INVARIANT SUBSPACES OF POSITIVE QUASINILPOTENT OPERATORS ON ORDERED BANACH SPACES

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Abstract. In this paper we find invariant subspaces of certain positive quasinilpotent operators on Krein spaces and, more generally, on ordered Banach spaces with closed generating cones. In the later case, we use the method of minimal vectors. We present applications to Sobolev spaces, spaces of differentiable functions, and C*-algebras.

0. Introduction and notations

Lomonosov proved in [Lom73] that if an operator T on a Banach space is not a multiple of the identity and commutes with a non-zero compact operator, then there is a closed (proper non-zero) subspace which is invariant under every operator commuting with T. The following result shows that the situation is even better for positive operators on Banach lattices.

Theorem 0.1 ([AAB94]). *Let S and T be two positive commuting operators on a Banach lattice such that S is quasinilpotent and dominates a non-zero positive compact operator. Then T has a closed invariant subspace.*

Moreover, this invariant subspace can be chosen to be a closed order ideal. Several variations of this result can be found in [AAB93, AAB94, AAB98, AA02, Drn01]. In particular, $ST = TS$ may be replaced with $ST \leqslant TS$. In [AE98], the method of minimal vectors was used to show that the presence of a compact operator can be replaced with a weaker "localization" condition. In the present paper we extend some of these results beyond Banach lattices. We show that many of them remain valid in ordered Banach spaces with closed generating cones. We present applications to spaces of differentiable functions, C^*-algebras, and Sobolev spaces.

The paper is organized as follows. In Section 1, we consider invariant subspaces of operators on Krein spaces. In Sections 2, 3 and 4, we consider applications of the

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results of Section 1. In Section 5, we extend the method of minimal vectors to ordered Banach spaces with generating cones, and in Section 6 we present applications of this method.

We now introduce some notations from the theory of ordered Banach spaces; for further details we refer the reader to [AB85, AT07]. A vector space X over $\mathbb{R}$ is an ***ordered vector space*** if it is equipped with an order relation such that $a \leqslant b$ implies $a + c \leqslant b + c$ and $\lambda a \leqslant \lambda b$ for all $a, b, c \in X$ and $\lambda \in \mathbb{R}_+$. The ***positive cone*** of X is defined as $X_+ = \{x \in X : x \geqslant 0\}$. Furthermore, X is said to be a ***vector lattice*** if the order is a lattice order, hence $x \vee y$ and $x \wedge y$ exist for all x and y. In particular, $x^+ = x \vee 0$, $x^- = (-x) \vee 0$, and $|x| = x \vee (-x)$ are defined for every $x \in X$, and we have $x = x^+ - x^-$ and $|x| = x^+ + x^-$. We say that X is an ***ordered Banach space*** if it is a Banach space and an ordered vector space such that X_+ is norm-closed. If, in addition, $X = X_+ - X_+$, then we say that X is a ***Banach space with a generating closed cone***. A ***lattice-ordered Banach space*** is an ordered Banach space with the order being a lattice order. A lattice-ordered Banach space is a ***Banach lattice*** if $|x| \leqslant |y|$ implies $\|x\| \leqslant \|y\|$ for all $x, y \in X$.

A few examples can be mentioned. The classical spaces $L_p(\mu)$, where μ is a measure and $1 \leqslant p \leqslant \infty$, as well as $C_0(\Omega)$, where Ω is a locally compact Hausdorff space, are Banach lattices. However, there are many important ordered Banach spaces with generating cones which are not Banach lattices. One could mention, in particular, the spaces $C^k[0,1]$ of all k times continuously differentiable functions on $[0,1]$, as well as C*-algebras. Sobolev spaces $W^{k,p}(\Omega)$ are ordered Banach spaces. Moreover, we observe in Section 4 that for $k = 1$ they are lattice-ordered Banach spaces. However, they fail to be Banach lattices as the norm is not monotone on X_+.

Throughout this paper, X will usually stand for an ordered vector or Banach space. By a *subspace* of X we will always mean a linear subspace. By an *operator* on X we will understand a linear operator. If $a < b$ in X, we write $[a, b] = \{x \in X : a \leqslant x \leqslant b\}$. Sets of this form are called ***order intervals***. An operator T on X is ***positive*** if $Tx \geqslant 0$ whenever $x \geqslant 0$. In this case we write $T \geqslant 0$. We write $T \leqslant S$ if $S - T \geqslant 0$. If S and T are two operators on X, we say that T is ***dominated*** by S if $\pm Tx \leqslant Sx$ whenever $x \geqslant 0$. If X is a vector lattice, this is equivalent to $|Tx| \leqslant S|x|$ for every $x \in X$ because $\pm Tx = \pm(Tx^+ - Tx^-) \leqslant Sx^+ + Sx^- = S|x|$.

A subspace E of X is said to be ***invariant*** under a collection $\mathcal{S}$ of operators on X if $\{0\} \neq E \neq X$ and $T(E) \subseteq E$ for every $T \in \mathcal{S}$. For $A, B \subseteq X$ and $u \in X$ we write $u + B = \{u + b : b \in B\}$ and $A + B = \{a + b : a \in A, b \in B\}$. If A and B are two subsets of X, we say that A ***minorizes*** B if for each b in B there exists $a \in A$ such that $a \leqslant b$.

Suppose now that X is an ordered Banach space. It follows from $[a,b] = (a+X_+) \cap (b-X_+)$ that order intervals are closed sets. We will write $L(X)$ for the space of all bounded operators on X. Note that $L(X)$ is an ordered Banach space, so that if $T \leqslant S$ in $L(X)$, we write $[T,S] = \{R \in L(X) : T \leqslant R \leqslant S\}$ (in particular, all the operators in $[T,S]$ are bounded). The ***commutant*** of $T \in L(X)$ is the set $\{T\}' = \{S \in L(X) : TS = ST\}$. Following [AA02], we define the ***super left-commutant*** $\langle Q]$ and the ***super right-commutant*** of $[Q\rangle$ of Q as follows:

$$\langle Q] = \{T \geqslant 0 : TQ \leqslant QT\} \qquad \text{and} \qquad [Q\rangle = \{T \geqslant 0 : TQ \geqslant QT\}.$$

We will make use of the following well-known theorem (see, e.g., [AAB92]).

Theorem 0.2 (Lozanovski)**.** *Every positive operator between ordered Banach spaces with generating closed cones is bounded.*

Recall that a subspace E of a vector lattice X is said to be an (order) ideal if $a \in E$ and $|x| \leqslant |a|$ imply $x \in E$. If $A \subseteq X$, then $I(A)$ stands for the smallest ideal in X containing A. It is easy to see that

$$I(A) = \Big\{x \in X : |x| \leqslant \sum_{i=1}^{n} \lambda_i |a_i|, \lambda_1, \ldots, \lambda_n \in \mathbb{R} \text{ and } a_1, \ldots, a_n \in A\Big\}.$$

We now extend these concepts to ordered vector spaces. A subspace E in an ordered vector space is said to be an ***(order) ideal*** if

(i) $x \in E$ implies that there exists a positive $a \in E$ such that $x \leqslant a$, and
(ii) $\pm x \leqslant a \in E$ implies $x \in E$.

