Q is one-to-one and has dense range, as otherwise $\ker Q$ or $\overline{\operatorname{Range} Q}$ are Q-hyperinvariant. Following [AA02] we define the ***super left-commutant*** $\langle Q]$ and the ***super right-commutant*** of $[Q\rangle$ of Q as follows:

$$\langle Q] = \{T \geqslant 0 \,:\, TQ \leqslant QT\} \qquad [Q\rangle = \{T \geqslant 0 \,:\, TQ \geqslant QT\}$$

If $a < b$ in X, we write $[a,b] = \{x \in X \,:\, a \leqslant x \leqslant b\}$. A subspace $Y \subseteq X$ is an (order) ***ideal*** if $|y| \leqslant |x|$ and $x \in Y$ imply $y \in Y$. For $K \in \mathcal{L}(X)$ we say that K is ***dominated*** by Q if $|Kx| \leqslant Q|x|$ for every $x \in X$. Obviously, every operator in $[0,Q] = \big\{K \in \mathcal{L}(X) \,:\, 0 \leqslant K \leqslant Q\big\}$ is dominated by Q.

Definition 2. We say that a collection of operators $\mathcal{F}$ satisfies condition $(*)$ if there exists a closed ball $B(x_0,r)$ in X such that $x_0 \geqslant 0$ and $\|x_0\| > r$, and for every sequence (x_n) in $B(x_0,r) \cap [0,x_0]$ there exists a subsequence (x_{n_i}) and a sequence $K_i \in \mathcal{F}$ such that $K_i x_{n_i}$ converges to a non-zero vector.

Let $x_0 \in X_+$ with $\|x_0\| > 1$, put $B = B(x_0,1)$. Let $B + X_+$ be the algebraic sum of the two sets, i.e.,

$$B + X_+ = \{x + h \,:\, x \in B,\ h \geqslant 0\}.$$

Lemma 3. *For $z \in X$, the following are equivalent.*

(i) $z \in B + X_+$;
(ii) $z \geqslant x$ *for some* $x \in B$;
(iii) $x_0 \wedge z \in B$;
(iv) $\|(x_0 - z)^+\| \leqslant 1$.

Proof. The equivalence (i)$\Leftrightarrow$(ii) is trivial, (iii)$\Leftrightarrow$(iv) follows from the identity $a - a \wedge b = (a-b)^+$, and (iii)$\Rightarrow$(ii) because $z \geqslant x_0 \wedge z$. To show (ii)$\Rightarrow$(iii), suppose that $z \geqslant x$ for some $x \in B$. Then $x_0 \wedge x \leqslant x_0 \wedge z \leqslant x_0$, so that

$$0 \leqslant x_0 - x_0 \wedge z \leqslant x_0 - x_0 \wedge x \leqslant |x_0 - x|,$$

hence $\|x_0 - x_0 \wedge z\| \leqslant \|x_0 - x\| \leqslant 1$. □

Corollary 4. *The set $B + X_+$ is closed, convex, and does not contain the origin.*

Proof. The set $B + X_+$ is clearly convex. By Lemma 3, $0 \notin B + X_+$. Since the map $z \mapsto \|(x_0 - z)^+\|$ is continuous, $B + X_+$ is closed. □

Put $D = Q^{-1}(B + X_+)$. Then D is convex, closed, and doesn't contain the origin. Notice that D is non-empty because $\operatorname{Range} Q$ is dense.

Lemma 5. *If $z \in D$ then $|z| \in D$.*

Proof. Let $z \in D$, then $Qz \in B + X_+$. It follows from $z \leqslant |z|$ that $Qz \leqslant Q|z|$, so that $Q|z| \in B + X_+$. □

Let d be the distance from D to the origin. Fix positive real number ε, there exists $y \in D$ such that $\|y\| \leqslant (1+\varepsilon)d$. Since $\big\||y|\big\| = \|y\|$, by Lemma 5 we can assume without loss of generality that $y > 0$. We will say that y is a $(1+\varepsilon)$-***minimal vector*** for Q and $B + X_+$. Note that when X is reflexive, one can actually find a 1-minimal vector, or, simply, a minimal vector.

Note that if $z \in D \cap B(0,d)$ then $\lambda z \notin D$ whenever $0 \leqslant \lambda < 1$. It follows that $\lambda Qz \notin B + X_+$ for every $0 \leqslant \lambda < 1$, so that Qz belongs to the boundary $\partial(B + X_+)$ of $B + X_+$. Then

$$Q\big(B(0,d)\big) \cap (B + X_+) = Q\big(B(0,d) \cap D\big) \subseteq \partial(B + X_+).$$

In particular, $Q\big(B(0,d)\big)$ and the interior $(B+X_+)^\circ$ are two disjoint convex sets. Since the former of the two has non-empty interior, they can be separated by a continuous linear functional (see, e.g., [AB99, Theorem 5.5]). That is, there exists a functional f with $\|f\| = 1$ and a positive real number c such that $f_{|Q(B(0,d))} \leqslant c$ and $f_{|(B+X_+)^\circ} \geqslant c$. By continuity, $f_{|(B+X_+)} \geqslant c$. We say that f is a ***minimal functional*** for Q and B.

