LOMONOSOV'S THEOREM CANNOT BE EXTENDED TO CHAINS OF FOUR OPERATORS

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ABSTRACT. We show that the celebrated Lomonosov theorem cannot be improved by increasing the number of commuting operators. Specifically, we prove that if $T\colon \ell_1 \to \ell_1$ is the operator without a non-trivial closed invariant subspace constructed by C. J. Read, then there are three operators S_1, S_2 and K (non-multiples of the identity) such that T commutes with S_1, S_1 commutes with S_2, S_2 commutes with K, and K is compact. It is also shown that the commutant of T contains only series of T.

All Banach spaces in this note are assumed to be infinite dimensional and separable; all operators are linear and bounded. We say that an operator is a non-scalar operator if it is not a multiple of the identity operator.

One of the major results in the history of the Invariant Subspace Problem was obtained by V. Lomonosov in [L] who proved that if an operator T on a Banach space commutes with another non-scalar operator S and S commutes with a non-zero compact operator K, then T has an invariant subspace. Motivated by their study of the Invariant Subspace Problem for positive operators on Banach lattices, Y. A. Abramovich and C. D. Aliprantis have asked recently whether or not Lomonosov's theorem can be extended to chains of four or more operators. The purpose of this note is to answer this question in the negative. For our initial operator T we will take an operator without an invariant subspace on ℓ_1 coming from the famous construction of C. J. Read (see [R1]). Then we will produce three non-scalar operators S_1, S_2, and K with K compact (as a matter of fact K has rank one) such that $TS_1 = S_1T$, $S_1S_2 = S_2S_1$, and $S_2K = KS_2$. This will be done in Section 1.

After that, in Section 2, we will consider a related question of describing the commutant of the C. J. Read operator.

1. A chain from C. J. Read's operator to a rank-one operator

We begin with reminding the reader of the construction in [R1] that will be central for us. As in [R1], we denote the standard unit vectors of ℓ_1 by $(f_i)_{i=0}^{\infty}$. The symbol F denotes the linear subspace of ℓ_1, spanned by f_i's, and thus, F consists of eventually vanishing sequences.

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Let $\mathbf{d}=(a_1,b_1,a_2,b_2,\ldots)$ be a strictly increasing sequence of positive integers. Also let $a_0=1$, $v_0=0$, and $v_n=n(a_n+b_n)$ for $n\geqslant 1$. Then there is a unique sequence $(e_i)_{i=0}^{\infty}\subset F$ with the following properties:

0) $f_0=e_0$;

A) if integers r, n, and i satisfy $0<r\leqslant n$, $i\in[0,v_{n-r}]+ra_n$, we have $f_i=a_{n-r}(e_i-e_{i-ra_n})$;

B) if integers r, n, and i satisfy $1\leqslant r<n$, $i\in(ra_n+v_{n-r},(r+1)a_n)$, (respectively, $1\leqslant n$, $i\in(v_{n-1},a_n)$), then $f_i=2^{(h-i)/\sqrt{a_n}}e_i$, where $h=(r+\frac{1}{2})a_n$ (respectively, $h=\frac{1}{2}a_n$);

C) if integers r, n, and i satisfy $1\leqslant r\leqslant n$, $i\in[r(a_n+b_n),na_n+rb_n]$, then $f_i=e_i-b_ne_{i-b_n}$;

D) if integers r, n, and i satisfy $0\leqslant r<n$, $i\in(na_n+rb_n,(r+1)(a_n+b_n))$, then $f_i=2^{(h-i)/\sqrt{b_n}}e_i$, where $h=(r+\frac{1}{2})b_n$.

Indeed, since $f_i=\sum_{j=0}^{i}\lambda_{ij}e_j$ for each $i\geqslant 0$ and λ_{ii} is always nonzero, this linear relation is invertible. Further,

$$(*)\qquad \operatorname{lin}\{e_i\,:\,i=1,\ldots,n\}=\operatorname{lin}\{f_i\,:\,i=1,\ldots,n\}\text{ for every }n\geqslant 0.$$

In particular all e_i are linearly independent and also span F. Then C. J. Read defines $T\colon F\to F$ to be the unique linear map such that $Te_i=e_{i+1}$, and in Lemma 5.1 he proves that $\|Tf_i\|\leqslant 1$ for every $i\geqslant 0$ provided $\mathbf{d}$ increases sufficiently rapidly, i. e., satisfies several conditions of the form

$$\begin{aligned} a_n &\geqslant G(n,a_0,b_0,a_1,b_1,\ldots,a_{n-1},b_{n-1}),\text{ and}\\ b_n &\geqslant H(n,a_0,b_0,a_1,b_1,\ldots,a_{n-1},b_{n-1},a_n),\end{aligned}$$

where the G and H are some real-valued functions. It follows that T can be extended to a bounded operator on ℓ_1. Finally, C. .J. Read proves that this extension, which is still denoted by T, has no invariant subspaces provided $\mathbf{d}$ increases sufficiently rapidly.