Note that in a vector lattice this definition agrees with the usual one. Furthermore, if X is an ordered vector space and A is a convex subset of X, we will write

$$I_0(A) = \big\{x \in X : \pm x \leqslant \lambda a \text{ for some } \lambda \in \mathbb{R}_+ \text{ and some } a \in A\big\}.$$

Note that $I_0(A)$ need not contain A. The following three lemmas are elementary, we leave the proofs to the reader.

Lemma 0.3. *Suppose that X is an ordered vector space.*

(i) *If A is a convex subset of X then $I_0(A)$ is an ideal; $I_0(A)$ is contained in every ideal containing A.*
(ii) *A subset $E \subseteq X$ is an ideal iff $E = \bigcup_{a \in E_+} I_0(a)$.*

(iii) *Suppose that E and F are two ideals in X such that $I_0(a) \cap I_0(b)$ is an ideal whenever $0 \leqslant a \in E$ and $0 \leqslant b \in F$. Then $E \cap F$ is an ideal.*

Lemma 0.4. *Suppose that X is a vector lattice and $A \subseteq X$ such that either*

(i) *A is a convex subset of X_+, or*
(ii) *A is the range of a positive operator on X.*

Then $I_0(A) = I(A)$.

Lemma 0.5. *Let X be an ordered vector space with a generating cone and Q a positive operator on X. Then $I_0(\mathrm{Range}\, Q)$ is an ideal invariant under $\langle Q]$.*

Note that if X is a Banach lattice and E is an ideal in X then $\overline{E}$ is still an ideal. This remains true if X is a lattice-ordered Banach spaces with continuous lattice operations. Since various lattice operations can be expressed via each other, the continuity of any of the maps $x \mapsto x^+$, $x \mapsto x^-$, $x \mapsto |x|$, $x \mapsto x \wedge y$, and $x \mapsto x \vee y$ implies the continuity of the other maps and joint continuity of $x \vee y$ and $x \wedge y$.

Proposition 0.6. *Suppose that X is a lattice-ordered Banach space with continuous lattice operations and E is an ideal in X. Then $\overline{E}$ is again an ideal.*

Proof. Suppose that $x \in \overline{E}$. Take a sequence (x_n) in E such that $x_n \to x$. Then $E \ni |x_n| \to |x|$, hence $|x| \in \overline{E}$. Suppose now that $0 \leqslant y \leqslant x \in \overline{E}$. Again, take a sequence (x_n) in E such that $x_n \to x$. Then for every n we have $E \ni |x_n| \wedge y \to x \wedge y = y$, hence $y \in \overline{E}$. □

1. Positive quasinilpotent operators on Krein spaces

Suppose that X is an ordered Banach space (in particular, X_+ is closed). Recall that $u \in X_+$ is said to be a (strong) order ***unit*** if for every $x \in X$ there exists $\lambda > 0$ such that $x \leqslant \lambda u$ or, equivalently, if $I_0(u) = X$. We make use of the following standard lemma.

Lemma 1.1 (c.f. [AAB92, Lemma 3.2])**.** *Suppose that X is an ordered Banach space and $u \in X_+$. Then the following statements are equivalent:*

(i) *u is an order unit;*
(ii) *$u \in \mathrm{Int}\, X_+$;*
(iii) *$\lambda B_0 \subseteq [-u, u]$ for some $\lambda \in \mathbb{R}_+$, where B_0 is the unit ball of X.*

Proof. (iii)⇒(i) is trivial. To show (ii)⇒(iii), suppose that $u \in \operatorname{Int} X_+$. Then $u+\lambda B_0 \subseteq X_+$ for some $\lambda \in \mathbb{R}_+$. It follows that for every $x \in X$ with $\|x\| \leqslant \lambda$ we have $u \pm x \geqslant 0$, so that $-u \leqslant x \leqslant u$, hence $\lambda B_0 \subseteq [-u,u]$.

To show (i)⇒(ii), suppose that u is an order unit. Then $X = \bigcup_{n=1}^{\infty}[-nu,nu]$. By the Baire Category Theorem, $[-u,u]$ has non-empty interior, so that $B(a,\lambda) \subseteq [-u,u]$ for some $a \in X$ and $\lambda \in \mathbb{R}_+$. Now suppose that $x \in B(u,\lambda)$, then $a \pm (x-u) \in B(a,\lambda) \subseteq [-u,u]$. Now $-u \leqslant a+(x-u)$ implies $x+a \geqslant 0$, while $a-(x-u) \leqslant u$ implies $x-a \geqslant 0$. Adding these two inequalities together we get $x \geqslant 0$, so that $B(u,\lambda) \subseteq X_+$. □

An ordered Banach space with an order unit is said to be a ***Krein space***. Clearly, the positive cone in a Krein space is generating. We would like to mention the following result, which is a corollary of Krein's theorem [KR48], see also [AAB92, SW99, OT05].

Theorem 1.2. *If T is a positive operator on a Krein space then there is a closed subspace invariant under $\{T\}'$.*

Suppose that X is an ordered Banach space. Recall that $w \in X_+$ is said to be ***quasi-interior*** if $I_0(w)$ is dense in X. Clearly, every order unit is quasi-interior.

Lemma 1.3. *If X is a Krein space then every quasi-interior point is an order unit.*

Proof. Let $w \in X_+$ be quasi-interior. By Lemma 1.1 there exists $u \in X_+$ such that $B_0 \subseteq [-u,u]$. It follows that $2u + B_0 \subseteq [u,3u]$. Since $I_0(w)$ is dense, it meets $2u + B_0$. It follows that there exists $x \in B_0$ such that $u \leqslant 2u + x \leqslant \lambda w$ for some $\lambda \in \mathbb{R}_+$. Therefore, $w \geqslant \frac{1}{\lambda}u$, hence $I_0(w) \supseteq I_0(u) = X$. □

Theorem 1.4. *Suppose that Q is a positive quasinilpotent operator on a Krein space X. Then $\langle Q]$ has a non-dense invariant ideal.*

Proof. Since X is a Krein space, there is an order unit u in X_+. Let $w = Qu$. We claim that w is not quasi-interior. Indeed, otherwise it would be a unit by Lemma 1.3, so that $u \leqslant \lambda w$ for some $\lambda \in \mathbb{R}_+$. It follows that $Qu \geqslant \frac{1}{\lambda}u$, so that $Q^n u \geqslant \frac{1}{\lambda^n}u$, hence $\lambda^n Q^n u$ is contained in $u + X_+$. Since λQ is quasinilpotent, we have $\lambda^n Q^n u \to 0$. But $u + X_+$ is closed, hence $0 \in u + X_+$, a contradiction.

Let $x \in X$, then $\pm x \leqslant \lambda u$ for some $\lambda \in \mathbb{R}_+$, so that $\pm Qx \leqslant \lambda w$. It follows that $\operatorname{Range} Q \subseteq I_0(w)$, so that $I_0(\operatorname{Range} Q) = I_0(w)$, hence $I_0(\operatorname{Range} Q)$ is not dense. Now apply Lemma 0.5. □

2. Applications to $C(K)$ and $C^k(\overline{\Omega})$ spaces and to C*-algebras

In this section we apply the results of the preceding sections to unital uniform algebras. We will see that these algebras are Krein spaces, and apply Theorem 1.4 to find invariant closed subspaces for positive quasinilpotent operators on them. Moreover, these subspaces can be chosen to be the closures of non-dense invariant order ideals. It should be pointed out that in this setting one has to distinguish between the algebraic ideals and the order ideals. The non-unital case will be considered in Section 6.