Lemma 6. *If y is a $(1+\varepsilon)$-minimal vector and f is a minimal functional for Q and $B + X_+$, then the following are true.*

(i) *f is positive;*

(ii) *$f(x_0) \geqslant 1$;*

(iii) $\frac{1}{1+\varepsilon}f(Qy) \leqslant f(x_0 \wedge Qy) \leqslant f(Qy)$;
(iv) $\frac{1}{1+\varepsilon}\|Q^*f\|\|y\| \leqslant (Q^*f)(y) \leqslant \|Q^*f\|\|y\|$.

Proof. (i) Let $z \in X_+$ then $x_0 + \lambda z \in B + X_+$ for every positive real number λ. It follows that $f(x_0 + \lambda z) \geqslant c$, so that $f(z) \geqslant \bigl(c - f(x_0)\bigr)/\lambda \to 0$ as $\lambda \to +\infty$.

(ii) For every x with $\|x\| \leqslant 1$ we have $x_0 - x \in B$. It follows that $f(x_0 - x) \geqslant c$, so that $f(x_0) \geqslant c + f(x)$. Taking sup over all x with $\|x\| \leqslant 1$ we get $f(x_0) \geqslant c + \|f\| \geqslant 1$.

(iii) Since f is positive, it follows from $x_0 \wedge Qy \leqslant Qy$ that $f(x_0 \wedge Qy) \leqslant f(Qy)$. Notice that $y/(1+\varepsilon) \in B(0,d)$, so that $f(Qy)/(1+\varepsilon) \leqslant c$. On the other hand, by Lemma 3 we have $x_0 \wedge Qy \in B \subseteq B + X_+$, so that $f(x_0 \wedge Qy) \geqslant c \geqslant \frac{1}{1+\varepsilon}f(Qy)$.

(iv) We trivially have $(Q^*f)(y) \leqslant \|Q^*f\|\|y\|$. Observe that the hyperplane $Q^*f = c$ separates D and $B(0,d)$. Indeed, if $z \in B(0,d)$, then $(Q^*f)(z) = f(Qz) \leqslant c$, and if $z \in D$ then $Qz \in B + X_+$ so that $(Q^*f)(z) = f(Qz) \geqslant c$. For every z with $\|z\| \leqslant 1$ we have $dz \in B(0,d)$, so that $(Q^*f)(dz) \leqslant c$, it follows that $\bigl\|Q^*f\bigr\| \leqslant \frac{c}{d}$. On the other hand, for every $\delta > 0$ there exists $z \in D$ with $\|z\| \leqslant d + \delta$, then $(Q^*f)(z) \geqslant c \geqslant \frac{c}{d+\delta}\|z\|$, whence $\bigl\|Q^*f\bigr\| \geqslant \frac{c}{d+\delta}$. It follows that $\bigl\|Q^*f\bigr\| = \frac{c}{d}$. For every $z \in D$ we have $(Q^*f)(z) \geqslant c = d\bigl\|Q^*f\bigr\|$. It follows from $\|y\| \leqslant (1+\varepsilon)d$ that $(Q^*f)(y) \geqslant \frac{1}{1+\varepsilon}\bigl\|Q^*f\bigr\|\|y\|$. □

For each $n \geqslant 1$ choose a $(1+\varepsilon)$-minimal vector y_n for Q^n and $B + X_+$. We say that (y_n) is a $(1+\varepsilon)$-***minimal sequence*** for Q and $B + X_+$.

Lemma 7. *If Q is quasinilpotent, then (y_n) has a subsequence (y_{n_i}) such that $\frac{\|y_{n_i-1}\|}{\|y_{n_i}\|} \to 0$.*

Proof. Otherwise there would exist $\delta > 0$ such that $\frac{\|y_{n-1}\|}{\|y_n\|} > \delta$ for all n, so that $\|y_1\| \geqslant \delta\|y_2\| \geqslant \ldots \geqslant \delta^n\|y_{n+1}\|$. Since $Q^n y_{n+1} \in Q^{-1}(B + X_+)$ then

$$\bigl\|Q^n y_{n+1}\bigr\| \geqslant d \geqslant \frac{\|y_1\|}{1+\varepsilon} \geqslant \frac{\delta^n}{1+\varepsilon}\|y_{n+1}\|.$$

It follows that $\|Q^n\| \geqslant \delta^n/(1+\varepsilon)$, which contradicts the quasinilpotence of Q. □

