SESQUITRANSITIVE AND LOCALIZING OPERATOR ALGEBRAS

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Abstract. An algebra of operators on a Banach space X is said to be transitive if X has no nontrivial closed subspaces invariant under every member of the algebra. In this paper we investigate a number of conditions which guarantee that a transitive algebra of operators is "large" in various senses. Among these are the conditions of algebras being localizing or sesquitransitive. An algebra is *localizing* if there exists a closed ball $B \not\ni 0$ such that for every sequence (x_n) in B there exists a subsequence (x_{n_k}) and a bounded sequence (A_k) in the algebra such that $(A_k x_{n_k})$ converges to a non-zero vector. An algebra is *sesquitransitive* if for every non-zero $z \in X$ there exists $C > 0$ such that for every x linearly independent of z, for every non-zero $y \in X$, and every $\varepsilon > 0$ there exists A in the algebra such that $\|Ax - y\| < \varepsilon$ and $\|Az\| \leqslant C\|z\|$. We give an algebraic version of this definition as well, and extend Jacobson's density theorem to algebraically sesquitransitive rings.

1. Introduction, preliminaries and notation

Throughout this paper, X will be a real or complex Banach space, and $L(X)$ will denote the space of all continuous linear operators on X. If $T \in L(X)$, we say that T has an ***invariant subspace*** if there exists a closed non-zero proper subspace Y of X such that $T(Y) \subseteq Y$. We say that a subspace Y is ***hyperinvariant*** for T if Y is invariant under every operator in $\{T\}'$. Here $\mathcal{S}'$ is the commutant of a set $\mathcal{S} \subseteq L(X)$, that is, $\mathcal{S}' = \{A \in L(X) \mid \forall S \in \mathcal{S}\ AS = SA\}$. A subset $\mathcal{S} \subseteq L(X)$ is said to be ***transitive*** if $\mathcal{S}x$ is dense in X for every non-zero $x \in X$, where $\mathcal{S}x = \{Ax \mid A \in \mathcal{S}\}$. The symbol $\mathcal{A}$ will usually stand for a subalgebra of $L(X)$. We will write B_X and $B_{\mathcal{A}}$ for the closed unit balls of X and $\mathcal{A}$ respectively. It can be easily verified that $\mathcal{A}$ is transitive iff it has no common invariant subspaces. Furthermore, $\mathcal{A}$ is transitive iff $\overline{\mathcal{A}}^{\mathrm{WOT}}$ is transitive, where $\overline{\mathcal{A}}^{\mathrm{WOT}}$ stands for the closure of $\mathcal{A}$ in the weak operator topology (WOT).

It was proved in [Lom73] that if $T \in L(X)$ commutes with a non-zero compact operator, then T has an invariant subspace. If, in addition, X is a complex Banach space and T is not a multiple of the identity operator then T has a hyperinvariant subspace. Hooker [Hoo81] proved that in the real case T would still have a hyperinvariant subspace provided that, in addition, T doesn't satisfy a real-irreducible quadratic equation.

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However, in general there exist operators on real and complex Banach spaces with no invariant subspaces; see [Enf76, Read84].

Note that for an operator T, its commutant $\{T\}'$ is a WOT-closed algebra, and T has no hyperinvariant subspaces iff $\{T\}'$ is transitive. This naturally leads to the study of transitive algebras. It follows from [Enf76, Read84] that there exist transitive algebras of operators on Banach spaces which are not WOT-dense. However, there are several known conditions which, together with transitivity, guarantee that the algebra is WOT-dense. For example, every strictly transitive algebra (see Section 4 for the definition) is WOT-dense [Yood49, Ric50]. In the finite-dimensional case, the Burnside Theorem asserts that $M_n(\mathbb{C})$ contains no proper transitive subalgebras. Also an algebraic version of [Lom73] (see, e.g., [RR03]) asserts that if $\mathcal{A}$ is a transitive algebra of operators on a complex Banach space such that $\mathcal{A}$ contains a compact operator then $\overline{\mathcal{A}}^{\mathrm{WOT}} = L(X)$.

In this paper we study several conditions on an operator algebra $\mathcal{A}$ which, although do not necessarily imply that $\overline{\mathcal{A}}^{\mathrm{WOT}} = L(X)$, provide some information about the size of $\mathcal{A}$ by ensuring that $\mathcal{A}'$ is small (e.g., finite-dimensional). We also introduce several new conditions on algebras of operators.

Definition 1.1. We will say that an algebra $\mathcal{A}$ of operators on a Banach space X is ***localizing*** if there exists a closed ball B in X such that $0 \notin B$ and for every sequence (x_n) in B there is a subsequence (x_{n_i}) and a sequence (S_i) in $\mathcal{A}$ such that $\|S_i\| \leq 1$ and $(S_i x_{n_i})$ converges in norm to a nonzero vector.

It is easy to see that if T is an injective compact operator, then $\{T\}'$ is localizing.

The following theorem was obtained in [Tro04] using the method of minimal vectors [AE98, And03, CPS04].

Theorem 1.2. *Let T be a quasinilpotent operator on a Banach space X. If $\{T\}'$ is localizing then T has a hyperinvariant subspace.*

Theorem 1.2 easily extends to algebras of operators as follows.

Theorem 1.3. *Suppose that X is a Banach space and $\mathcal{A}$ is a transitive localizing subalgebra of $L(X)$. Then $\mathcal{A}'$ contains no non-zero quasinilpotent operators.*

Proof. Suppose $T \in \mathcal{A}'$ is non-zero and quasinilpotent. It follows from $\mathcal{A} \subseteq \{T\}'$ that $\{T\}'$ is localizing, so that T has a hyperinvariant subspace. This contradicts transitivity of $\mathcal{A}$. $\square$

In Section 2 we investigate SC-algebras, i.e., the algebras where the unit ball is relatively compact in the strong operator topology (SOT). In particular, we show that if $\{T\}'$ is an SC-algebra and $\{T^*\}'$ is localizing then T^* has an invariant subspace.

