A THEOREM OF KREIN REVISITED

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ABSTRACT. M. Krein proved in [KR48] that if T is a continuous operator on a normed space leaving invariant an open cone, then its adjoint T^* has an eigenvector. We present generalizations of this result as well as some applications to C^*-algebras, operators on ℓ_1, operators with invariant sets, contractions on Banach lattices, the Invariant Subspace Problem, and von Neumann algebras.

M. Krein proved in [KR48, Theorem 3.3] that if T is a continuous operator on a normed space leaving invariant a non-empty open cone, then its adjoint T^* has an eigenvector. Krein's result has an immediate application to the Invariant Subspace Problem because of the following observation. If T is a bounded operator on a Banach space and not a multiple of the identity, and $T^*f = \lambda f$, then the kernel of f is a closed non-trivial subspace of codimension 1 which is invariant under T. Moreover, $\overline{\mathrm{Range}(\lambda I - T)}$ is a closed nontrivial subspace which is proper (it is contained in the kernel of f) and hyperinvariant for T, that is, it is invariant under every operator commuting with T.

Several proofs and modifications of Krein's theorem appear in the literature, see, e.g., [AAB92, Theorems 6.3 and 7.1] and [S99, p. 315]. We prove yet another version of Krein's Theorem: if T is a positive operator on an ordered normed space in which the unit ball has a dominating point, then T^* has a positive eigenvector. We deduce the original Krein's version of the theorem from this, as well as several applications and related results. In particular, we show that if a bounded operator T on a Banach space satisfies any of the following conditions, then T^* has an eigenvector. Moreover, if the condition holds for a commutative family of operators, then the family of the adjoint operators has a common eigenvector.

- T leaves invariant a cone with an interior point;
- T is a positive operator on a unital C^*-algebra;

2000 *Mathematics Subject Classification.* Primary: 46B40, 47B60, 47B65; Secondary: 47A15, 47B48, 46L05, 46L10.

Key words and phrases. Krein theorem, ordered normed space, cone with interior point, positive eigenvector, invariant cone, invariant subspace, invariant ideal, invariant set, adjoint operator, C^*-algebra, von Neumann algebra.

The research of the first author was supported by NSF grant DMS-9970369.

Rocky Mountain Journal of Mathematics, **35**(1), 2005, 195–210.

- T is an operator on ℓ_1 such that entries of its matrix satisfy $t_{kk} \pm t_{kj} \geqslant \sum_{i \neq k} |t_{ik} \pm t_{ij}|$ for some k and for all $j \neq k$;
- T leaves invariant a convex set whose interior is non-void and doesn't contain zero;
- T is a contraction with a fixed point;
- T is a positive contraction on a Banach lattice and $Te > e$ for some $e > 0$.

We also show that under the last condition T has a closed invariant order ideal. Finally, we prove a non-commutative version of this result for rearrangement invariant operator spaces arising from von Neumann algebras.

Throughout the paper X denotes a real or complex normed space, X^* the dual of X, T a bounded linear operator on X, and B_X the closed unit ball of X.

Definition 1. We call a subset K of a normed space X a ***cone*** if K is closed under addition and non-negative scalar multiplication, and there exists a non-zero vector $x \in K$ such that $-x \notin K$.

Our definition of a cone is a most general one. In the literature such objects are sometimes called ***wedges***, while for a cone it is often assumed in addition that $x \in K$ implies $-x \notin K$ for *every* non-zero x. This additional condition ensures that the relation on X defined via "$x \leqslant y$ iff $y - x \in K$" is a linear order relation, and, vice versa, every linear order relation defines a cone satisfying this condition, namely, the cone X_+ of all non-negative elements. We will still use the symbol "$\leqslant$", even though in our case $x > 0$ and $x < 0$ may happen simultaneously. However, this does not create any problems, and, naturally, everything we do is still valid for the more restrictive definitions of a cone. See [KR48] for a discussion on definitions and properties of cones.

Given a closed cone K in a normed space X, we will call X an ***ordered normed space*** with respect to the (semi)order relation determined by K. Notice that K coincides with the cone X_+ of all non-negative elements of X. A linear operator is said to be ***positive*** if $T(X_+) \subseteq X_+$. For $f \in X^*$ we write $f \geqslant 0$ or $f \in X^*_+$ if $f(x) \geqslant 0$ whenever $x \geqslant 0$. Clearly, X^*_+ is a w^*-closed cone in X^*. It can be easily verified that if T is a positive bounded operator on X then T^* is a positive operator on X^*, that is, $T^*(X^*_+) \subseteq X^*_+$. It is known (see, e.g. [KR48]) that if K is a closed cone, then K and $-K$ can be (non-strictly) separated by a continuous functional, or, equivalently, there exists a non-trivial positive functional in X^*.