Throughout this section we will assume, without loss of generality, that all integers a_i and b_i are even. We are going to construct non-scalar operators S_1, S_2, and K such that K has rank one and commutes with S_2, S_2 commutes with S_1, and S_1 commutes with T. In fact, we take $S_1=T^2$, so that the equality $TS_1=S_1T$ is automatic. Define S_2 on F via

$$S_2e_i=\begin{cases} e_i & \text{if } i \text{ is even;}\\ 0 & \text{otherwise.}\end{cases}$$

We claim that

$$S_2f_i=\begin{cases} f_i & \text{if } i \text{ is even;}\\ 0 & \text{otherwise.}\end{cases}$$

To prove this we consider all possible cases:

0) In this case $S_2f_0=S_2e_0=e_0=f_0$;

A) Since a_n is even then

$$S_2 f_i = a_{n-r}(S_2 e_i - S_2 e_{i-ra_n}) = \begin{cases} a_{n-r}(e_i - e_{i-ra_n}) = f_i & \text{if } i \text{ is even;} \\ 0 & \text{otherwise.} \end{cases}$$

B) In this case

$$S_2 f_i = 2^{(h-i)/\sqrt{a_n}} S_2 e_i = \begin{cases} 2^{(h-i)/\sqrt{a_n}} e_i = f_i & \text{if } i \text{ is even;} \\ 0 & \text{otherwise.} \end{cases}$$

C) Since b_n is even, we have

$$S_2 f_i = S_2 e_i - b_n S_2 e_{i-b_n} = \begin{cases} e_i - b_n e_{i-b_n} = f_i & \text{if } i \text{ is even;} \\ 0 & \text{otherwise.} \end{cases}$$

D) Finally, in this case

$$S_2 f_i = 2^{(h-i)/\sqrt{b_n}} S_2 e_i = \begin{cases} 2^{(h-i)/\sqrt{b_n}} e_i = f_i & \text{if } i \text{ is even;} \\ 0 & \text{otherwise.} \end{cases}$$

In particular, S_2 is bounded on F and can be extended to ℓ_1. For every $i \geqslant 0$ we have

$$T^2 S_2 e_i = \begin{cases} T^2 e_i = e_{i+2} & \text{if } i \text{ is even;} \\ 0 & \text{otherwise.} \end{cases}$$

On the other hand

$$S_2 T^2 e_i = S_2 e_{i+2} = \begin{cases} e_{i+2} & \text{if } i \text{ is even;} \\ 0 & \text{otherwise,} \end{cases}$$

so that $T^2 S_2 x = S_2 T^2 x$ for every $x \in F$. Since F is dense in ℓ_1, it follows that T^2 and S_2 commute on ℓ_1.

Finally, define K on ℓ_1 via $K f_0 = f_0$ and $K f_i = 0$ for all $i > 0$. Then K is a bounded rank one operator on ℓ_1, and K commutes with S_2.

Note that if m divides a_n and b_n for every n, then, similarly to the previous construction, we could take for S_1 the operator T^m instead of T^2. It follows from Lomonosov's theorem that T^m has an invariant subspace (confer [R1, Lemma 6.4]).

In [R2] C. J. Read presents as a modification of his original example a quasinilpotent operator on ℓ_1 without closed nontrivial invariant subspaces. The same argument as above provides a chain of four commuting operators connecting this operator to a compact operator.

2. Commutants of C. J. Read's operators

Let $\mathfrak{M}$ denote the collection of all real infinite matrices $A = (a_{ij})_{i,j=0}^{\infty}$. Elements of $\mathfrak{M}$ can be added entry-wise, the zero and identity matrices are defined in the natural way. Let $\mathcal{F}$ denote the subfamily of $\mathfrak{M}$ consisting of the matrices with finite number of nonzero entries in every column and every row. For A, B, and C in $\mathfrak{M}$ we say that $AB = C$ if for all $i, j \in \mathbb{N}$ we have $a_{ij} = \sum_{k=0}^{\infty} b_{ik} c_{kj}$ and the series converge absolutely. Though AB may not exist in general, AB exists if A or B belongs to $\mathcal{F}$.

We can also define the action of a matrix on a sequence: we say that $Ax = y$ for $A \in \mathfrak{M}$ and $x, y \in \mathbb{R}^{\mathbb{N}}$ if $y_i = \sum_{j=0}^{\infty} a_{ij} x_j$ for every $i \geqslant 0$. Again, if $A \in \mathcal{F}$, then Ax is defined for every $x \in \mathbb{R}^{\mathbb{N}}$. Let $A^{(j)}$ denote the j-th column of $A \in \mathfrak{M}$, then $A^{(j)} = Af_j$. Finally, $(A)_{ij}$ or a_{ij} will denote the (i,j)-th entry of $A \in \mathfrak{M}$.