Clearly, for every compact Hausdorff space K, the space $C(K)$ is a Krein space with $\mathbb{1}$ being an order unit. Hence, Theorem 1.4 yields the following.

Corollary 2.1. *If Q is a positive quasinilpotent operator on $C(K)$ where K is a compact Hausdorff space, then $\langle Q]$ has a closed invariant ideal.*

Let Ω be an open connected subset of $\mathbb{R}^n$. Recall that the space $C^k(\overline{\Omega})$ consists of all real-values functions f on Ω such that $D_\alpha f$ is bounded and uniformly continuous on Ω whenever $|\alpha| \leqslant k$. With the norm defined by $\|f\| = \sum_{|\alpha|\leqslant k}\|D_\alpha f\|_{\sup}$, the set $C^k(\overline{\Omega})$ is an ordered Banach space, see, e.g., [KJF77]. Clearly, $\mathbb{1}$ is an order unit in $C^k(\overline{\Omega})$, hence $C^k(\overline{\Omega})$ is a Krein space. Therefore, by Theorem 1.4 we have the following.

Corollary 2.2. *If Q is a positive quasinilpotent operator on $C^k(\overline{\Omega})$ then $\langle Q]$ has an invariant non-dense order ideal. In particular, it has a closed invariant subspace.*

Before we proceed to C*-algebras, we would like to make a remark about applicability of our techniques to complex Banach spaces. In all our previous results we assumed X to be an ordered Banach space over real scalars. By a complex ordered Banach space we understand the complexification $X_c = X + iX$ of an ordered Banach space X over $\mathbb{R}$ (see [AA02] on complexifications of ordered Banach spaces). An operator $T \in L(X_c)$ is said to be positive if $T(X_+) \subseteq X_+$, where X_+ is the positive cone of X. Suppose that X_+ is generating in X. Then every positive operator T in $L(X_c)$ is the complexification of its restriction to X, that is, $T = (T_{|X})_c$. It is easy to see that if a subspace V of X is invariant under $S \in L(X)$, then $V + iV$ is a subspace of X_c invariant under S_c. Hence, in order to prove that a positive operator T on X_c has an invariant subspace, it suffices to find an invariant subspace for the restriction of T to X.

Let $\mathcal{A}$ be a C*-algebra. Then $\mathcal{A}$ is a Banach space over $\mathbb{C}$. Observe that its self-adjoint part $\mathcal{A}_{\rm sa}$ is a real Banach space. Recall that for $x \in \mathcal{A}_{\rm sa}$ we write $x \geqslant 0$ when $\sigma(x) \subseteq [0, +\infty)$. With this order, the positive cone $\mathcal{A}_+$ is closed and generating (generally, $\mathcal{A}_{\mathrm{sa}}$ is not a lattice). Furthermore, $\mathcal{A}$ can be viewed as the complexification of $\mathcal{A}_{\mathrm{sa}}$. Suppose that T is a positive operator on $\mathcal{A}$. By the preceding paragraph, if its restriction to $\mathcal{A}_{\mathrm{sa}}$ has an invariant subspace in $\mathcal{A}_{\mathrm{sa}}$, then the T has an invariant subspace in $\mathcal{A}$. Hence, it suffices to look for invariant subspaces of positive operators on $\mathcal{A}_{\mathrm{sa}}$.

If, in addition, $\mathcal{A}$ is unital, then $\mathcal{A}_{\mathrm{sa}}$ is a Krein space. Indeed, suppose that $x \in \mathcal{A}_{\mathrm{sa}}$ with $\|x\| \leqslant 1$. Then $\sigma(\pm x) \subseteq [-1,1]$ and the Spectral Mapping Theorem yields $\sigma(e \pm x) \subseteq [0,2] \subseteq \mathbb{R}_+$, so that $\pm x \leqslant e$ and, therefore, $x \in [-e,e]$. It follows from Lemma 1.1 that $\mathcal{A}_{\mathrm{sa}}$ is a Krein space. Hence, Theorem 1.4 yields the following result.

Theorem 2.3. *If Q is a positive quasinilpotent operator on a unital C^*-algebra then $\langle Q]$ has an invariant non-dense order ideal. In particular, it has a closed invariant subspace.*

3. Applications to Sobolev spaces $W^{k,p}(\Omega)$

Throughout this section we assume that Ω is a bounded open subset of $\mathbb{R}^N$, $1 \leqslant p \leqslant \infty$, and $k \in \mathbb{N}$. The Sobolev space $W^{k,p}(\Omega)$ is defined as the set of all those functions $f \in L_p(\Omega)$ for which the weak partial derivatives $D_\alpha f$ exist and are in $L_p(\Omega)$ for each multi-index α with $|\alpha| \leqslant k$. The norm on $W^{k,p}(\Omega)$ is given by

$$\|f\| = \begin{cases} \Big[\sum\limits_{|\alpha|\leqslant k} \big\|D_\alpha f\big\|_{L_p}^p\Big]^{\frac{1}{p}} & \text{if } p<\infty, \\ \sum\limits_{|\alpha|\leqslant k} \big\|D_\alpha f\big\|_{L_\infty} & \text{if } p=\infty, \end{cases}$$

where $\|\cdot\|_{L_p}$ is the norm in $L_p(\Omega)$. As usually, we will write $\mathbb{1}$ for χ_Ω. For more details on Sobolev spaces we refer the reader to [GT77, PW03, KJF77].

We equip $W^{k,p}(\Omega)$ with the a.e. order, i.e., the order inherited from $L_p(\Omega)$. Note that X_+ is norm-closed. Indeed, let (f_n) be a sequence in X_+ such that $f_n \to f$ in $W^{k,p}(\Omega)$. Then $\|f_n - f\|_{L_p} \to 0$, so that $f \geqslant 0$ because the positive cone of $L_p(\Omega)$ is closed. Thus, $W^{k,p}(\Omega)$ is an ordered Banach space. It is well known that as a Banach space $W^{k,p}(\Omega)$ is isomorphic to $L_p(0,1)$ when $1 < p < \infty$, see, e.g., [PW03, Theorem 11]. However, we will see that the order structure of $W^{k,p}(\Omega)$ is very different from that of $L_p(\Omega)$.

Remark 3.1. Observe that the norm of Sobolev spaces is generally not monotone. For example, let $f \in W^{1,1}[0,1]$ be defined as follows: $f(t) = t$ for all $t \in [0,1]$. Then $0 \leqslant f \leqslant \mathbb{1}$, but $\|f\| > \|\mathbb{1}\|$.

The question of existence of invariant subspaces of positive operators on Sobolev spaces was investigated in [IM04], and a variant of Theorem 0.1 for $W^{1,p}(\Omega)$ was proved there.

If $p = \infty$ then $\mathbb{1}$ is an order unit in $W^{k,\infty}(\Omega)$. It follows that $W^{k,\infty}(\Omega)$ is a Krein space, so that if Q is a positive quasinilpotent operator on $W^{k,\infty}(\Omega)$ then $\langle Q]$ has an invariant non-dense ideal by Theorem 1.4. For this reason, from now on we assume that $p < \infty$.