In Section 3 we introduce ***quasi-localizing*** algebras by replacing the condition $\|S_i\| \leq 1$ in Definition 1.1 with inequalities $\|S_i z_{n_i}\| < C\|z_{n_i}\|$ for a subsequence of a given sequence (z_n). We show that Theorem 1.3 remains valid for quasi-localizing algebras.

Motivated by the quasi-localizing property, in Section 4 we define an algebra to be ***sesquitransitive*** if for every non-zero $z \in X$ there exists $C > 0$ such that for every x linearly independent of z, for every non-zero $y \in X$, and every $\varepsilon > 0$ there exists A in the algebra such that $\|Ax - y\| < \varepsilon$ and $\|Az\| \leqslant C\|z\|$. We say that $\mathcal{A}$ is ***uniformly sesquitransitive*** if C can be chosen to be independent of z. We prove that sesquitransitive algebras have trivial commutant. We show in Section 5 that the Burnside theorem and [Lom73] remain valid in the real case if transitivity is replaced with sesquitransitivity.

2. SC-algebras with localizing adjoint

In this section we make use of the following fixed point theorem due to Ky Fan [Fan52]. Recall that if Ω is a topological space and $C\colon \Omega \to \mathcal{P}(\Omega)$ is a point-to-set map from Ω to the power set of Ω, then C is said to be ***upper semi-continuous*** if for every $x_0 \in \Omega$ and every open set U such that $C(x_0) \subseteq U$ there is a neighborhood V of x_0 such that $C(x) \subseteq U$ whenever $x \in V$.

Theorem 2.1 ([Fan52])**.** *Let K be a compact convex set in a locally convex space, and suppose that C is an upper semi-continuous point-to-set map from K to closed convex non-empty subsets of K. Then there is $x_0 \in K$ with $x_0 \in C(x_0)$.*

Recall that the original proof of the main result in [Lom73] involved the following fact.

Lemma 2.2 ([Lom73])**.** *Let X be a real or complex Banach space, $\mathcal{S}$ a convex transitive subset of $L(X)$, and K a non-zero compact operator. Then there exists $A \in \mathcal{S}$ such that AK has a non-zero fixed vector.*

The following theorem goes along the same lines. Suppose that X is a dual Banach space, i.e., $X = Y^*$ for some Banach space Y. The ***weak* operator topology*** on $L(X)$ is defined as follows: a net (A_α) converges to A in W*OT if $\langle (A_\alpha - A)x, \xi\rangle \to 0$

for all $x \in X$ and $\xi \in Y$. It is known that the norm closed unit ball $B_{L(X)}$ of $L(X)$ is W*OT-compact. It follows easily that if $\mathcal{A}$ is a W*OT-closed subalgebra of $L(X)$, then $B_{\mathcal{A}}$ is also W*OT-compact. If $\mathcal{B}$ a is a subalgebra of $L(Y)$, we write $\overset{*}{\mathcal{B}} = \{A^* \mid A \in \mathcal{B}\}$ and call it the algebra of adjoints of $\mathcal{B}$.

Following [Lom80] we say that an algebra of operators is an ***SC-algebra*** if its unit ball is SOT-relatively compact. It is easy to see that if $\mathcal{A}$ is an SC-algebra then the map $A \in \mathcal{A} \mapsto Ax \in X$ is compact for every $x \in X$. It was shown in [Lom80] that if T is an essentially normal operator on a Hilbert space such that neither $\{T\}'$ nor $\{T^*\}'$ is an SC-algebra, then T has an invariant subspace. However, in the following theorem we use the SC condition in order to prove existence of invariant subspaces.

Theorem 2.3. *Suppose that X is a dual complex Banach space, $X = Y^*$, and $\mathcal{A}$ is a transitive localizing W*OT-closed algebra in $L(X)$ such that $\mathcal{A} = \overset{*}{\mathcal{B}}$ for some SC-algebra $\mathcal{B}$ in $L(Y)$. Then for every non-zero adjoint operator T in $\mathcal{A}'$ there exists $A \in \mathcal{A}$ such that AT has a non-zero fixed vector. Furthermore, T has an invariant subspace.*

Proof. Let T be a non-zero operator in $\mathcal{A}'$ such that $T = S^*$ for some $S \in L(Y)$. Since $\mathcal{A}$ is transitive and $\ker T$ and $\overline{\operatorname{Range} T}$ are $\mathcal{A}$-invariant, T is one-to-one and has dense range.

Let B be a ball as in Definition 1.1. We claim that there exists $r > 0$ such that for every $x \in B$ we have $rB_{\mathcal{A}}(Tx) \cap B \neq \varnothing$, that is, there exists $A \in \mathcal{A}$ such that $\|A\| \leqslant r$ and $ATx \in B$. Indeed, if this were false, then for every n we would find $x_n \in B$ such that $\|A\| \geqslant n$ whenever $A \in \mathcal{A}$ and $ATx_n \in B$. We can choose a subsequence (x_{n_i}) and a sequence of contractions (S_i) in $\mathcal{A}$ such that $S_i x_{n_i} \to w \neq 0$. It follows that $S_i T x_{n_i} = T S_i x_{n_i} \to Tw$. Since $\mathcal{A}$ is transitive, we can find $R \in \mathcal{A}$ such that $RTw \in \operatorname{Int} B$. It follows that for all sufficiently large i we have $RS_iTx_{n_i} \in \operatorname{Int} B$, so that $\|RS_i\| \geqslant n_i \to +\infty$ by our choice of x_{n_i}. But $\|RS_i\| \leqslant \|R\|$ is bounded, contradiction.