Theorem 8. *Suppose that Q is a positive quasinilpotent operator, one-to-one, with dense range. If the set of all operators dominated by Q satisfies $(*)$, then there exists a common nontrivial invariant subspace for $\langle Q]$. Moreover, if $[0,Q]$ satisfies $(*)$, then there exists a common nontrivial invariant closed ideal for $\langle Q]$.*

Proof. Suppose that that the set of all operators dominated by Q satisfies $(*)$, show that there exists a common nontrivial invariant subspace for $\langle Q]$. Let $B(x_0, r)$ be the ball given by $(*)$, without loss of generality $r = 1$. Fix $\varepsilon > 0$, for every $n \geqslant 1$ choose a $(1+\varepsilon)$-minimal vector y_n and a minimal functional f_n for Q^n and $B+X_+$. By Lemma 7 there is a subsequence (y_{n_i}) such that $\frac{\|y_{n_i-1}\|}{\|y_{n_i}\|} \to 0$. Since $\|f_{n_i}\|=1$ for all i, we can assume (by passing to a further subsequence), that (f_{n_i}) weak*-converges to some $g \in X^*$. By Lemma 6(ii) we have $f_n(x_0) \geqslant 1$ for all n, it follows that $g(x_0) \geqslant 1$. In particular, $g \neq 0$.

Consider the sequence $(x_0 \wedge Q^{n_i-1}y_{n_i-1})_{i=1}^{\infty}$. The terms of this sequence are positive, and by Lemma 3 they are contained in B, so that, by passing to yet a further subsequence, if necessary, we find a sequence (K_i) such that K_i is dominated by Q for all i and $K_i(x_0 \wedge Q^{n_i-1}y_{n_i-1})$ converges to some vector $w \neq 0$.

Show that $g(Tw)=0$ for every $T \in \langle Q]$. Suppose $T \in \langle Q]$. It follows from Lemma 6(iv) that $\bigl(Q^{*n_i}f_{n_i}\bigr)(y_{n_i}) \neq 0$ for every i, so that $X = \operatorname{span}\{y_{n_i}\} \oplus \ker\bigl(Q^{*n_i}f_{n_i}\bigr)$. Then one can write $Ty_{n_i-1} = \alpha_i y_{n_i} + r_i$, where α_i is a scalar and $r_i \in \ker\bigl(Q^{*n_i}f_{n_i}\bigr)$. We claim that $\alpha_i \to 0$. Indeed,

$$\bigl(Q^{*n_i}f_{n_i}\bigr)\bigl(Ty_{n_i-1}\bigr) = \alpha_i\bigl(Q^{*n_i}f_{n_i}\bigr)(y_{n_i}), \tag{1}$$

so that $\alpha_i \geqslant 0$. Now by Lemma 6(iv) we have

$$\bigl(Q^{*n_i}f_{n_i}\bigr)\bigl(Ty_{n_i-1}\bigr) \geqslant \frac{\alpha_i}{1+\varepsilon}\bigl\|Q^{*n_i}f_{n_i}\bigr\|\|y_{n_i}\|. \tag{2}$$

On the other hand,

$$\bigl(Q^{*n_i}f_{n_i}\bigr)\bigl(Ty_{n_i-1}\bigr) \leqslant \bigl\|Q^{*n_i}f_{n_i}\bigr\| \cdot \|T\| \cdot \|y_{n_i-1}\|. \tag{3}$$

It follows from (2) and (3) that $\alpha_i \leqslant (1+\varepsilon)\|T\|\frac{\|y_{n_i-1}\|}{\|y_{n_i}\|}$, so that $\alpha_i \to 0$. Since K_i is dominated by Q and $TQ \leqslant QT$, we have

$$\Bigl|f_{n_i}\bigl(TK_i(x_0 \wedge Q^{n_i-1}y_{n_i-1})\bigr)\Bigr| \leqslant f_{n_i}\Bigl(T\bigl|K_i(x_0 \wedge Q^{n_i-1}y_{n_i-1})\bigr|\Bigr) \leqslant$$
$$f_{n_i}\bigl(TQ(x_0 \wedge Q^{n_i-1}y_{n_i-1})\bigr) \leqslant f_{n_i}\bigl(TQ(Q^{n_i-1}y_{n_i-1})\bigr) \leqslant f_{n_i}\bigl(Q^{n_i}Ty_{n_i-1}\bigr).$$

It follows from (1) that $f_{n_i}\bigl(Q^{n_i}Ty_{n_i-1}\bigr) = \alpha_i f_{n_i}\bigl(Q^{n_i}y_{n_i}\bigr)$. Further, Lemma 6(iii) yields

$$\alpha_i f_{n_i}\bigl(Q^{n_i}y_{n_i}\bigr) \leqslant \alpha_i(1+\varepsilon)f_{n_i}\bigl(x_0 \wedge Q^{n_i}y_{n_i}\bigr) \leqslant \alpha_i(1+\varepsilon)\bigl(\|x_0\|+1\bigr)$$

because $\|f_{n_i}\|=1$ and $x_0 \wedge Q^{n_i}y_{n_i} \in B$. Thus,

$$f_{n_i}\bigl(TK_i(x_0 \wedge Q^{n_i-1}y_{n_i-1})\bigr) \to 0.$$

On the other hand,

$$TK_i(x_0 \wedge Q^{n_i-1}y_{n_i-1}) \to Tw$$

in norm. Since $f_{n_i} \xrightarrow{w^*} g$, we conclude that $g(Tw)=0$.