Lemma 2. *Suppose that X is a real normed space and $e \in X$ with $\|e\| = 1$. If $f \in X^*$ then $f(e) = \|f\|$ if and only if $f(x) \leqslant f(e)$ for all $x \in B_X$.*

Proof. If $f(e) = \|f\|$ then $f(x) \leqslant |f(x)| \leqslant \|f\|\|x\| = f(e)$ whenever $\|x\| = 1$. Conversely, suppose $f(x) \leqslant f(e)$ for all x of norm one. Since $-f(x) = f(-x) \leqslant f(e)$, we have $|f(x)| \leqslant f(e)$, so that $\|f\| \leqslant f(e)$. Finally, $f(e) = |f(e)| \leqslant \|f\|$. □

Definition 3. If X is an ordered normed space and $e \in B_X$, we say that e ***dominates the unit ball of*** X if $x \leqslant e$ for all $x \in B_X$. We write $B_X \leqslant e$ then.

In this case it follows immediately from Lemma 2 that every positive functional attains its norm on e. In the proof of the following theorem we use techniques developed in the proof of a special case of Krein's theorem in [AAB92, AA02].

Theorem 4. *Suppose that X is an ordered real normed space and $e \in X$ such that $\|e\| = 1$ and $B_X \leqslant e$. If T is a positive operator on X then T^* has a positive eigenvector. Moreover, if Γ is a commutative family of positive operators on X, then their adjoints have a common positive eigenvector.*

Proof. Let $\mathcal{S} = \{f \in X_+^* \ : \ f(e) = 1\}$. Since $\mathcal{S} = X_+^* \cap \{f \in X^* \ : \ f(e) = 1\}$ then $\mathcal{S}$ is w^*-closed. Furthermore, if $f \in \mathcal{S}$ then $\|f\| = f(e) = 1$ by Lemma 2, so that $\mathcal{S} \subseteq B_X$, hence is w^*-compact. For $T \geqslant 0$ and $f \in \mathcal{S}$ we define

$$F_T(f) = \frac{f + T^*f}{[f + T^*f](e)} = \frac{f + T^*f}{1 + (T^*f)(e)}.$$

Since $T^* \geqslant 0$ then $F_T(f) \geqslant 0$. Clearly, $\big[F_T(f)\big](e) = 1$, so that $F_T(f) \in \mathcal{S}$, hence $F_T\colon \mathcal{S} \to \mathcal{S}$. It can be easily verified that F_T is w^*-to-w^*-continuous. Indeed, if $f_\alpha \xrightarrow{w^*} f$, then for all $x \in X$ we have

$$\left[F_T(f_\alpha)\right](x) = \frac{f_\alpha(x) + (T^*f_\alpha)(x)}{1 + (T^*f_\alpha)(e)} \to \frac{f(x) + (T^*f)(x)}{1 + (T^*f)(e)} = \left[F_T(f)\right](x)$$

because T^* is w^*-to-w^*-continuous. By the Fixed Point Theorem there exists $h \in \mathcal{S}$ such that $F_T(h) = h$, i.e., $\frac{h+T^*h}{1+(T^*h)(e)} = h$ so that $T^*h = \big((T^*h)(e)\big)h$, hence h is an eigenvector of T^*.

Let Γ be a commutative family of positive operators on X. For $T \in \Gamma$ denote A_T the set of the fixed points of F_T in $\mathcal{S}$. It can be easily verified that $f \in \mathcal{S}$ belongs to A_T if and only if f is an eigenvector of T^*. Clearly, A_T is w^*-closed, hence w^*-compact. We claim that $\{A_T\}_{T\in\Gamma}$ has the finite intersection property, this would imply that it has non-empty intersection, and, therefore, the family $\{T^*\}_{T\in\Gamma}$ has a common eigenvector in $\mathcal{S}$. We prove the claim by induction on the size of the set. Suppose that $\bigcap_{T\in\Gamma_0} A_T \neq \varnothing$ for every n-element subset $\Gamma_0 \subseteq \Gamma$. Let Γ_0 be an n-element subset of Γ and $S \in \Gamma$, show that $\bigcap_{T\in\Gamma_0\cup\{S\}} A_T \neq \varnothing$. Pick $f \in \bigcap_{T\in\Gamma_0} A_T$, then for each T in Γ_0 there exists $\lambda_T \geqslant 0$ such that $T^*f = \lambda_T f$. Let $C_T = \ker(\lambda_T I - T^*) \cap \mathcal{S}$, then C_T is a convex w^*-closed subset of A_T, hence w^*-compact. It follows that $C = \bigcap_{T\in\Gamma_0} C_T$ is convex and w^*-compact. Furthermore, $C \neq \varnothing$ as $f \in C$. If $T \in \Gamma_0$ and $h \in C_T$, then