Let $X = W^{k,p}(\Omega)$ such that Ω is regular (i.e., $\partial\Omega$ is of class $\mathcal{C}^{0,1}$, see, e.g., [GT77] for the precise definition of regular domains) and $p > N/k$ or $p = N/k = 1$. The classical Sobolev embedding theorem asserts that in this case there exists $C > 0$ such that $\|f\|_\infty \leqslant C\|f\|$ for every $f \in X$. It follows that the unit ball of X is contained in $[-C\mathbb{1}, C\mathbb{1}]$. Combining this observation with Lemma 1.1, we obtain the following result.

Theorem 3.2. *If Ω is regular and $p > N/k$ or $p = N/k = 1$ then $W^{k,p}(\Omega)$ is a Krein space.*

In particular, $W^{1,p}[0,1]$ is a Krein space for any $1 \leqslant p < \infty$. The next result now follows immediately from Theorems 0.2, 3.2, 1.2, and 1.4.

Theorem 3.3. *Suppose that Ω is regular and $p > N/k$ or $p = N/k = 1$. If Q is a positive operator on $W^{k,p}(\Omega)$ then Q is bounded and there is a closed subspace invariant under $\{Q\}'$. If, in addition, Q is quasinilpotent, then $\langle Q]$ has a non-dense invariant ideal.*

4. A special case: $W^{1,p}(\Omega)$

In this section we consider the case $k = 1$. The following fact from [GT77] shows that $W^{1,p}(\Omega)$ is a sublattice of $L_p(\Omega)$.

Lemma 4.1 ([GT77, Lemma 7.6])**.** *If $f \in W^{1,p}(\Omega)$ then f^+, f^-, and $|f|$ are also in $W^{1,p}(\Omega)$ and*

$$\frac{\partial f^+}{\partial x_i} = \begin{cases} \frac{\partial f}{\partial x_i} & \text{if } f > 0 \\ 0 & \text{if } f \leqslant 0 \end{cases}, \quad \frac{\partial f^-}{\partial x_i} = \begin{cases} 0 & \text{if } f \geqslant 0 \\ \frac{\partial f}{\partial x_i} & \text{if } f < 0 \end{cases}, \quad \frac{\partial |f|}{\partial x_i} = \begin{cases} \frac{\partial f}{\partial x_i} & \text{if } f > 0 \\ 0 & \text{if } f = 0 \\ -\frac{\partial f}{\partial x_i} & \text{if } f < 0 \end{cases}$$

for each $i = 1, \dots, N$ (the equalities are a.e.).

As usually, $|f|$, f^+, and f^- are defined pointwise. Hence, $W^{1,p}(\Omega)$ is a vector lattice and $\big\||f|\big\| = \|f\|$ for every $f \in W^{1,p}(\Omega)$ (recall that $W^{1,p}(\Omega)$ is generally not a Banach lattice by Remark 3.1). In particular, the positive cone of $W^{1,p}(\Omega)$ is generating. Together with Theorem 0.2 this immediately yields the following.

Corollary 4.2. *Every positive operator on $W^{1,p}(\Omega)$ is bounded.*

Theorem 4.3. *The Sobolev space $W^{1,p}(\Omega)$ is a lattice-ordered Banach space with continuous lattice operations.*

Proof. Let $X = W^{1,p}(\Omega)$. In view of the preceding discussion, we only need to verify the continuity of the lattice operations in X. Take (f_n) in X such that $f_n \to f$ for some $f \in X$. It follows that $\|f_n - f\|_{L_p} \to 0$ and $\big\|\frac{\partial f_n}{\partial x_i} - \frac{\partial f}{\partial x_i}\big\|_{L_p} \to 0$ for each i. Since $L_p(\Omega)$ is a Banach lattice, $\|f_n^+ - f^+\|_{L_p} \to 0$. It is left to show that $\big\|\frac{\partial f_n^+}{\partial x_i} - \frac{\partial f^+}{\partial x_i}\big\|_{L_p} \to 0$ for every i. Consider the following subsets of Ω: $A^+ = \{f > 0\}$, $A^- = \{f < 0\}$, and $A^0 = \{f = 0\}$. Also, for every n define $A_n^+ = \{f_n > 0\}$ and $A_n^{0-} = \{f_n \leqslant 0\}$. In order to show that

$$\int_\Omega \Big|\frac{\partial f_n^+}{\partial x_i} - \frac{\partial f^+}{\partial x_i}\Big|^p \to 0,$$

we split it into integrals over the following six sets: $A^+ \cap A_n^+$, $A^+ \cap A_n^{0-}$, $A^- \cap A_n^+$, $A^- \cap A_n^{0-}$, $A^0 \cap A_n^+$, and $A^0 \cap A_n^{0-}$. In fact, by Lemma 4.1, the integrand vanishes a.e. on $A^- \cap A_n^{0-}$ and $A^0 \cap A_n^{0-}$, so that there is nothing to do about these two sets.

Next, we consider $A^+ \cap A_n^+$ and $A^0 \cap A_n^+$. Lemma 4.1 yields that $\frac{\partial f_n^+}{\partial x_i} = \frac{\partial f_n}{\partial x_i}$ a.e. on A_n^+ and $\frac{\partial f^+}{\partial x_i} = \frac{\partial f}{\partial x_i}$ a.e. on A^+. Further, on A^0 we have $\frac{\partial f^-}{\partial x_i} = 0$, hence $\frac{\partial f}{\partial x_i} = \frac{\partial f^+}{\partial x_i}$ a.e.. Therefore,

$$\int_{(A^+\cup A^0)\cap A_n^+} \Big|\frac{\partial f_n^+}{\partial x_i} - \frac{\partial f^+}{\partial x_i}\Big|^p = \int_{(A^+\cup A^0)\cap A_n^+} \Big|\frac{\partial f_n}{\partial x_i} - \frac{\partial f}{\partial x_i}\Big|^p \leqslant \int_\Omega \Big|\frac{\partial f_n}{\partial x_i} - \frac{\partial f}{\partial x_i}\Big|^p \to 0.$$

For the remaining two sets, we show first that the sequences $m(A^+ \cap A_n^{0-})$ and $m(A^- \cap A_n^+)$ tend to zero as $n \to \infty$, where m stands for the Lebesgue measure on Ω. Indeed, since $\|f_n - f\|_{L_p} \to 0$, by passing to a subsequence we may assume that $f_n \xrightarrow{\text{a.e.}} f$ on Ω. In particular, $f_n \xrightarrow{\text{a.e.}} f$ on A^+. By Egoroff's theorem we can find $B \subseteq A^+$ such that $f_n \to f$ uniformly on B and $m(B) \geqslant m(A^+) - \varepsilon$. Since $A^+ = \bigcup_{k=1}^\infty \{f > \frac{1}{k}\}$, we have $m\big(\{f > \frac{1}{k}\}\big) > m(A^+) - \varepsilon$ for some k. It follows that $f_n > 0$ on $\{f > \frac{1}{k}\} \cap B$ for all sufficiently large n, so that $A_n^{0-} \cap \{f > \frac{1}{k}\} \cap B = \varnothing$. Therefore, $m(A^+ \cap A_n^{0-}) \to 0$. Similarly, we show that $m(A^- \cap A_n^+) \to 0$. Now Lemma 4.1 yields