Define a set function $C\colon B \to \mathcal{P}(B)$ via $x \in B \mapsto C(x) = B \cap rB_{\mathcal{A}}(Tx)$. By the preceding argument $C(x)$ is non-empty. Clearly, $C(x)$ is convex. Observe also that $C(x)$ is weak* closed for every $x \in B$ because $B_{\mathcal{A}}(Tx)$ is weak* compact as the image of the W*OT-compact set $B_{\mathcal{A}}$ under the map $A \in L(X) \mapsto ATx \in X$ which is W*OT-w*-continuous.

We will show that C is weak* upper semi-continuous. Suppose not; then there exists $x_0 \in B$ and a weak* open set U such that $C(x_0) \subseteq U$, but for every weak* neighborhood α of x_0 there exists $x_\alpha \in \alpha$ such that $C(x_\alpha)$ is not contained in U. Pick any $y_\alpha \in C(x_\alpha) \setminus U$. Let Λ be the set of all weak* neighborhoods of x_0, ordered by the reverse inclusion. The collections $(x_\alpha)_{\alpha\in\Lambda}$ and $(y_\alpha)_{\alpha\in\Lambda}$ can be viewed as nets indexed by Λ, and $x_\alpha \xrightarrow{w^*} x_0$. Since B is weak* compact, by passing to a sub-net if necessary we can assume that $y_\alpha \xrightarrow{w^*} y_0$ for some $y_0 \in B$. Also, $y_\alpha \notin U$ for every α implies $y_0 \notin U$. Note that $y_\alpha \in C(x_\alpha)$ implies that there exists $A_\alpha \in rB_{\mathcal{A}}$ such that $y_\alpha = A_\alpha T x_\alpha$. For every α we have $A_\alpha = F_\alpha^*$ for some F_α in $\mathcal{B}$. Since $\mathcal{B}$ is an SC-algebra, we can assume that $F_\alpha \xrightarrow{SOT} F_0$ for some $F_0 \in L(Y)$. It follows that $A_\alpha \xrightarrow{W^*OT} A_0$, where $A_0 = F_0^*$, so that $A_0 \in rB_{\mathcal{A}}$. Let $\xi \in Y$. We have $SF_\alpha\xi \to SF_0\xi$ in norm (recall that $T = S^*$), so that

$$\langle y_\alpha, \xi\rangle = \langle A_\alpha T x_\alpha, \xi\rangle = \langle x_\alpha, SF_\alpha\xi\rangle \to \langle x_0, SF_0\xi\rangle = \langle A_0 T x_0, \xi\rangle.$$

It follows that $y_0 = A_0 T x_0$, so that $y_0 \in C(x_0) \subseteq U$, contradiction.

Since the map $C\colon B \to \mathcal{P}(B)$ is upper semi-continuous in the weak* topology, it has a fixed point by Ky Fan's Theorem, i.e., there exists $x \in B$ such that $x \in C(x)$. That is, there exists $A \in rB_{\mathcal{A}}$ such that $x = ATx$.

Thus, the fixed space of AT, defined by $F = \ker(I - AT)$, is non-trivial and closed. If T is not invertible then F is proper and we are done. If T is invertible, pick any $\lambda \in \sigma(T)$ and put $S = \lambda I - T$. Then S is not invertible, so that the preceding reasoning yield that S has an invariant subspace. Clearly, it will be also invariant under T. □

Now we can prove a version of Theorem 1.2 for non-quasinilpotent operators.

Corollary 2.4. *Suppose that T is an adjoint operator on a dual complex Banach space. If $\{T\}'$ is localizing and is the algebra of adjoints of an SC-algebra, then T has an invariant subspace.*

Proof. If T has a hyperinvariant subspace then there is nothing to prove. Otherwise, $\{T\}'$ is transitive, and we get the result by Theorem 2.3. □

Note that the hypotheses of Corollary 2.4 are satisfied for a one-to-one compact operator on a reflexive Banach space.

Remark 2.5. Observe that Theorem 2.3 and Corollary 2.4 remain valid for real spaces provided that T doesn't satisfy an irreducible quadratic equation. Indeed, the problem with T occurs only in the last statement in Theorem 2.3 and the last paragraph of its proof. Suppose that $\lambda I - T$ is invertible for all $\lambda \in \mathbb{R}$. Pick any $\alpha + \beta i \in \sigma(T)$, and put $S = \beta^2 I + (T - \alpha I)^2$. It can be easily verified that S is non-zero and not invertible.

Hence, there exists $A \in \mathcal{A}$ such that $F := \ker(I - AS)$ is proper and non-trivial. Clearly, F is T-invariant.

3. Minimal vectors of quasi-localizing algebras

In this section we consider another generalization of Theorem 1.3.

Definition 3.1. Let X be a Banach space, and $\mathcal{A}$ a subalgebra of $L(X)$. We say that $\mathcal{A}$ is ***quasi-localizing*** if there exists a ball B in X not containing the origin, such that given two sequences (x_n) in B and (z_n) in $X \setminus \{0\}$, there exist subsequences (x_{n_i}) and (z_{n_i}), a sequence (S_i) in $\mathcal{A}$, and a real $C > 0$ such that $\|S_i z_{n_i}\| \leqslant C\|z_{n_i}\|$ for all i and $(S_i x_{n_i})$ converges to a non-zero vector w.

Note that we do not require that S_i's are uniformly bounded. It is easy to see that every localizing algebra is quasi-localizing.