Let Y be the linear span of $\langle Q]w$, that is, $Y = \operatorname{lin}\{Tw : T \in \langle Q]\}$. Since $\langle Q]$ is a multiplicative semigroup, Y is invariant under every $T \in \langle Q]$. It follows from

$0 \neq Qw \in Y$ that Y is non-zero. Finally, $\overline{Y} \neq X$ because $g(Tw) = 0$ for all $T \in \langle Q]$, so that $Y \subseteq \ker g$.

Suppose now that $[0, Q]$ satisfies $(*)$. Then the vector w constructed in the previous argument is positive. Let E be the ideal generated by $\langle Q]w$, that is

$$E = \big\{ y \in X \;:\; |y| \leqslant Tw \text{ for some } T \in \langle Q] \big\}.$$

Then E is non-trivial since $w \in E$, it is easy to see that E is invariant under $\langle Q]$. Since g is a positive functional, then g vanishes on E, hence $\overline{E} \neq X$. □

Remark 9. Notice that in the proof we don't really need $(*)$ to hold for every sequence in $B(x_0, 1) \cap [0, x_0]$, but only for a certain subsequence of $(x_0 \wedge Q^n y_n)$, where (y_n) is a $(1+\varepsilon)$-minimal sequence.

Corollary 10. *If Q is a quasinilpotent positive operator, one-to-one, with dense range, and there exists $x_0 \in X$ such that $[0, x_0]$ is compact, then $\langle Q]$ has a common invariant non-trivial closed ideal.*

Proof. The statement follows immediately from Theorem 8 because $[0, Q]$ satisfies $(*)$ with $K_i = Q$. □

We say that $x_0 \in X_+$ is an ***atom*** if every element of $[0, x_0]$ is a scalar multiple of x_0. It was shown in [Drn00] that if Q is a positive quasinilpotent operator on a Banach lattice with an atom, and $S \in [Q\rangle$, then Q and S have a common non-trivial invariant closed ideal. From Corollary 10 we deduce a similar statement for $\langle Q]$.

Corollary 11. *If Q is a one-to-one quasinilpotent positive operator with dense range on a Banach lattice with an atom, then $\langle Q]$ has a non-trivial common invariant closed ideal.*

Theorem 12. *Suppose that*

(i) *Q is a positive and quasinilpotent operator, one-to-one, with dense range;*
(ii) *$\mathcal{F}$ is a collection of positive contractive operators satisfying $(*)$, and*
(iii) *$\mathcal{S}$ is a semigroup of operators such that $TK \in \langle Q]$ for every $T \in \mathcal{S}$ and $K \in \mathcal{F}$.*

Then $\mathcal{S}$ has a common non-trivial invariant subspace. Moreover, if $\mathcal{S}$ consists of positive operators then it has a common non-trivial invariant closed ideal.

Proof. Let $B = B(x_0, r)$ be the ball mentioned in $(*)$, without loss of generality $r = 1$. Fix $\varepsilon > 0$, for every $n \geqslant 1$ choose a $(1+\varepsilon)$-minimal vector y_n and a minimal functional f_n for Q^n and $B + X_+$. By Lemma 7 there is a subsequence (y_{n_i}) such that $\frac{\|y_{n_i-1}\|}{\|y_{n_i}\|} \to 0$.

Since $\|f_{n_i}\| = 1$ for all i, we can assume (by passing to a further subsequence), that (f_{n_i}) weak*-converges to some $g \in X^*$. By Lemma 6(ii) we have$f_n(x_0) \geqslant 1$ for all n, it follows that $g(x_0) \geqslant 1$. In particular, $g \neq 0$.

Consider the sequence $(x_0 \wedge Q^{n_i-1}y_{n_i-1})_{i=1}^{\infty}$. The terms of this sequence are positive, and by Lemma 3 they are contained in B, so that, by passing to yet a further subsequence, if necessary, we find a sequence (K_i) in $\mathcal{F}$ such that $K_i(x_0 \wedge Q^{n_i-1}y_{n_i-1})$ converges to some vector $w > 0$.