$$T^*F_S(h) = \frac{T^*h + T^*S^*h}{1 + (S^*h)(e)} = \frac{\lambda_T h + S^*(\lambda_T h)}{1 + (S^*h)(e)} = \lambda_T F_S(h),$$

so that $F_S(h) \in C_T$. It follows that $F_S(C_T) \subseteq C_T$, so that $F_S(C) \subseteq C$. The Fixed Point Theorem implies that F_S has a fixed point in C, hence $A_S \cap C \neq \varnothing$. Since $C \subseteq \bigcap_{T\in\Gamma_0} A_T$, this proves the claim. □

Theorem 5. *If T is a continuous operator on a real normed space, leaving invariant a cone with an interior point, then T^* has a positive eigenvector. Moreover, a commutative collection of such operators has a common positive eigenvector.*

Proof. Let Γ be a commutative family of bounded operators on X, C a cone in X such that $T(C) \subseteq C$ for each $T \in \Gamma$, and e an interior point of C. Without loss of generality, C is closed, $\|e\| > 1$, and $e + B_X \subseteq C$. Let C_0 be the cone spanned by $e + B_X$, that is, $C_0 = \{\alpha(e + x) \,:\, \alpha \geqslant 0,\, \|x\| \leqslant 1\}$. Put $W = (C_0 - e) \cap (e - C_0)$. Note that $e + B_X \subseteq C_0$ so that $B_X \subset C_0 - e$. Also, $B_X = -B_X \subseteq e - C_0$, so that $B_X \subseteq W$. Furthermore, W is bounded. Indeed, if $w \in W$ then $w = \alpha_1(e + x_1) - e = e - \alpha_2(e + x_2)$ for some $\alpha_1, \alpha_2 > 0$ and $x_1, x_2 \in B_X$. It follows that $\alpha_1 x_1 + \alpha_2 x_2 = \big(2 - (\alpha_1 + \alpha_2)\big)e$. Thus, $\big|2 - (\alpha_1 + \alpha_2)\big|\|e\| = \|\alpha_1 x_1 + \alpha_2 x_2\| \leqslant \alpha_1 + \alpha_2$. If $\alpha_1 + \alpha_2 > 2$ then $\big((\alpha_1 + \alpha_2) - 2\big)\|e\| - (\alpha_1 + \alpha_2) \leqslant 0$, so that $\alpha_1 + \alpha_2 \leqslant \frac{2\|e\|}{\|e\|-1}$. It follows that $\alpha_1 \leqslant \alpha_1 + \alpha_2 \leqslant \max\big\{2, \frac{2\|e\|}{\|e\|-1}\big\}$ Finally, since $\|w\| \leqslant \alpha_1(\|e\| + 1) + \|e\|$, it follows that W is bounded. Thus, W is the unit ball of a norm, which is equivalent to the original norm of X. In the new norm e will be of norm one. Finally, e dominates W with respect to the order defined by C. Now apply Theorem 4. □

Remark 6. One can easily see that Theorem 5 is equivalent to the original theorem of Krein. Indeed, if T leaves invariant a non-empty open cone, then Theorem 5 states that T^* has an eigenvector. Conversely, suppose that T leaves invariant a cone with an interior point. Let x be an interior point of the cone, then $f(x + Tx) \geqslant f(x) > 0$ for every positive functional $f \neq 0$, so that $x + Tx$ is again an interior point of the cone. It follows that $I + T$ leaves invariant the interior of the cone, so that $(I + T)^*$

has an eigenvector by Krein's theorem. This yields the existence of an eigenvector for T^*.

Next, we discuss some applications of Theorems 4 and 5.

Recall that an element e in a Banach lattice E is called a ***strong unit*** if for every positive $x \in E$ there exists a natural number n such that $x \leqslant ne$. It is known (see [AB85, p. 188] for details) that a Banach lattice with a strong unit is an AM-space with unit up to an equivalent norm. But in an AM-space the unit dominates the unit ball. Therefore, Theorems 4 yields the following result.