Recall some notation and terminology from the method of minimal vectors that will be needed in the proof of Theorem 3.2 (we refer the reader to [AE98] and [Tro04] for details). Suppose that $x_0 \in X$ and $r < \|x_0\|$, so that the closed ball $B = B(x_0, r)$ doesn't contain the origin. Suppose that Q is a one-to-one operator with dense range, and $\varepsilon > 0$. Let $d = \operatorname{dist}(0, Q^{-1}B)$. Choose $y \in Q^{-1}B$ such that $\|y\| \leqslant (1+\varepsilon)d$. Such a y is called a $(1+\varepsilon)$***-minimal vector***. Using the Hahn-Banach Theorem, find $f \in X^*$ of norm one, such that $f_{|B} \geqslant c$ and $f_{|QB(0,d)} \leqslant c$ for some $c > 0$. We call f a ***minimal functional***. It is easy to see that $f(x_0) \geqslant r$ and that the hyperplane $Q^* f = c$ separates (non-strictly) $B(0,d)$ and QB. It follows easily that

$$(Q^* f)(y) \geqslant (1+\varepsilon)^{-1} \|Q^* f\| \|y\|. \tag{1}$$

Repeating the preceding procedure with Q replaces with Q^n for every $n \in \mathbb{N}$, we produce y_n and f_n. Thus, we end up with sequences (y_n) and (f_n) such that y_n is a $(1+\varepsilon)$-minimal vector and f_n is a minimal functional for Q^n and B.

Since every localizing algebra is quasi-localizing, the following theorem is a generalization of Theorem 1.3. The proof is similar to that of the main theorem of [Tro04].

Theorem 3.2. *Suppose that X is a Banach space and $\mathcal{A}$ is a transitive quasi-localizing subalgebra of $L(X)$. Then $\mathcal{A}'$ contains no non-zero quasinilpotent operators.*

Proof. We present a proof for the case of a real Banach space. The complex case can be obtained by straightforward modifications. Suppose that Q is a non-zero quasinilpotent operator in $\mathcal{A}'$. Without loss of generality, $\mathcal{A}$ is unital. Since $\mathcal{A}$ is transitive, Q is one-to-one and has dense range. Let $B = B(x_0, r)$ be the ball as in Definition 3.1. Fix

$\varepsilon > 0$. Let (y_n) and (f_n) be the sequences of $(1+\varepsilon)$-minimal vectors and minimal functionals for B and (Q^n). Then there is a subsequence (n_i) such that $\frac{\|y_{n_i-1}\|}{\|y_{n_i}\|} \to 0$. Indeed, otherwise there would exist $\delta > 0$ such that $\frac{\|y_{n-1}\|}{\|y_n\|} > \delta$ for all n, so that $\|y_1\| \geqslant \delta\|y_2\| \geqslant \ldots \geqslant \delta^n\|y_{n+1}\|$. Since $Q^n y_{n+1} \in Q^{-1}B$, it follows from the definition of y_1 that

$$\big\|Q^n y_{n+1}\big\| \geqslant \frac{\|y_1\|}{1+\varepsilon} \geqslant \frac{\delta^n}{1+\varepsilon}\|y_{n+1}\|.$$

It follows that $\|Q^n\| \geqslant \frac{\delta^n}{1+\varepsilon}$, which contradicts the quasinilpotence of Q.

Since $\|f_{n_i}\| = 1$ for all i, we can assume (by passing to a further subsequence) that (f_{n_i}) weak*-converges to some $g \in X^*$. Since $f_n(x_0) \geqslant r$ for all n, it follows that $g(x_0) \geqslant r$, hence $g \neq 0$.

Observe that the sequence $(Q^{n_i-1}y_{n_i-1})_{i=1}^{\infty}$ is contained in B. Since $\mathcal{A}$ is quasi-localizing, by passing to yet a further subsequence if necessary, we find a sequence (S_i) in $\mathcal{A}$ such that $\|S_i y_{n_i-1}\| \leqslant C\|y_{n_i-1}\|$ and $S_i Q^{n_i-1} y_{n_i-1} \to w \neq 0$. Put

$$Y = \mathcal{A}Qw = \big\{TQw \mid T \in \mathcal{A}\big\}.$$

One can easily verify that Y is a linear subspace of X invariant under $\mathcal{A}$. Since Q is one-to-one, we have $0 \neq Qw$. Since $\mathcal{A}$ is transitive, Y is dense in X. On the other hand, we will show that $Y \subseteq \ker g$, which would lead to a contradiction.

Let $T \in \mathcal{A}$, we will show that $g(TQw) = 0$. It follows from (1) that

$$\begin{aligned}\frac{\big|f_{n_i}(Q^{n_i}TS_i y_{n_i-1})\big|}{f_{n_i}(Q^{n_i}y_{n_i})} &= \frac{\big|(Q^{*n_i}f_{n_i})(TS_i y_{n_i-1})\big|}{(Q^{*n_i}f_{n_i})(y_{n_i})} \\ &\leqslant \frac{\|Q^{*n_i}f_{n_i}\|\|TS_i y_{n_i-1}\|}{(1+\varepsilon)^{-1}\|Q^{*n_i}f_{n_i}\|\|y_{n_i}\|} \leqslant \frac{\|T\|\cdot C\|y_{n_i-1}\|}{(1+\varepsilon)^{-1}\|y_{n_i}\|} \to 0\end{aligned}$$

Since $\|f_{n_i}\| = 1$ and $Q^{n_i}y_{n_i} \in B$, we have

$$f_{n_i}(Q^{n_i}y_{n_i}) \leqslant \|Q^{n_i}y_{n_i}\| \leqslant \|x_0\| + r,$$

it follows that $f_{n_i}(Q^{n_i}TS_i y_{n_i-1}) \to 0$. On the other hand,

$$f_{n_i} \xrightarrow{w^*} g \qquad \text{and} \qquad Q^{n_i}TS_i y_{n_i-1} = TQS_i Q^{n_i-1}y_{n_i-1} \to TQw,$$

therefore $g(TQw) = 0$. □

Recall that an operator S on a Banach space X is ***strictly singular*** if the restriction of S to any infinite-dimensional subspace of X fails to be an isomorphism. It is easy to see that every compact operator is strictly singular, and that strictly singular operators form a norm closed two-sided algebraic ideal in $L(X)$. There is an example of a strictly singular operator without invariant subspaces [Read99]. See [LT77] for further details on strictly singular operators.