$$\int_{A^+\cap A_n^{0-}} \Big|\frac{\partial f_n^+}{\partial x_i} - \frac{\partial f^+}{\partial x_i}\Big|^p = \int_{A^+\cap A_n^{0-}} \Big|\frac{\partial f}{\partial x_i}\Big|^p \to 0,$$

and

$$\begin{aligned}\int_{A^-\cap A_n^+}\Bigl|\frac{\partial f_n^+}{\partial x_i}-\frac{\partial f^+}{\partial x_i}\Bigr|^p &= \int_{A^-\cap A_n^+}\Bigl|\frac{\partial f_n}{\partial x_i}\Bigr|^p \\ &\leqslant 2^p\int_{A^-\cap A_n^+}\Bigl|\frac{\partial f_n}{\partial x_i}-\frac{\partial f}{\partial x_i}\Bigr|^p+2^p\int_{A^-\cap A_n^+}\Bigl|\frac{\partial f}{\partial x_i}\Bigr|^p \\ &\leqslant 2^p\int_{\Omega}\Bigl|\frac{\partial f_n}{\partial x_i}-\frac{\partial f}{\partial x_i}\Bigr|^p+2^p\int_{A^-\cap A_n^+}\Bigl|\frac{\partial f}{\partial x_i}\Bigr|^p\to 0.\end{aligned}$$

It follows that $\bigl\|\frac{\partial f_n^+}{\partial x_i}-\frac{\partial f^+}{\partial x_i}\bigr\|_{L_p}\to 0$. □

Combining Theorem 4.3 with Proposition 0.6, we immediately obtain the following result.

Corollary 4.4. *The closure of every ideal in $W^{1,p}(\Omega)$ is again an ideal.*

Recall that Theorem 0.1 asserts that an operator on a Banach lattice satisfying a certain condition has an invariant closed ideal. As we mentioned before, a variant of Theorem 0.1 for $W^{1,p}(\Omega)$ proved in [IM04] states that an operator on $W^{1,p}(\Omega)$ satisfying the same condition has a closed invariant subspace. In fact, this subspace is produced in [IM04] as the closure of a certain ideal. In view of Corollary 4.4, this closure is itself an ideal. Hence, the following result holds.

Corollary 4.5. *Let S and T be two positive commuting operators on a $W^{1,p}(\Omega)$ such that S is quasinilpotent and dominates a non-zero positive compact operator. Then T has a closed invariant ideal.*

As in [IM04], instead of S being quasinilpotent, it is sufficient that S be locally quasinilpotent at a positive vector.

Theorem 3.3 and Corollary 4.4 together yield the following result.

Theorem 4.6. *Suppose that Ω is regular and $p>N$ or $p=N=1$. If Q is a positive operator on $W^{1,p}(\Omega)$ then Q is bounded and there is a closed subspace invariant under $\{Q\}'$. If, in addition, Q is quasinilpotent, then $\langle Q]$ has a closed invariant ideal.*

5. Minimal vectors in spaces with a generating cone

The method of minimal vectors was originally developed in [AE98] to prove the existence of invariant subspaces for certain classes of operators on Hilbert spaces. The method was later extended to Banach space in [JKP03, And03, CPS04, Tro04, LRT]. Following [LRT], we say that a collection $\mathcal{F}$ of linear bounded operators on a Banach space X ***localizes*** a subset $A\subseteq X$ if for every sequence (x_n) in A there is a subsequence

(x_{n_i}) and a sequence (K_i) in $\mathcal{F}$ such that the sequence $(K_i x_{n_i})$ converges to a non-zero vector. It was proved in [Tro04] that if T is a quasinilpotent operator on a Banach space X such that the unit ball of the commutant of T localizes a closed ball in X, then T has a hyperinvariant subspace. This implies Lomonosov's theorem [Lom73] for quasinilpotent operators.

In [AT05], the method of minimal vectors was used to produce several generalizations of Theorem 0.1 and of other related results. In this section we show that the technique developed in [AT05] extends naturally to ordered Banach spaces with generating cones. In fact, it is even simpler and more transparent in this more general setting, and the proofs are shorter. The crucial tool in extending the method is the following well-known theorem (see, e.g., [AAB92]).

Theorem 5.1 (Krein and Šmulian)**.** *Let X be a Banach space with a generating closed cone X_+. There exists a constant $\Delta > 0$ such that for each $x \in X$ there exist $x_1, x_2 \in X_+$ such that $x = x_1 - x_2$ and $\|x_i\| \leqslant \Delta\|x\|$, $i = 1, 2$.*

The smallest such Δ will be denoted $\Delta(X)$. We will write $B(u, r)$ for the closed ball of radius r centered at $u \in X$. We will also write $B_u = B(u, 1)$. In particular, B_0 stands for the unit ball of X. Suppose that Q is a positive bounded operator on an ordered Banach space X, $u \in X_+$, and $C \in \mathbb{R}_+$. Following [AT05], we say that $y \in X$ is a ***C-minimal vector*** for Q and u if $y \geqslant 0$, $y \in Q^{-1}(B_u + X_+)$ and $\|y\| \leqslant C \operatorname{dist}\big\{0, Q^{-1}(B_u + X_+)\big\}$.

Lemma 5.2. *Suppose that X is a Banach space with a generating closed cone with $\Delta = \Delta(X)$ and Q is a positive operator on X such that $I_0(\operatorname{Range} Q)$ is dense in X. Suppose that $u \in X_+$ such that $0 \notin \overline{B_u + X_+}$. Then there exists a (2Δ)-minimal vector for Q and u.*

Proof. Note first that Q is bounded by Theorem 0.2. Since $I_0(\operatorname{Range} Q)$ is dense, it meets B_u. Let $x \in B_u \cap I_0(\operatorname{Range} Q)$. Then $x \leqslant Qh$ for some h, so that $Qh \in B_u + X_+$, hence $\operatorname{Range} Q \cap (B_u + X_+)$ is non-empty. Put $D = Q^{-1}(B_u + X_+)$, then D is non-empty. Since Q is continuous, $0 \notin \overline{D}$. It follows that $d := \operatorname{dist}\{D, 0\} > 0$. Find $z \in D$ such that $\|z\| \leqslant 2d$. By Theorem 5.1 there exists $y \in X_+$ such that $z \leqslant y$ and $\|y\| \leqslant 2d\Delta$. It follows from $Qz \in B_u + X_+$ and $Q \geqslant 0$ that $Qz \leqslant Qy$, so that $Qy \in B_u + X_+$, hence y is a (2Δ)-minimal vector. □

Proposition 5.3. *Suppose that Q is a positive bounded operator on an ordered Banach space X, $u \in X_+$ such that $0 \notin \overline{B_u + X_+}$, and $C \in \mathbb{R}_+$. Suppose that y is a C-minimal* *vector for Q and u. Then there exists $f \in X^*$ (called* ***a minimal functional****) such that*

(i) *f is positive, and $f(u) \geqslant 1 = \|f\|$;*
(ii) *There exists $c > 0$ such that $f_{|Q(B(0,d))} \leqslant c$ and $f_{|B_u+X_+} \geqslant c$, where $d = \operatorname{dist}\{0, Q^{-1}(B_u + X_+)\}$;*
(iii) *$\frac{1}{C}\|Q^*f\|\|y\| \leqslant (Q^*f)(y) \leqslant \|Q^*f\|\|y\|$;*
(iv) *$f(Qy) \leqslant C\|u\|$;*
(v) *$\|Q^*f\| \leqslant \frac{C^2\|u\|}{\|y\|}$.*

Proof. (i), (ii), and (iii) are proved exactly as the corresponding statements in [AT05, Lemma 6]. Note that $C^{-1}y \in B(0,d)$, hence $C^{-1}Qy \in Q\big(B(0,d)\big)$. Since $u \in B_u + X_+$, (ii) and (i) imply $C^{-1}f(Qy) \leqslant c \leqslant f(u) \leqslant \|u\|$, which proves (iv). Finally, combining (iii) and (iv) we get (v). □

For the rest of this section we will make the following assumptions.