Suppose that $T \in \mathcal{S}$. It follows from Lemma 6(iv) that $\big(Q^{*n_i}f_{n_i}\big)(y_{n_i}) \neq 0$ for every i, so that $X = \operatorname{span}\{y_{n_i}\} \oplus \ker\big(Q^{*n_i}f_{n_i}\big)$. Then one can write $TK_iy_{n_i-1} = \alpha_i y_{n_i} + r_i$, where α_i is a scalar and $r_i \in \ker\big(Q^{*n_i}f_{n_i}\big)$. We claim that $\alpha_i \to 0$. Indeed,

$$\big(Q^{*n_i}f_{n_i}\big)\big(TK_iy_{n_i-1}\big) = \alpha_i\big(Q^{*n_i}f_{n_i}\big)(y_{n_i}), \tag{4}$$

so that $\alpha_i \geqslant 0$. Now by Lemma 6(iv) we have

$$\big(Q^{*n_i}f_{n_i}\big)\big(TK_iy_{n_i-1}\big) \geqslant \frac{\alpha_i}{1+\varepsilon}\big\|Q^{*n_i}f_{n_i}\big\|\|y_{n_i}\|. \tag{5}$$

On the other hand,

$$\big(Q^{*n_i}f_{n_i}\big)\big(TK_iy_{n_i-1}\big) \leqslant \big\|Q^{*n_i}f_{n_i}\big\| \cdot \|T\| \cdot \|y_{n_i-1}\|. \tag{6}$$

It follows from (5) and (6) that $\alpha_i \leqslant (1+\varepsilon)\|T\|\frac{\|y_{n_i-1}\|}{\|y_{n_i}\|}$, so that $\alpha_i \to 0$. Notice that

$$0 \leqslant f_{n_i}\big(QTK_i(x_0 \wedge Q^{n_i-1}y_{n_i-1})\big) \leqslant f_{n_i}\big(QTK_iQ^{n_i-1}y_{n_i-1}\big) \leqslant f_{n_i}\big(Q^{n_i}TK_iy_{n_i-1}\big)$$

because $TK \in \langle Q]$. It follows from (4) that

$$f_{n_i}\big(Q^{n_i}TK_iy_{n_i-1}\big) = \alpha_i f_{n_i}\big(Q^{n_i}y_{n_i}\big).$$

Further, Lemma 6(iii) yields

$$\alpha_i f_{n_i}\big(Q^{n_i}y_{n_i}\big) \leqslant \alpha_i(1+\varepsilon)f_{n_i}\big(x_0 \wedge Q^{n_i}y_{n_i}\big) \leqslant \alpha_i(1+\varepsilon)\big(\|x_0\|+1\big)$$

because $\|f_{n_i}\| = 1$ and $x_0 \wedge Q^{n_i}y_{n_i} \in B$. Thus,

$$f_{n_i}\big(QTK_i(x_0 \wedge Q^{n_i-1}y_{n_i-1})\big) \to 0.$$

On the other hand,

$$QTK_i(x_0 \wedge Q^{n_i-1}y_{n_i-1}) \to QTw$$

in norm. Since $f_{n_i} \xrightarrow{w^*} g$, we conclude that $g(QTw) = 0$.

Let Y be the linear span of $\mathcal{S}w$, that is, $Y = \operatorname{lin}\{Tw : T \in \mathcal{S}\}$. Then Y is invariant for all operators in $\mathcal{S}$. Since Q has dense range, Q^* is one-to-one, so that $Q^*g \neq 0$. We have $\overline{Y} \neq X$ because $(Q^*g)(Tw) = 0$ for all $T \in \mathcal{S}$. Finally, if $Y = \{0\}$, then $Tw = 0$ for all $T \in \mathcal{S}$, then the span of w is invariant under every operator in $\mathcal{S}$.

Suppose now that all the operators in $\mathcal{S}$ are positive. Let E be the ideal generated by $\mathcal{S}w$, that is

$$E = \bigl\{y \in X \,:\, |y| \leqslant Tw \text{ for some } T \in \mathcal{S}\bigr\}.$$

It is easy to see that E is invariant under $\mathcal{S}$. Since Q^*g is a positive functional, then g vanishes on E, hence $\overline{E} \neq X$. If E is non-trivial, we are done. Suppose that $E = \{0\}$. Then, in particular, $Tw = 0$ for every $T \in \mathcal{S}$. But then every operator on $\mathcal{S}$ vanishes on the ideal F generated by w:

$$F = \bigl\{y \in X \,:\, |y| \leqslant \lambda w \text{ for some real number} \lambda > 0\bigr\},$$

hence F is $\mathcal{S}$-invariant. Further, $w \in F$ so that F is non-zero. Finally, $\overline{F} \neq X$ as otherwise every operator in $\mathcal{S}$ is zero. □

Corollary 13. *Suppose that Q is a positive quasinilpotent operator, one-to-one, with dense range. Suppose that the set of all contractions in $\langle Q]$ satisfies $(*)$. Then $\langle Q]$ has a common non-trivial invariant closed ideal.*

Proof. Notice that $\langle Q]$ is a semigroup and apply Theorem 12 with $\mathcal{F} = \bigl\{K \in \langle Q] : \|K\| \leqslant 1\bigr\}$ and $\mathcal{S} = \langle Q]$. □

Next, we are going to discuss some applications. Recall that a positive operator T on a vector lattice is said to be ***interval preserving*** if $T[0,x] = [0,Tx]$ for every $x \geqslant 0$.