It is easy to see that the spectrum of a strictly singular operator consists of eigenvalues and zero. If it has eigenvalues, then every eigenspace is a hyperinvariant subspace. Otherwise, it is quasinilpotent. Therefore, from Theorem 3.2 we can immediately deduce the following result.

Corollary 3.3. *If T is strictly singular and $\{T\}'$ is quasi-localizing, then T has a hyperinvariant subspace.*

Recall that a Banach space is said to be ***hereditarily indecomposable*** if no closed subspace of it can be written as a direct sum of two infinite-dimensional closed subspaces, [GM93]. Every operator on a hereditarily indecomposable Banach spaces is of the form $\lambda I + S$, where S is strictly singular.

Corollary 3.4. *If $\mathcal{A}$ is a transitive quasi-localizing subalgebra on a hereditarily indecomposable Banach space then $\mathcal{A}'$ is trivial.*

Proof. Let $T \in \mathcal{A}'$, then $T = \mu I + S$ for some strictly singular operator S. It follows that $S \in \mathcal{A}'$. Furthermore, given any scalar λ, then $\lambda I - S \in \mathcal{A}'$, so that $\ker(\lambda I - S)$ is invariant under $\mathcal{A}$, hence trivial. Therefore, S has no eigenvalues. It follows that S is quasinilpotent, and Theorem 3.2 yields $S = 0$. $\square$

Suppose now that $\mathcal{S}$ is a collection of one-to-one operators with dense range. Again, fix a ball $B = B(x_0, r)$ with $0 \notin B$, fix $\varepsilon > 0$ and for each $A \in \mathcal{S}$ choose a $(1+\varepsilon)$-minimal vector y_A for A and B.

Proposition 3.5. *Suppose that $\mathcal{S}$, B, ε, and $(y_A)_{A\in\mathcal{S}}$ are as above. If there are nets (z_α) in X and (A_α) in $\mathcal{S}$ such that $\frac{\|z_\alpha\|}{\|y_{A_\alpha}\|} \to 0$ while $A_\alpha z_\alpha$ converges to some $w \neq 0$, then $\mathcal{S}'$ is non-transitive.*

Proof. For $A \in \mathcal{S}$ let f_A be a minimal functional for A and B. By passing to a sub-net if necessary, we can assume that $f_{A_\alpha} \xrightarrow{w^*} g$ for some $g \in X^*$. Again, $g \neq 0$ because $g(x_0) \geqslant r$.

Put $Y = \mathcal{S}'w$. Then, clearly, Y is invariant under $\mathcal{S}'$ and non-trivial as $w \in Y$. We will show that $Y \subseteq \ker g$, this will imply that Y is not dense in X. Let $T \in \mathcal{S}'$, then it follows from (1) that

$$\frac{\left|f_{A_\alpha}(A_\alpha T z_\alpha)\right|}{f_{A_\alpha}(A_\alpha y_{A_\alpha})} = \frac{\left|(A_\alpha^* f_{A_\alpha})(T z_\alpha)\right|}{(A_\alpha^* f_{A_\alpha})(y_{A_\alpha})} \leqslant (1+\varepsilon)\frac{\|A_\alpha^* f_{A_\alpha}\|\|T z_\alpha\|}{\|A_\alpha^* f_{A_\alpha}\|\|y_{A_\alpha}\|} \leqslant (1+\varepsilon)\frac{\|T\|\|z_\alpha\|}{\|y_{A_\alpha}\|} \to 0.$$

Since

$$0 \leqslant f_{A_\alpha}(A_\alpha y_{A_\alpha}) \leqslant \|f_{A_\alpha}\|\|A_\alpha y_{A_\alpha}\| \leqslant \|x_0\| + r,$$

it follows that $f_{A_\alpha}(A_\alpha T z_\alpha) \to 0$. On the other hand, since

$$f_{A_\alpha} \xrightarrow{w^*} g \qquad \text{and} \qquad A_\alpha T z_\alpha = T A_\alpha z_\alpha \to Tw,$$

it follows that $g(Tw) = 0$. □

Consider the condition in Proposition 3.5. We can assume without loss of generality (by scaling) that $\|A_\alpha\| = 1$ for all α. Then (z_α) cannot converge to zero, as this would imply $A_\alpha z_\alpha \to 0$. Thus, it is necessary that $\|y_{A_\alpha}\| \to \infty$. This leads to the following question.

Question. Under what conditions on $\mathcal{S}$ is the set $\{y_A \mid A \in \mathcal{S}, \|A\| = 1\}$ unbounded?

4. Sesquitransitivity

Recall that a set $\mathcal{S} \subseteq L(X)$ is said to be n***-transitive*** for $n \in \mathbb{N}$ if for every linearly independent n-tuple $x_1, \dots, x_n$ in X, for every n-tuple $y_1, \dots, y_n$ in X, and for every $\varepsilon > 0$ there exists $A \in \mathcal{S}$ such that $\|Ax_i - y_i\| < \varepsilon$, $i = 1, \dots, n$. Motivated by the notion of quasi-localizing algebras, we introduce sesquitransitive sets of operators.

Definition 4.1. We say that a set $\mathcal{S}$ in $L(X)$ is ***uniformly sesquitransitive*** if there exists a constant $C > 0$ such that for every linearly independent x and z in X, for every $y \in X$, and for every $\varepsilon > 0$ there exists $A \in \mathcal{S}$ such that $\|Ax - y\| < \varepsilon$ and $\|Az\| \leqslant C\|z\|$. We say that $\mathcal{S}$ is ***sesquitransitive*** if for every non-zero $z \in X$ there is a positive real $C = C(z)$ such that for every x linearly independent of z, for every $y \in X$, and every $\varepsilon > 0$ there exists $A \in \mathcal{S}$ such that $\|Ax - y\| < \varepsilon$ and $\|Az\| \leqslant C\|z\|$.