(1)
(i) *X is a Banach space with a generating closed cone;*
(ii) *$u \in X_+$ such that $0 \notin \overline{B_u + X_+}$;*
(iii) *$A \subseteq X_+$ such that A minorizes $(B_u + X_+) \cap X_+$;*
(iv) *Q is a positive quasinilpotent operator on X.*

The goal of this section is to show that under these assumptions plus some extra conditions, $\langle Q]$ has a closed invariant subspace. Before we proceed, we would like to make a few comments on these assumptions.

For A one can always take, e.g., the entire set $(B_u + X_+) \cap X_+$. However, in what follows, we will be localizing A, so that we will be interested in having A as small as possible. If X is a vector lattice, one can always take $A = \big[u \wedge \big(B_u + X_+\big)\big] \cap X_+$. In [AT05], where X was a Banach lattice, the set $B_u \cap [0,u]$ served as A.

Clearly, $0 \notin \overline{B_u + X_+}$ implies $\|u\| > 1$. The converse is also true if the norm is monotone on X_+, i.e., when $0 \leqslant x \leqslant y$ implies $\|x\| \leqslant \|y\|$. In particular, this holds if X is a Banach lattice. The following example shows that the converse fails in general.

Example. Let $X = \mathbb{R}^2$ ordered so that X_+ is the first quadrant. Norm X so that the unit ball is the absolute convex hull of $(0,1)$ and $(2,2)$. Let $u = (0,2)$, then $\|u\| = 2$. However, $(-2,0) \in B_u$ and $(2,0) \in X_+$, so that $(0,0) \in B_u + X_+$.

Finally, note that under assumptions (1), Theorem 0.2 guarantees that Q is bounded.

Proposition 5.4. *Suppose that X, u, and Q are as in* (1) *and $I_0(\operatorname{Range} Q)$ is dense in X. Put $\Delta = \Delta(X)$. Then for every $n \in \mathbb{N}$ there exists a positive (2Δ)-minimal* *vector y_n and a minimal functional f_n for Q^n and u. Furthermore, there exists a subsequence (n_i) such that $\frac{\|y_{n_i-1}\|}{\|y_{n_i}\|} \to 0$ and $f_{n_i} \xrightarrow{w^*} g$ for some non-zero $g \in X_+^*$.*

Proof. The existence of sequences of minimal vectors (y_n) and minimal functionals (f_n) follows from Lemma 5.2 and Proposition 5.3. Suppose that there exists $\delta > 0$ such that $\frac{\|y_{n-1}\|}{\|y_n\|} > \delta$ for all n, so that $\|y_1\| \geqslant \delta\|y_2\| \geqslant \ldots \geqslant \delta^n\|y_{n+1}\|$. Let again $D = Q^{-1}\big(B_u + X_+\big)$. Then $Q^n y_{n+1} \in D$, so that

$$\big\|Q^n y_{n+1}\big\| \geqslant \operatorname{dist}\big(0, D\big) \geqslant \frac{\|y_1\|}{2\Delta} \geqslant \frac{\delta^n}{2\Delta}\|y_{n+1}\|.$$

It follows that $\|Q^n\| \geqslant \frac{\delta^n}{2\Delta}$, which contradicts the quasinilpotence of Q. Hence, $\frac{\|y_{n_i-1}\|}{\|y_{n_i}\|} \to 0$ for some subsequence (y_{n_i}). Since $\|f_n\| = 1$ for every n and the unit ball of X^* is weak*-compact, we can assume by passing to a further subsequence that $f_{n_i} \xrightarrow{w^*} g$ for some $g \in X^*$. Clearly, $g \geqslant 0$. Since $f_n(u) \geqslant 1$ for each n, it follows that $g(u) \geqslant 1$, hence $g \neq 0$. □

Theorem 5.5. *Suppose that X, u, A, and Q are as in* (1). *If the set of all operators dominated by Q localizes A then $\langle Q]$ has an invariant closed subspace. Furthermore, if $[0, Q]$ localizes A then $\langle Q]$ has a non-dense invariant ideal.*

Proof. Let $\Delta = \Delta(X)$. By Theorem 0.2, every operator in $\langle Q]$ and $[0, Q]$ is bounded. In view of Lemma 0.5 we may assume without loss of generality that $I_0(\operatorname{Range} Q)$ is dense in X. Let (y_{n_i}), (f_{n_i}), and g be as in Proposition 5.4. Then $Q^{n_i-1}y_{n_i-1} \in (B_u + X_+) \cap X_+$ for each i. It follows that there exists $a_i \in A$ such that $a_i \leqslant Q^{n_i-1}y_{n_i-1}$. By passing to a further subsequence, we can find a sequence (K_i) such that Q dominates K_i for every i and $K_i a_i$ converges to some $w \neq 0$. Then $\pm K_i a_i \leqslant Q a_i \leqslant Q^{n_i} y_{n_i-1}$. For every $T \in \langle Q]$ we have

$$\begin{aligned} \pm f_{n_i}(TK_i a_i) \leqslant f_{n_i}(Q^{n_i} T y_{n_i-1}) &= (Q^{*n_i} f_{n_i})(T y_{n_i-1}) \\ &\leqslant \|Q^{*n_i} f_{n_i}\| \|T\| \|y_{n_i-1}\| \leqslant \frac{4\Delta^2\|u\|}{\|y_{n_i}\|}\|T\|\|y_{n_i-1}\| \end{aligned}$$

by Propositions 5.3(v). It follows from $\frac{\|y_{n_i-1}\|}{\|y_{n_i}\|} \to 0$ that $f_{n_i}(TK_i a_i) \to 0$. On the other hand, $f_{n_i}(TK_i a_i) \to g(Tw)$. Thus, $g(Tw) = 0$ for every $T \in \langle Q]$.

If $Tw = 0$ for all $T \in \langle Q]$ then the one-dimensional subspace spanned by w is invariant under $\langle Q]$. Suppose $Tw \neq 0$ for some $T \in \langle Q]$; let Y be the linear span of $\langle Q]w$. Then Y is non-trivial and invariant under $\langle Q]$. Finally, $Y \subseteq \ker g$ implies $\overline{Y} \neq X$, hence $\overline{Y}$ is a required subspace.