Lemma 14. *An operator T on a Banach lattice is one-to-one and interval preserving if and only if*

(i) $\mathrm{Range}\, T$ *is an ideal and*
(ii) $x \geqslant 0 \quad \Leftrightarrow \quad Tx \geqslant 0.$

Proof. Suppose that T is one-to-one and interval preserving. In particular, T is positive, hence $x \geqslant 0$ implies $Tx \geqslant 0$. If $Tx \geqslant 0$ then $Tx = |Tx| \leqslant T|x| \in T\bigl[0,|x|\bigr]$, so that $x \in \bigl[0,|x|\bigr]$, hence $x \geqslant 0$. To see that $\mathrm{Range}\, T$ is an ideal, suppose that $|y| \leqslant Tx$ for some $x, y \in X$. Then $y \in [-Tx, Tx] = T[-x,x]$, so that $y \in \mathrm{Range}\, T$.

Conversely, suppose that T satisfies (i) and (ii). Fix $x \geqslant 0$, let $z \in [0,Tx]$. It follows from (i) that $z = Ty$ for some $y \in X$. Further, $0 \leqslant y \leqslant x$ by (ii), so that $z \in T[0,x]$, hence $[0,Tx] \subseteq T[0,x]$. The inclusion $T[0,x] \subseteq [0,Tx]$ is trivial. Finally, it follows immediately from (ii) that T is one-to-one. □

Theorem 15. *Suppose that X is a Banach lattice, and there exists $x_0 \in X_+$ with $\|x_0\| > 1$ such that the set $B(x_0, 1) \cap [0, x_0]$ has a least element. If Q is a one-to-one interval preserving quasinilpotent operator on X then $\langle Q]$ has a common invariant closed ideal.*

Proof. Let h be the least element of $B(x_0, 1) \cap [0, x_0]$. Clearly, $h > 0$. By Lemma 14, $\operatorname{Range} Q$ is an ideal in X. Since $\operatorname{Range} Q$ is a common invariant ideal for $\langle Q]$, we may assume without loss of generality that $\operatorname{Range} Q$ is dense. Notice that Q^n is one-to-one and interval preserving for every $n \geqslant 0$. Again, by Lemma 14, $\operatorname{Range} Q^n$ is an ideal and $x \geqslant 0 \Leftrightarrow Q^n x \geqslant 0$. Suppose that $0 < z \in Q^{-n}(B + X_+)$, then $Q^n z \geqslant h$. It follows that $h \in \operatorname{Range} Q^n$. Then $0 \leqslant Q^{-n} h \leqslant z$. Therefore, $y_n = Q^{-n} h$ is a minimal vector for Q^n. Then $Q^n y_n = h$ for every n. Now Theorem 8 and Remark 9 complete the proof. □

Corollary 16. *Suppose that Ω is a locally compact topological space and Q is a one-to-one interval preserving quasinilpotent operator on $C_0(\Omega)$. Then $\langle Q]$ has a common invariant closed ideal.*

Proof. Take any positive $x_0 \in C_0(\Omega)$ with $\|x_0\| > 1$, then $(x_0 - \mathbf{1})^+$ is the least element of $B(x_0, 1) \cap [0, x_0]$. Now apply Theorem 15. □

Let X be the space $C_0(\Omega)$ for a locally compact topological space Ω, or the space $L_p(\mu)$ for some measure space and $1 \leqslant p \leqslant +\infty$. An operator T on X is called a ***weighted composition operator*** if it is a product of a multiplication operator and a composition operator. That is $Tx = w \cdot (x \circ \tau)$ for every $x \in X$, so that $(Tx)(t) = w(t)x\big(\tau(t)\big)$ for every $t \in \Omega$. We will denote this operator $C_{w,\tau}$. In the case $X = C_0(\Omega)$ one usually assumes that $w \in C(\Omega)$ and $\tau \colon \Omega \to \Omega$ is a continuous map, while in the case $X = L_p(\mu)$ one would take $w \in L_\infty(\mu)$ and τ a measurable transformation of the underlying measure space. In either case, if $w \geqslant 0$ then T is a positive operator. Notice that if $0 \leqslant v \leqslant w$ then $0 \leqslant C_{v,\tau} \leqslant C_{w,\tau}$.