Clearly, the following implications hold.

2-transitivity $\Rightarrow$ uniform sesquitransitivity $\Rightarrow$ sesquitransitivity $\Rightarrow$ transitivity.

Remark 4.2. It can be easily verified that a uniformly sesquitransitive algebra is quasi-localizing for any ball B not containing the origin and for every non-zero w in Definition 3.1. Indeed, suppose $\mathcal{A}$ is uniformly sesquitransitive with constant C. Let B be any ball centered at x_0 of radius r with $r > \|x_0\|$, and let w be any non-zero vector in B. We claim that $\mathcal{A}$ is quasi-localizing for this ball B and w with constant $\widetilde{C} = C \wedge \frac{\|w\|+1}{\|x_0\|+r}$. Indeed, given a sequence (x_n) in B, and a sequence (z_n) in $X \setminus \{0\}$. Fix $n \in \mathbb{N}$. If x_n and z_n are linearly independent, then we can find $A_n \in \mathcal{A}$ such that $\|A_n x_n - w\| < \frac{1}{n}$ and $\|A_n z_n\| \leqslant C\|z_n\|$. On the other hand, if $z_n = \lambda x_n$ then transitivity of $\mathcal{A}$ implies that there is $A_n \in \mathcal{A}$ such that $\|A_n x_n - w\| < \frac{1}{n}$, so that

$$\|A_n z_n\| = |\lambda|\|A_n x_n\| \leqslant |\lambda|\bigl(\|w\| + \tfrac{1}{n}\bigr) \leqslant |\lambda|\bigl(\|w\| + 1\bigr)\frac{\|x_n\|}{\|x_0\| + r} \leqslant \widetilde{C}\|z_n\|.$$

It is known (see, e.g., [RT05]) that the commutant of a 2-transitive algebra is trivial. The following theorem extends this fact to sesquitransitive algebras. It can be viewed as a counterpart of Theorem 3.2.

Proposition 4.3. *If X is a Banach space and $\mathcal{A}$ is a sesquitransitive subalgebra of $L(X)$, then $\mathcal{A}'$ is trivial.*

Proof. Suppose that $\mathcal{A}$ is sesquitransitive, but there exists $S \in \mathcal{A}'$ such that S is not a multiple of the identity. Then we can find a non-zero $z \in X$ such that Sz is not a multiple of z. Put $x = Sz$. Let $C = C(z)$ in the definition of sesquitransitivity. Choose $y \notin \operatorname{Range} S$ such that $\|y\| > C\|S\|\|z\|$. Then sesquitransitivity of $\mathcal{A}$ implies that for every $n \in \mathbb{N}$ there exists $A_n \in \mathcal{A}$ such that $\|A_n x - y\| \leqslant \frac{1}{n}$ and $\|A_n z\| \leqslant C\|z\|$. It follows that $A_n x \to y$, so that $\|A_n x\| \to \|y\|$. However,

$$\|A_n x\| = \|A_n S z\| = \|S A_n z\| \leqslant \|S\| \cdot C\|z\|,$$

so that $\|y\| \leqslant C\|S\|\|z\|$; a contradiction. $\square$

Next, we consider the algebraic version of sesquitransitivity. Recall that a set $\mathcal{S}$ of linear maps on a vector space is called ***strictly transitive*** if for every two non-zero vectors x and y there exists $A \in \mathcal{S}$ such that $Ax = y$. One says that $\mathcal{S}$ is ***strictly n-transitive*** for $n \in \mathbb{N}$ if for every n linearly independent vectors $x_1, \dots, x_n$, for n vectors $y_1, \dots, y_n$ there exists $A \in \mathcal{S}$ such that $Ax_i = y_i$, $i = 1, \dots, n$.

Definition 4.4. We will say that $\mathcal{S}$ is ***algebraically sesquitransitive*** if for any two non-zero linearly independent vectors x_1 and x_2 there exists a non-zero vector z such that for every non-zero y there exists $A \in \mathcal{S}$ such that $Ax_1 = y$ and $Ax_2 \neq z$.

It should be immediately clear that

$$\text{strict 2-transitivity} \;\Rightarrow\; \text{algebraic sesquitransitivity} \;\Rightarrow\; \text{strict transitivity}.$$

Algebraic sesquitransitivity is similar to sesquitransitivity in the sense that we can send x_1 to any prescribed destination, while keeping some control over the image of x_2. At the first glance it might seem that algebraic sesquitransitivity is just slightly stronger than strict transitivity. However, we will see that for rings it actually implies strict 2-transitivity (hence, we, in fact, have complete control over x_2).

Recall that a set $\mathcal{S}$ of operators is ***strictly dense*** if it is strictly n-transitive for every $n \in \mathbb{N}$. Jacobson's Density Theorem asserts that every strictly 2-transitive ring of linear maps on a vector space over any field is strictly dense. The following is a generalization of Jacobson's Density Theorem.

Theorem 4.5. *Suppose that X is a vector space over an arbitrary field, and $\mathcal{R}$ is a sub-ring of $L(X)$. If $\mathcal{R}$ is algebraically sesquitransitive then it is strictly dense in $L(X)$.*

Proof. It suffices to show that $\mathcal{R}$ is strictly 2-transitive, then the result would follows from Jacobson's Density Theorem. Suppose that $\mathcal{R}$ is not strictly 2-transitive.