Suppose now that $[0,Q]$ localizes A. Then we can assume that the vector w constructed in the preceding argument is positive. Put $E=I_0\big(\langle Q]w\big)$. Then E is an ideal by Lemma 0.3(i); E is invariant under $\langle Q]$, and E is non-trivial as $w\in E$. Finally, $g\geqslant 0$ implies $E\subseteq\ker g$, hence $\overline{E}\neq X$. ☐

Theorem 5.6. *Suppose that X, u, A, and Q are as in* (1)*, and $I_0(\operatorname{Range}Q)$ is dense. Suppose also that $\mathcal{F}$ is a family of positive contractive operators on X such that $\mathcal{F}$ localizes A and $\mathcal{S}\subseteq L(X)$ is a semigroup such that $\mathcal{S}\mathcal{F}\subseteq\langle Q]$. Then $\mathcal{S}$ has an invariant closed subspace. Furthermore, if $\mathcal{S}$ consists of positive operators then it has a non-dense invariant ideal.*

Proof. Again, let (y_{n_i}), (f_{n_i}), and g be as in Proposition 5.4 with $\Delta=\Delta(X)$. Since $Q^{n_i-1}y_{n_i-1}\in(B_u+X_+)\cap X_+$ for every i, we can find $a_i\in A$ such that $a_i\leqslant Q^{n_i-1}y_{n_i-1}$. By passing to a further subsequence, we can find a sequence (K_i) in $\mathcal{F}$ such that K_ia_i converges to some $w\neq 0$. Let $T\in\mathcal{S}$. Since $TK_i\in\langle Q]$, Propositions 5.3(v) yields

$$0\leqslant f_{n_i}(QTK_ia_i)\leqslant f_{n_i}(Q^{n_i}TK_iy_{n_i-1})$$
$$\leqslant\|Q^{*n_i}f_{n_i}\|\|T\|\|y_{n_i-1}\|\leqslant\frac{4\Delta^2\|u\|}{\|y_{n_i}\|}\|T\|\|y_{n_i-1}\|\to 0$$

since $\frac{\|y_{n_i-1}\|}{\|y_{n_i}\|}\to 0$. It follows that $g(QTw)=0$ for every $T\in\mathcal{S}$.

Let Y be the linear span of $\mathcal{S}w$; then Y is invariant under $\mathcal{S}$. Since $I_0(\operatorname{Range}Q)$ is dense and $g\geqslant 0$ we have $Q^*g\neq 0$. This yields $\overline{Y}\neq X$ because $Tw\in\ker Q^*g$ for every $T\in\mathcal{S}$. Finally, if $Y=\{0\}$, then $Tw=0$ for each $T\in\mathcal{S}$, so that $\operatorname{span}w$ is a one-dimensional subspace invariant under $\mathcal{S}$.

Suppose now that $\mathcal{S}$ consists of positive operators. Then w can be chosen to be positive. Let A be the convex hull of $\mathcal{S}w$ and put $E=I_0(A)$; then E is an ideal by Lemma 0.3(i) and E is invariant under $\mathcal{S}$. It follows from $Q^*g\geqslant 0$ that Q^*g vanishes on E, hence $\overline{E}\neq X$. If $E\neq\{0\}$, we are done. Otherwise, $Tw=0$ for each $T\in\mathcal{S}$. Put $F=I_0(w)$; then F is a non-trivial ideal; F is invariant under $\mathcal{S}$ since every $T\in\mathcal{S}$ vanishes on F. Also, $\overline{F}\neq X$ as otherwise every operator in $\mathcal{S}$ is zero. ☐

Corollary 5.7. *Suppose that X, u, A, and Q are as in* (1)*. If the set of all contractions in $\langle Q]$ localizes A then $\langle Q]$ has a non-dense invariant ideal.*

Proof. By Lemma 0.5 we may assume that $I_0(\operatorname{Range}Q)$ is dense. Now apply Theorem 5.6 with $\mathcal{F}=\big\{K\in\langle Q]\,:\,\|K\|\leqslant 1\big\}$ and $\mathcal{S}=\langle Q]$. ☐

Remark 5.8. In view of Proposition 0.6, if X is a lattice-ordered Banach space with continuous operations then the invariant ideals in Theorems 5.5 and 5.6 and Corollary 5.7 can be taken to be closed.

6. Applications of minimal vector technique

Non-unital uniform algebras. Consider $\mathcal{A}_{\mathrm{sa}}$ for a non-unital C*-algebra $\mathcal{A}$. An important special case is the space $C_0(\Omega)$ of continuous functions on a locally compact Hausdorff space which vanish at infinity; equipped with sup-norm.

These spaces are ordered Banach spaces with closed generating cones (but not Krein spaces). Hence, Theorems 5.5 and 5.6, as well as Corollary 5.7, guarantee that if Q is a positive quasinilpotent operator on any of these spaces satisfying the conditions described in the theorems, then $\langle Q]$ has a common invariant closed subspace.

Observe that for every $u \in \mathcal{A}_+$ with $\|u\| > 1$ we automatically have $0 \notin \overline{B_u + \mathcal{A}_+}$ (this is one of the assumptions in (1)). Indeed, suppose that $0 \in \overline{B_u + \mathcal{A}_+}$, then there exist sequences (x_n) in B_0 and $h_n \in \mathcal{A}_+$ such that $u + x_n + h_n \to 0$. It follows that $\|u + h_n\| - \|x_n\| \to 0$. However, in a C*-algebra, $0 \leqslant a \leqslant b$ implies $\|a\| \leqslant \|b\|$; see, e.g., [SW99, VI.3.2.ii]. Hence $\|u + h_n\| - \|x_n\| \geqslant \|u\| - 1 > 0$, a contradiction.

Sobolev spaces. Finally, we present an application of the localization technique developed in Section 5 to $W^{1,p}(\Omega)$. In the following theorem, we do not require that Ω is regular or that p and N are related.

Being a lattice-ordered space, $W^{1,p}(\Omega)$ clearly has a generating cone. Combining Corollaries 4.2 and 4.4 with Theorem 5.5 and Corollary 5.7, we obtain the following result.

Theorem 6.1. *Let $X = W^{1,p}(\Omega)$ and suppose that Q is a positive quasinilpotent operator on X, $u \in X_+$ with $0 \notin \overline{B_u + X_+}$, and $A \subseteq X_+$ such that A minorizes $(B_u + X_+) \cap X_+$.*

(i) *If the set of all operators dominated by Q localizes A then $\langle Q]$ has an invariant closed subspace.*

(ii) *if either $[0, Q]$ or the set of all contractions in $\langle Q]$ localizes A, then $\langle Q]$ has an invariant closed ideal.*

Next, we show that the assumptions in Theorem 6.1 are not as restrictive as they might seem.

Lemma 6.2. *If $X = W^{k,p}(\Omega)$ and $u \in X_+$ with $\|u\|_{L_p} > 1$ then $0 \notin \overline{B_u + X_+}$.*

Proof. Suppose not. Then there exist sequences (x_n) in the unit ball of X and (h_n) in X_+ such that $u + x_n + h_n \to 0$. This implies $\|u + x_n + h_n\|_{L_p} \to 0$, so that $\|u + h_n\|_{L_p} - \|x_n\|_{L_p} \to 0$. However, $\|u + h_n\|_{L_p} - \|x_n\|_{L_p} \geqslant \|u\|_{L_p} - 1 > 0$, a contradiction. $\square$

Remark 6.3. Let X be a lattice ordered Banach space such that $\big\||x|\big\| = \|x\|$ for every $x \in X$ (for example, $X = W^{1,p}(\Omega)$). Let $u \in X_+$ such that $0 \notin \overline{B_u + X_+}$. Then, of course, $\|u\| > 1$. Let $A = \{y^+ : y \in B_u\}$. We claim that then A is norm-bounded, $A \subseteq (B_u + X_+) \cap X_+$ so that $0 \notin A$, and A minorizes $(B_u + X_+) \cap X_+$.