Suppose that Ω is compact, then Krein's Theorem asserts that every positive operator on $C(\Omega)$ has an invariant subspace ([KR48], see also [AAB92, OT]). Further, suppose $Q = C_{w,\tau}$ is positive and quasinilpotent[1] operator on $C(\Omega)$. Then the weight function $w(t)$ has to vanish at some $t_0 \in \Omega$. Indeed, otherwise it would be bounded

[1]Kitover [Kit79] found a necessary and sufficient condition for a weighted composition operator on $C(\Omega)$ to be quasinilpotent

below by a constant $m > 0$, and then Q would dominate a multiple of a composition operator $x \mapsto m(x \circ \tau)$, which would contradict the quasinilpotence of Q. Let

$$E = \big\{ y \in X \,:\, |y| \leqslant Qx \text{ for some } x \geqslant 0 \big\}.$$

It is easy to see that E is an ideal, invariant under $\langle Q]$. But E is contained in the closed ideal $\big\{ x \in C(\Omega) \,:\, x(t_0) = 0 \big\}$, hence E is not dense in $C(\Omega)$. Thus, *if Ω is compact and Q is a positive quasinilpotent weighted composition operator on $C(\Omega)$, then $\langle Q]$ has a common non-trivial closed ideal.*

In general, when Ω is just locally compact but not compact, the previous arguments does not apply. However, we have the following.

Theorem 17. *Suppose that Q is positive quasinilpotent weighted composition operator on $C_0(\Omega)$. Then $\langle Q]$ has a common closed invariant ideal.*

Proof. Suppose that $Q = C_{w,\tau}$, where $w\colon \Omega \to \mathbb{R}$ and $\tau\colon \Omega \to \Omega$ are continuous. Without loss of generality, $w \geqslant 0$ and $\|w\| > 1$. In view of Theorem 8 it suffices to show that $[0, Q]$ satisfies $(*)$. Find $u \in C_0(\Omega)$ such that $0 \leqslant u \leqslant w$ and $\|u\| > 1$. There exists a compact set $D \subseteq \Omega$ such that $u(t) < 1$ whenever $t \in D^C$. Since τ is continuous, the set $\tau(D)$ is also compact. Choose $x_0 \in C_0(\Omega)$ so that $x_0(s) = 2$ whenever $s \in \tau(D)$.

Pick any $0 \leqslant x \in B(x_0, 1)$. Let $O = \big\{ t \in \Omega \,:\, x \circ \tau(t) \neq 0 \big\}$. Observe that O is open and $D \subseteq O$. For each $t \in \Omega$, put

$$v(t) = \begin{cases} \frac{(u(t)-1)^+}{x \circ \tau(t)} & \text{if } t \in O; \\ 0 & \text{otherwise.} \end{cases}$$

Observe that v is continuous. Indeed, v is clearly continuous on O. Suppose that $t_0 \in O^C$. Since D is a compact subset of O and v vanishes off D, it follows that

$$\lim_{t \to t_0,\, t \in O} v(t) = \lim_{t \to t_0,\, t \in O \setminus D} v(t) = 0.$$

Observe also that if $t \in D$ then $x \circ \tau(t) \geqslant 1$, so that $v(t) \leqslant \big(u(t) - 1\big)^+$. If $t \in D^C$ then $v(t) = 0$. Thus, $0 \leqslant v \leqslant (u - \mathbf{1})^+ \leqslant w$. In particular, $0 \leqslant C_{v,\tau} \leqslant Q$. For every $t \in O$ we have $(C_{v,\tau}x)(t) = v(t)x(\tau(t)) = \big(u(t) - 1\big)^+$. On the other hand, if $t \in O^C$ then $(C_{v,\tau}x)(t) = 0 = \big(u(t) - 1\big)^+$ since $t \in D^C$. Thus, $C_{v,\tau}x = (u - \mathbf{1})^+ \neq 0$.

Now, suppose that (x_i) is a sequence in $B(x_0, 1) \cap [0, x_0]$. By the preceding argument, for each i we can find a continuous function v_i such that $0 \leqslant C_{v_i,\tau} \leqslant Q$ and $C_{v_i,\tau}x_i = (u - \mathbf{1})^+$. Hence, we can take $n_i = i$ and $K_i = C_{v_i,\tau}$ in $(*)$. Thus, $[0, Q]$ satisfies $(*)$, and then Theorem 8 finishes the proof. ☐