Using a standard argument we will show that there exist linearly independent vectors $x_1, x_2 \in X$ such that if $Ax_1 = 0$ for some $A \in \mathcal{R}$, then $Ax_2 = 0$. Indeed, otherwise, for every two linearly independent vectors x_1 and x_2 we would find operators $A, B \in \mathcal{R}$ such that $Ax_1 = 0$ and $Ax_2 \neq 0$, and $Bx_1 \neq 0$ and $Bx_2 = 0$. Furthermore, since $\mathcal{R}$ is strictly transitive, for every y_1 and y_2 in X we would find $C, D \in \mathcal{R}$ such that $C(Ax_2) = y_2$ and $D(Bx_1) = y_1$. Let $S = CA + DB \in \mathcal{R}$, then $Sx_1 = y_1$ and $Sx_2 = y_2$, so that $\mathcal{R}$ is strictly 2-transitive, contradiction.

By Definition 4.4 there exists $z \in X$ such that for every non-zero y there exists $A \in \mathcal{R}$ such that $Ax_1 = y$ and $Ax_2 \neq z$.

Define a linear operator $T \in L(X)$ by $T(Ax_1) = Ax_2$ for every $A \in \mathcal{R}$. It can be easily verified that T is well defined and commutes with every operator in $\mathcal{R}$. Together with strict transitivity of $\mathcal{R}$ this yields that T is a bijection. Therefore, one can find a non-zero $y \in X$ such that $Ty = z$. Then there exists $A \in \mathcal{R}$ such that $Ax_1 = y$ and $Ax_2 \neq z$. However, $Ax_2 = TAx_1 = Ty = z$, contradiction. $\square$

We would like to mention that strict n-semitransitivity introduced in [RT] is another generalization of strict n-transitivity. In a result similar to Theorem 4.5, [RT] shows that every strictly 2-semitransitive ring is strictly dense.

5. Transitive and sesquitransitive algebras in real spaces

It is easy to see that if $\dim X < \infty$ then a subalgebra of $L(X)$ is transitive iff it is strictly transitive. Recall that a classical theorem of Burnside asserts that $M_n(\mathbb{C})$ has no proper transitive subalgebras (see, e.g., [RR00]). Clearly, this is false in the real case: the algebra generated by the rotation through $\pi/2$ in $L(\mathbb{R}^2)$ is transitive but proper. In this section we establish several analogues of the Burnside Theorem as well as of [Lom73] for algebras on real Banach spaces.

Recall that a unital algebra is a ***division*** algebra if every non-zero element in it is invertible. It was proved by Rickart [Ric60, Theorem 1.7.6.] that every real normed division algebra is algebraically isomorphic to either $\mathbb{R}$, $\mathbb{C}$, or $\mathbb{H}$. Here $\mathbb{H}$ stands for the quaternion algebra.

Suppose that $\mathcal{A}$ is a transitive subalgebra of $M_n(\mathbb{R})$. It follows from Schur's Lemma and from Wedderburn-Artin Theorem that $\mathcal{A}'$ is algebraically isomorphic to either $\mathbb{R}$, $\mathbb{C}$, or $\mathbb{H}$. Furthermore, $\mathcal{A}'' = \mathcal{A}$, and $\mathcal{A}$ is (algebraically isomorphic to) $M_n(\mathbb{R})$, $M_{\frac{n}{2}}(\mathbb{C})$, or $M_{\frac{n}{4}}(\mathbb{H})$, respectively.

Next, we consider some consequences of these facts for infinite dimensional real Banach spaces. In particular, we consider a version of [Lom73] for algebras of operators. Namely, it follows from [Lom73] that if X is a complex Banach space then *every transitive subalgebra of $L(X)$ containing a compact operator is WOT-dense in $L(X)$*, see [RR03, Theorem 8.23]. This statement fails in real Banach spaces. We will prove a version of this statement for transitive algebras in real Banach spaces.

Proposition 5.1. *Suppose that $\mathcal{A}$ is a transitive algebra of operators on a real Banach space X, and $\mathcal{A}'$ has a finite-dimensional invariant subspace. Then $\mathcal{A}'$ is algebraically isomorphic to $\mathbb{R}$, $\mathbb{C}$, or $\mathbb{H}$.*

Proof. Suppose that M is a finite-dimensional subspace invariant under $\mathcal{A}'$. Consider the restriction map $\Phi\colon \mathcal{A}' \to L(M)$ given by $\Phi(T) = T_{|M}$. For every non-zero $T \in \mathcal{A}'$, transitivity of $\mathcal{A}$ implies that $\ker T$ and, therefore, $\ker T_{|M}$ are trivial. It follows that Φ is one-to-one. Moreover, $\Phi(T)$ is invertible. Hence, $\Phi(\mathcal{A}')$ is a division algebra. Now Rickart's Theorem implies that $\Phi(\mathcal{A}')$ and, therefore, $\mathcal{A}'$ is isomorphic to $\mathbb{R}$, $\mathbb{C}$, or $\mathbb{H}$. $\square$

Theorem 5.2. *Suppose that X is a real Banach space and $\mathcal{A}$ is a transitive subalgebra of $L(X)$ containing a compact operator. Then $\mathcal{A}'$ is algebraically isomorphic to either $\mathbb{R}$, $\mathbb{C}$, or $\mathbb{H}$.*

Proof. By Lemma 2.2, $\mathcal{A}$ contains a compact operator K with eigenvalue 1. Then the corresponding eigenspace $M := \ker(K - I)$ is a finite-dimensional subspace invariant under $\mathcal{A}'$. Now the conclusion follows from Proposition 5.1. $\square$

Proposition 5.3. *Suppose that V is a finite-dimensional vector space over $\mathbb{R}$, $\mathcal{A}$ is a transitive subalgebra of $L(V)$, and $T \in \mathcal{A}'$ such that T is not a multiple of the identity. Then $\{T\}'$ is least possible, that is, $\{T\}'$ is the algebra generated by T and $\mathcal{A}$.*