Corollary 7. *The adjoint of a positive operator on a Banach lattice with a strong unit has a positive eigenvector.*

In particular, the adjoint of a positive operator on a $C(\Omega)$ space, where Ω is a compact Hausdorff space, has a positive eigenvector. A direct proof of this fact can also be found in [AAB98].

The case of complex normed spaces can often be reduced to the real case as follows. Suppose that X_c is a complexification of a real ordered normed space X, every element of X_c can be written in the form $x + iy$ for some $x, y \in X$. If T is a positive operator on X, then its complexification $T_c\colon X_c \to X_c$ defined by $T_c(x + iy) = Tx + iTy$ will be referred as a positive operator on X_c. Notice that T coincides with the restriction of T_c to X. Suppose that $T^*f = \lambda f$ for some $f \in X^*$ and $\lambda \in \mathbb{R}$, then we can extend f to a continuous linear functional f_c on X_c via $f_c(x+iy) = f(x)+if(y)$. Then $T_c^* f_c = \lambda f_c$. Indeed,

$$[T_c^* f_c](x + iy) = f_c\bigl(T_c(x + iy)\bigr) = f_c(Tx + iTy) = \\ f(Tx) + if(Ty) = (T^*f)(x) + i(T^*f)(y) = \lambda f(x) + i\lambda f(y) = \lambda f_c(x + iy).$$

Thus, Theorems 4 and 5 are applicable to complex normed spaces.

For example, we can apply our technique to positive operators on C^*-algebras. A C^*-algebra $\mathcal{A}$ can be viewed as the complexification of the real Banach space $\mathcal{A}_{\mathrm{sa}}$ of its self-adjoint elements. Recall that a self-adjoint element a in $\mathcal{A}$ is positive if $\sigma(a) \subset \mathbb{R}_+$. If $\mathcal{A}$ has unit e and x is a self-adjoint element of $\mathcal{A}$ such that $\|x\| \leqslant 1$, then the Spectral Mapping Theorem implies that $\sigma(e - x) \subseteq [0, 2]$, hence $x \leqslant e$. It follows that e dominates the unit ball of $\mathcal{A}_{\mathrm{sa}}$. Theorem 4 immediately yields the following result.

Corollary 8. *If T is a positive operator on a unital C^*-algebra, then T^* has a positive eigenvector.*

Let $(e_j)_{j=1}^{\infty}$ denote the standard unit basis of $X = \ell_1$, while $(e_i^*)_{i=1}^{\infty}$ stands for the dual basis of X^*. Recall that every bounded operator T on ℓ_1 can be written as an infinite matrix with entries $t_{ij} = \langle e_i^*, Te_j\rangle$.

Theorem 9. *Suppose that T is a bounded operator on ℓ_1 with matrix (t_{ij}), and suppose that there exists an index k such that*

$$t_{kk} \pm t_{kj} \geqslant \sum_{i \neq k} |t_{ik} \pm t_{ij}| \tag{1}$$

for each $j \neq k$. Then T^* has a positive eigenvector.

Proof. Without loss of generality $k = 1$. Let K be the cone spanned by $e_1 + B_X$. It is easy to see that K is spanned by the set $\{e_1 \pm e_i\}_{i=2}^{\infty}$. We claim that $K = \{(x_i) : x_1 \geqslant \sum_{i=2}^{\infty} |x_i|\}$. Indeed, it is easy to see that the later set is closed under addition and positive scalar multiplication, hence it is a cone. Furthermore, it contains $e_1 \pm e_i$ for each $i \geqslant 2$, so that it contains K. Finally, if a non-zero sequence (x_i) satisfies $x_1 \geqslant \sum_{i=2}^{\infty} |x_i|$ then $\frac{x}{x_1} - e_1 \in B_X$, so that (x_i) is contained in K.

Clearly, e_1 dominates the unit ball of X with respect to the order induced by K. The condition $t_{11} \pm t_{1j} \geqslant \sum_{i=2}^{\infty} |t_{i1} \pm t_{ij}|$ means that

$$T(e_1 \pm e_j) = Te_1 \pm Te_j = (\text{the 1st column of } T) \pm (\text{the } j\text{-th column of } T) \in K$$

for every $j \geqslant 2$, it follows that $T(K) \subseteq K$. Theorem 4 finishes the proof. ☐