A similar statement for $L_p(\mu)$ spaces fails, there is an example (see, e.g., [MN91]) of a positive quasinilpotent weighted composition operator on $L_p[0,1]$ (actually, a weighted translation) with no closed invariant ideals. It is worth pointing out why the methods that we use in $C_0(\Omega)$ spaces don't work in $L_p(\mu)$ spaces. We cannot use Theorem 15 like we do in Corollary 16 because balls in $L_p(\mu)$ have no infimum. In order to use Theorem 8 like we did in Theorem 17, we need to show that $[0,Q]$ satisfies $(*)$. For simplicity consider $Q = C_{\omega,\tau}$ on $L_1[0,1]$ and assume that $x_0 = w = \mathbf{1}$ (the general case can be reduced to this). We would need to show that for every sequence (x_n) in $B(\mathbf{1}, 1-\varepsilon) \cap [0,\mathbf{1}]$ there exists a subsequence (x_{n_i}) and a uniformly bounded sequence of weights $k_i \in L_\infty[0,1]$ with $k_i x_{n_i}$ converging in norm to a non-zero function h. Let (A_n) be a sequence of independent events in $[0,1]$, each of measure ε, and let x_n be the characteristic function of the complement of A_n. Since for every subsequence (n_i) and every i_0 the set $\bigcup_{i\geqslant i_0} A_{n_i}$ has measure one, and $k_i x_{n_i}$ vanishes on A_{n_i}, it follows that $h = 0$ a.e.

The authors would like to thank G. Androulakis and N. Tomczak-Jaegermann for enlightening discussions.

References

[AA02] Y. A. Abramovich and C. D. Aliprantis, *An invitation to operator theory*, Graduate Studies in Mathematics, vol. 50, American Mathematical Society, Providence, RI, 2002. MR **2003h:**47072

[AAB92] Y. A. Abramovich, C. D. Aliprantis, and O. Burkinshaw, *Positive operators on Kreĭn spaces*, Acta Appl. Math. **27** (1992), no. 1-2, 1–22, Positive operators and semigroups on Banach lattices (Curaçao, 1990). MR **93h:**47045

[AAB93] ———, *Invariant subspaces of operators on ℓ_p-spaces*, J. Funct. Anal. **115** (1993), no. 2, 418–424. MR **94h:**47009

[AAB94] ———, *Invariant subspace theorems for positive operators*, J. Funct. Anal. **124** (1994), no. 1, 95–111. MR **95e:**47006

[AAB98] ———, *The invariant subspace problem: some recent advances*, Rend. Istit. Mat. Univ. Trieste **29** (1998), no. suppl., 3–79 (1999), Workshop on Measure Theory and Real Analysis (Italian) (Grado, 1995). MR **2000f:**47062

[AB99] Charalambos D. Aliprantis and Kim C. Border, *Infinite-dimensional analysis, a hitchhiker's guide*, second ed., Springer-Verlag, Berlin, 1999. MR **2000k:**46001

[AE98] Shamim Ansari and Per Enflo, *Extremal vectors and invariant subspaces*, Trans. Amer. Math. Soc. **350** (1998), no. 2, 539–558. MR **98d:**47019

[And03] George Androulakis, *A note on the method of minimal vectors*, Trends in Banach spaces and operator theory (Memphis, TN, 2001), Contemp. Math., vol. 321, Amer. Math. Soc., Providence, RI, 2003, 29–36. MR 1 978 805

[CPS04] I. Chalendar, J.R. Partington, and M. Smith, *Approximation in reflexive Banach spaces and applications to the invariant subspace problem*, Proc. American Math. Soc., **132** (2004), 1133–1142.

[Drn00] Roman Drnovšek, *Triangularizing semigroups of positive operators on an atomic normed Riesz space*, Proc. Edinburgh Math. Soc. (2) **43** (2000), no. 1, 43–55. MR **2001i:**47062

[JKP03] Il Bong Jung, Eungil Ko, and Carl Pearcy, *On quasinilpotent operators*, Proc. Amer. Math. Soc. **131** (2003), 2121–2127. MR 1 963 758

[Kit79] A. K. Kitover, *The spectrum of automorphisms with weight, and the Kamowitz-Scheinberg theorem*, Funktsional. Anal. i Prilozhen. **13** (1979), no. 1, 70–71. MR **80f:**46053

[KR48] M. G. Kreĭn and M. A. Rutman, *Linear operators leaving invariant a cone in a Banach space*, Uspehi Matem. Nauk (N. S.) **3** (1948), no. 1(23), 3–95. MR **10:**256c

[Lom73] V. I. Lomonosov, *Invariant subspaces of the family of operators that commute with a completely continuous operator*, Funkcional. Anal. i Priložen. **7** (1973), no. 3, 55–56. MR **54:**8319

[MN91] Peter Meyer-Nieberg, *Banach lattices*, Springer-Verlag, Berlin, 1991. MR **93f:**46025

[OT] Timur Oikhberg and Vladimir G. Troitsky, *A theorem of Krein revisited*, Rocky Mountain Journal of Mathematics, to appear.

[Tr04] Vladimir G. Troitsky, *Minimal vectors in arbitrary Banach spaces*, Proc. American Math. Soc., **132** (2004), 1177–1180.

Department of Mathematics, Texas A&M University, College Station, TX 77843-3368. USA.

Department of Mathematical and Statistical Sciences, University of Alberta, Edmonton, AB, T6G 2G1. Canada.

E-mail address: anisca@math.tamu.edu, vtroitsky@math.ualberta.ca