Proof. We reduce the real case to the complex case as in [Sir05]. It follows from Proposition 5.1 that by replacing T with $\lambda I + \mu T$ for some $\mu, \lambda \in \mathbb{R}$ we can assume that $T^2 = -I$. Define a complex structure on V by putting $ix = Tx$ for $x \in V$. One can easily check that with this complex scalar multiplication V becomes a vector space over $\mathbb{C}$; denote it by $V_{\mathbb{C}}$. Observe that V and $V_{\mathbb{C}}$ coincide as sets. Note that an operator $S \in L(V)$ belongs to $L(V_{\mathbb{C}})$ iff it is complex-linear, or, equivalently, if $ST = TS$. Hence, $L(V_{\mathbb{C}}) = \{T\}'$. It follows from $T \in \mathcal{A}'$ that we can view $\mathcal{A}$ as a subset of $L(V_{\mathbb{C}})$. Note that $\mathcal{A}$ is still transitive (as the definition of transitivity doesn't involve scalar multiplication). However, $\mathcal{A}$ need not be closed under complex multiplication in $L(V_{\mathbb{C}})$. Let $\widetilde{\mathcal{A}} = \{A + iB \mid A, B \in \mathcal{A}\}$ in $L(V_{\mathbb{C}})$. Then $\widetilde{\mathcal{A}}$ is a transitive subalgebra of $L(V_{\mathbb{C}})$. Hence, by the Burnside Theorem, $\mathcal{A}$ is all of $L(V_{\mathbb{C}})$, so that $\{T\}' = \widetilde{\mathcal{A}} = \big\{A + TB \mid A, B \in \mathcal{A}\big\}$. □

Finally, we will prove that the Burnside Theorem and the algebraic version of [Lom73] remain valid over real scalars if transitivity is replaced with sesquitransitivity.

Proposition 5.4. *Suppose that $\mathcal{A}$ is a subalgebra of $M_n(\mathbb{R})$. If $\mathcal{A}$ is sesquitransitive or algebraically sesquitransitive then $\mathcal{A} = M_n(\mathbb{R})$.*

Proof. If $\mathcal{A}$ is algebraically sesquitransitive then it is strictly n-transitive by Theorem 4.5, hence $\mathcal{A} = M_n(\mathbb{R})$. Now, if $\mathcal{A}$ is sesquitransitive then $\mathcal{A}'$ is trivial by Proposition 4.3, so that $\mathcal{A} = \mathcal{A}'' = M_n(\mathbb{R})$. □

The following is well known for complex Banach spaces.

Lemma 5.5. *Suppose that K is a compact operator on a real Banach space X such that K has a non-zero fixed vector. Then the uniformly closed subalgebra of $L(X)$ generated by K contains an idempotent of finite rank.*

Proof. It follows immediately from the hypotheses that 1 is an eigenvalue of K. Since K is compact, so is its complexification K_c on X_c. Let Z be the spectral subspace of K corresponding to $\{1\}$. It follows from $\sigma(K_{c|Z}) = \{1\}$ that $K_{c|Z}$ is invertible. Since $K_{c|Z}$ is compact, it follows that Z is finite-dimensional. Using the usual Functional Calculus, we can find the canonical spectral projection onto Z. Recall that we can write this projection as $f(K)$, where f is the characteristic function of an open subset U of $\mathbb{C}$ such that $U \cap \sigma(K) = \{1\}$.

Let $\mathcal{A}$ be the uniformly closed algebra generated by K_c in $L(X_c)$. It follows from Theorem 5.4(a) of [Con90] that $\sigma_{\mathcal{A}}(K) = \sigma(K)$, so that $f(K) \in \mathcal{A}$. It is left to show that $f(K)$ is actually a real operator, that is, that $f(K)$ leaves X invariant.

Again, by Function Calculus we can write $f(K) = \frac{1}{2\pi i}\int_\Gamma R(\lambda; K)\,d\lambda$, where $R(\lambda; K)$ is the resolvent of K at λ and the integration is done over a circle centered at 1 and contained in U. Observe that for $x \in X$ we have the following relation for the complex conjugates in X_c: $R(\overline{\lambda}; K)x = \overline{R(\lambda; K)x}$. Indeed, direct verification shows that if $x = (\lambda - K)(y + iz)$ then $x = (\overline{\lambda} - K)(y - iz)$ for $y, z \in X$. It follows that $f(K)x = \frac{1}{2\pi i}\int_\Gamma R(\lambda; K)x\,d\lambda$ belongs to X. □

Theorem 5.6. *If X is a real Banach space then every sesquitransitive subalgebra $\mathcal{A}$ of $L(X)$ containing a compact operator is WOT-dense in $L(X)$.*

Proof. Without loss of generality we assume that $\mathcal{A}$ is uniformly closed. Lemma 2.2 yields that there is a compact operator K in $\mathcal{A}$ such that K has a non-zero fixed vector. It follows from Lemma 5.5 that $\mathcal{A}$ contains an idempotent operator P of finite rank. Let $Y = \operatorname{Range} P$, then $\dim Y < \infty$.

We will show that the restriction algebra $P\mathcal{A}P$ is still sesquitransitive on Y. Indeed, let $z \in Y$, then there exists C such that for all x, y in X such that x and z are linearly independent and for every $\varepsilon > 0$ there is $A \in \mathcal{A}$ such that $\|Ax - y\| < \varepsilon$ and $\|Az\| \leqslant C\|z\|$. In particular, when $x, y \in Y$ we have

$$\|PAPx - y\| = \|P(Ax - y)\| \leqslant \|P\|\varepsilon$$

and $\|PAPz\| \leqslant \|P\| \cdot C\|z\|$.

Proposition 5.4 implies that the restriction of $P\mathcal{A}P$ is all of $L(Y)$. It follows that $P\mathcal{A}P$ and, therefore, $\mathcal{A}$ contains an operator of rank one. Now a standard argument (see, e.g., the proof of Lemma 7.4.5 in [RR00]) shows that $\mathcal{A}$ contains all operators of finite rank, hence is WOT-dense in $L(X)$. □

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