The space $\mathcal{B}(\ell_1)$ of all (bounded) operators on ℓ_1 can be naturally embedded in $\mathfrak{M}$: if $R \in \mathcal{B}(\ell_1)$, then $r_{ij} = (Rf_j)_i$. Obviously, the sum of operators in $\mathcal{B}(\ell_1)$ corresponds to the sum of matrices in $\mathfrak{M}$. Moreover, the action of R on an element of ℓ_1 is in accord with the definition of the action of a matrix on a sequence: if $x = (x_1, x_2, \dots) \in \ell_1$ then $x = \sum_{j=0}^{\infty} x_j f_j$, so that $Rx = \sum_{j=0}^{\infty} x_j Rf_j$, and $(Rx)_i = \sum_{j=0}^{\infty} x_j (Rf_j)_i = \sum_{j=0}^{\infty} r_{ij} x_j$. Also, the product of two operators in $\mathcal{B}(\ell_1)$ corresponds to the product of two matrices: if $R, P \in \mathcal{B}(\ell_1)$, then $RPf_j = RP^{(j)} = R \sum_{k=0}^{\infty} p_{kj} f_k = \sum_{k=0}^{\infty} p_{kj} Rf_k$, so that $(RP)_{ij} = (RPf_j)_i = \sum_{k=0}^{\infty} p_{kj} r_{ik}$. Notice that two operators in $\mathcal{B}(\ell_1)$ commute if and only if they commute as matrices in $\mathfrak{M}$. Finally, the identity and zero operators correspond to the identity and zero matrices respectively.

Let S be the right shift operator, i.e., $Sf_j = f_{j+1}$. Given a formal power series $p(t) = \sum_{n=0}^{\infty} p_n t^n$, the matrix $p(S) = p_0 I + p_1 S + p_2 S^2 + \dots$ belongs to $\mathfrak{M}$ and is of the form

$$\begin{pmatrix} p_0 & 0 & 0 & 0 & \dots \\ p_1 & p_0 & 0 & 0 & \dots \\ p_2 & p_1 & p_0 & 0 & \dots \\ p_3 & p_2 & p_1 & p_0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}. \tag{1}$$

On the other hand, every matrix of this form is obviously a series of S.

Lemma 1. *If $A \in \mathfrak{M}$ commutes with S, then $A = p(S)$ for some series p.*

Proof. Since

$$(AS)_{ij} = \sum_{k=0}^{\infty} a_{ik} s_{kj} = a_{i,j+1}$$

and

$$(SA)_{ij} = \sum_{k=0}^{\infty} s_{ik} a_{kj} = \begin{cases} a_{i-1,j} & \text{if } i \geqslant 1; \\ 0 & \text{if } i = 0 \end{cases}$$

for every pair (i, j), it follows that A is of the form (1). $\square$

Consider $Q \in \mathfrak{M}$ such that $Qf_j = Q^{(j)} = e_j$ for every $j \geqslant 0$. It follows from $(*)$ that $Q \in \mathcal{F}$. It also follows from $(*)$ that Q is invertible, and $Q^{-1} \in \mathcal{F}$. Further, we can define "change of basis" map $A \in \mathfrak{M} \mapsto \tilde{A} = Q^{-1} A Q$. Since $Q, Q^{-1} \in \mathcal{F}$, this map is defined for every $A \in \mathfrak{M}$, one-to-one, onto, and $A = Q \tilde{A} Q^{-1}$. Clearly, $\tilde{A}$ describes the action of A in terms of the e_i's. Finally, $AB = BA$ if and only if $\tilde{A}\tilde{B} = \tilde{B}\tilde{A}$ for $A, B \in \mathfrak{M}$.

Recall that T denotes the Read operator introduced in the previous section. Since $Te_j = e_{j+1}$ for every $j \geqslant 0$, then $TQf_j = Qf_{j+1}$, so that $Q^{-1} T Q f_j = f_{j+1}$ which implies

$\tilde{T} = S$. Suppose $RT = TR$ for some $R \in \mathcal{B}(\ell_1)$, then $\tilde{R}S = S\tilde{R}$, so that $\tilde{R} = p(S)$ for some series $p(t) = \sum_{n=0}^{\infty} p_n t^n$ by Lemma 1. Therefore, $R = Q\tilde{R}Q^{-1} = Qp(S)Q^{-1} = p(QSQ^{-1}) = p(T)$. Since every bounded operator of the form $\sum_{n=0}^{\infty} p_n T^n$ commutes with T, we have proved the following proposition:

Proposition 2. *The commutant of T is the set of all bounded operators of the form $\sum_{n=0}^{\infty} p_n T^n$.*

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We remark in conclusion that in [TV] we study the modulus $|T|$ of the Read operator of [R2], and we prove that, unlike T, the modulus $|T|$ does have an invariant subspace.

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