Indeed, suppose that $y \in B_u$. We have $y^+ = \frac{y+|y|}{2}$, hence $\|y^+\| \leqslant \|y\| \leqslant \|u\| + 1$, so that A is bounded. Also, $y^+ = y + y^- \in B_u + X_+$, so that $A \subseteq (B_u + X_+) \cap X_+$. Now, let $z \in (B_u + X_+) \cap X_+$. Then $z = y + h$ for some $y \in B_u$ and $h \geqslant 0$. It follows from $z \geqslant 0$ and $z \geqslant y$ that $z \geqslant y^+$, hence A minorizes $(B_u + X_+) \cap X_+$.

We now show that Theorem 6.1 implies an analogue of Theorem 0.1 for $W^{1,p}(\Omega)$.

Theorem 6.4. *Suppose that Ω is a regular domain and Q is a positive quasinilpotent operator on $X = W^{1,p}(\Omega)$ and $0 \leqslant K \leqslant Q$ for a non-zero compact operator K. Then $\langle Q]$ has a closed invariant ideal.*

Proof. Since $K \neq 0$ we have $K\mathbb{1} \neq 0$ as, otherwise, K would vanish on all bounded functions, but bounded functions are dense in X. It follows that there exists $\varepsilon > 0$ such that $0 \notin \overline{K\big(B(\mathbb{1}, \varepsilon)\big)}$. Let $u = \frac{1}{\varepsilon}\mathbb{1}$, then $0 \notin \overline{K(B_u)}$. By scaling u even further we may also assume that $\|u\|_{L_p} > 1$, hence $0 \notin \overline{B_u + X_+}$ by Lemma 6.2.

As in Remark 6.3, take $A = \{y^+ : y \in B_u\}$; then A minorizes $(B_u + X_+) \cap X_+$. Since u is a multiple of $\mathbb{1}$, it follows from Lemma 4.1 that $\|y^+ - u\| \leqslant \|y - u\| \leqslant 1$ for every $y \in B_u$; hence $A \subseteq B_u$.

Observe that $[0, Q]$ localizes A. Indeed, let (x_n) be a sequence in A. Since $K \in [0, Q]$ and K is compact, there is a subsequence (x_{n_i}) such that $Kx_{n_i} \to w$ for some w. Clearly, $w \in \overline{K(A)} \subseteq \overline{K(B_u)}$, hence $w \neq 0$. Now apply Theorem 6.1. $\square$

References

[AA02] Y.A. Abramovich and C.D. Aliprantis, *An invitation to operator theory*, Graduate Studies in Mathematics, vol. 50, Amer. Math. Soc., Providence, RI, 2002.

[AAB92] Y.A. Abramovich, C.D. Aliprantis, and O. Burkinshaw, *Positive operators on Kreĭn spaces*, Acta Appl. Math. **27** (1992), no. 1-2, 1–22, Positive operators and semigroups on Banach lattices (Curaçao, 1990).

[AAB93] ______, *Invariant subspaces of operators on ℓ_p-spaces*, J. Funct. Anal. **115** (1993), no. 2, 418–424.

[AAB94] ———, *Invariant subspace theorems for positive operators*, J. Funct. Anal. **124** (1994), no. 1, 95–111.

[AAB98] ———, *The invariant subspace problem: some recent advances*, Rend. Istit. Mat. Univ. Trieste **29** (1998), no. suppl., 3–79 (1999), Workshop on Measure Theory and Real Analysis (Italian) (Grado, 1995).

[AB85] C.D. Aliprantis and O. Burkinshaw, *Positive operators*, Academic Press Inc., Orlando, Fla., 1985.

[AE98] S. Ansari and P. Enflo, *Extremal vectors and invariant subspaces*, Trans. Amer. Math. Soc. **350** (1998), no. 2, 539–558.

[And03] G. Androulakis, *A note on the method of minimal vectors*, Trends in Banach spaces and operator theory (Memphis, TN, 2001), Contemp. Math., vol. 321, Amer. Math. Soc., Providence, RI, 2003, pp. 29–36.

[AT07] C.D. Aliprantis and R. Tourky, *Cones and order*, Graduate Texts in Mathematics, Amer. Math. Soc., Providence, RI, 2007..

[AT05] R. Anisca and V.G. Troitsky, *Minimal vectors of positive operators*, Indiana Univ. Math. J. **54** (2005), no. 3, 861–872.

[CPS04] I. Chalendar, J.R. Partington, and M. Smith, *Approximation in reflexive Banach spaces and applications to the invariant subspace problem*, Proc. Amer. Math. Soc. **132** (2004), no. 4, 1133–1142.

[Dix77] J. Dixmier, *C^*-algebras.* North-Holland Mathematical Library, Vol. 15. North-Holland Publishing Co., Amsterdam-New York-Oxford, 1977.

[Drn01] R. Drnovšek, *Common invariant subspaces for collections of operators*, Integral Equations Operator Theory **39** (2001), no. 3, 253–266.

[GT77] D. Gilbarg and N.S. Trudinger, *Elliptic partial differential equations of second order*, Springer-Verlag, Berlin, 1977, Grundlehren der Mathematischen Wissenschaften, Vol. 224.

[IM04] M.C. Isidori and A. Martellotti, *Invariant subspaces for compact-friendly operators in Sobolev spaces*, Positivity **8** (2004), no. 2, 109–122.

[JKP03] I.B. Jung, E. Ko, and C. Pearcy, *On quasinilpotent operators*, Proc. Amer. Math. Soc. **131** (2003), no. 7, 2121–2127.

[KJF77] A. Kufner, O. John, and S. Fučík, *Function spaces*, Noordhoff International Publishing, Leyden, 1977, Monographs and Textbooks on Mechanics of Solids and Fluids; Mechanics: Analysis.

[KR48] M.G. Kreĭn and M.A. Rutman, *Linear operators leaving invariant a cone in a Banach space*, Uspehi Matem. Nauk (N. S.) **3** (1948), no. 1(23), 3–95.

[Lom73] V.I. Lomonosov, *Invariant subspaces of the family of operators that commute with a completely continuous operator*, Funkcional. Anal. i Priložen. **7** (1973), no. 3, 55–56.

[LRT] V.I. Lomonosov, H. Radjavi, and V.G. Troitsky, *Sesquitransitive and localizing operator algebras*, preprint.

[OT05] T. Oikhberg and V.G. Troitsky, *A theorem of Krein revisited*, Rocky Mountain J. of Math. **35** (2005), no. 1, 195–210.

[PW03] A. Pełczyński and M. Wojciechowski, *Sobolev spaces*, Handbook of the geometry of Banach spaces, Vol. 2, North-Holland, Amsterdam, 2003, 1361–1423.

[SW99] H.H. Schaefer and M.P. Wolff, *Topological vector spaces*, second ed., Springer-Verlag, New York, 1999.

[Tro04] V.G. Troitsky, *Minimal vectors in arbitrary Banach spaces*, Proc. Amer. Math. Soc. **132** (2004), no. 4, 1177–1180.

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