Example 10. Let K be as in the preceding proof, and let $\mathcal{C}$ be the set of all operators on ℓ_1 preserving K. Clearly, the adjoint of every operator in $\mathcal{C}$ has an eigenvector. By construction, $\mathcal{C}$ is itself a cone and a multiplicative semi-group in $\mathcal{L}(\ell_1)$. It is easy to see that $\mathcal{C}$ is closed in the strong operator topology (and, being a convex set, it is also closed in the weak operator topology). Finally, we claim that $\mathcal{C}$ has non-empty interior with respect to the norm topology of $\mathcal{L}(\ell_1)$. For example, put $S = (s_{ij})$ such that s_{ij} equals 1 if $i = j = 1$ and 0 otherwise. We claim that S is an interior point of $\mathcal{C}$. Indeed, suppose that $R = (r_{ij})$ such that $\|R\| < \frac{1}{5}$, and let $T = S + R$. Show that $T \in \mathcal{C}$. Note that $\sum_{i=1}^{\infty} |r_{ij}| = \|Re_j\| < \frac{1}{5}$ for every $j \geqslant 1$. It follows that $t_{11} \pm t_{1j} = 1 + r_{11} \pm r_{1j} \geqslant 1 - \frac{1}{5} - \frac{1}{5} = \frac{3}{5}$ for every $j > 1$. On the other hand,

$$\sum_{i=2}^{\infty} |t_{i1} \pm t_{ij}| = \sum_{i=2}^{\infty} |r_{i1} \pm r_{ij}| \leqslant \sum_{i=2}^{\infty} |r_{i1}| + \sum_{i=2}^{\infty} |r_{ij}| < \tfrac{2}{5}.$$

Hence, T satisfies (1) and, therefore, $T \in \mathcal{C}$.

Corollary 11. *If T is an operator on a real Banach space leaving invariant a convex set whose interior is non-void and doesn't contain zero, then then T^* has an eigenvector. Moreover, a commutative collection of such operators has a common eigenvector.*

Proof. Apply Theorem 5 to the cone generated by the invariant set. □

Krein's theorem gives a natural insight and provides a simple solution to Exercise VII.5.10 of [DS58], even though at the first glance the statement seems to have no connection to order structures.

Proposition 12 ([DS58, Exercise VII.5.10])**.** *If $\|T\| = 1$ and T has a non-zero fixed point, then T^* has an eigenvector.*

Proof. Suppose that $\|T\| = 1$ and $Te = e$ for some e of norm one. Then the set $e + B_X$ is invariant under T, so it the cone generated by this set. Clearly, this cone is proper and has non-void interior. Now apply Theorem 5. □

This approach can be generalized as follows.

Definition 13. If X is an ordered normed space, we say that it has ***monotone norm*** if $0 \leqslant x \leqslant y$ implies $\|x\| \leqslant \|y\|$.

Theorem 14. *Suppose that T is a positive operator on an ordered normed space with monotone norm such that $\|T\| = 1$ and $Te \geqslant e$ for some $e > 0$. Then T^* has a positive eigenvector.*

Proof. Without loss of generality we can assume $\|e\| = 1$. Since the norm is monotone, we have $(e + X_+) \cap B_X^\circ = \varnothing$, where B_X° stands for the open unit ball of X. Hence, $X_+ \cap (B_X^\circ - e) = \varnothing$, so that the two sets can be separated by a positive functional f. Then f is non-negative on $e - B_X$. Let K be the closed cone generated by X_+ and $e - B_X$. Since f is non-negative on K, it is, indeed, a proper cone.

If $x \in B_X$ then $e - x \in K$, so that e dominates B_X in the (semi)order induced on X by K. It is given that $T(X_+) \subseteq X_+ \subseteq K$. Furthermore, if $x \in B_X$ then $Tx \in B_X$, and we have $T(e - x) = (Te - e) + (e - Tx) \in X_+ + (e - B_X) \subseteq K$, so that $T(e - B_X) \subseteq K$. It follows that $T(K) \subseteq K$. Now apply Theorem 4 to the order induced by K. □

Notice that the condition $\|T\| = 1$ in Proposition 12 cannot be dropped. Indeed, for any $\alpha > 1$, let T be α times the left shift on ℓ_p, $1 \leq p < \infty$, that is,

$T(x_1,x_2,x_3,\dots)=(\alpha x_2,\alpha x_3,\dots)$. Then $\|T\|=\alpha$ and $(1,\alpha^{-1},\alpha^{-2},\dots)$ is a fixed point of T. Nevertheless, T^* clearly has no eigenvectors.

It follows immediately that under the hypothesis of Theorem 14 the operator T has an invariant subspace of co-dimension one. In fact, we will show that if $Te>e$ then there is a closed face of the positive cone of X which is invariant under T. Recall that $E\subset X_+$ (X_+ is the positive cone of X) is called a face of X_+ if E is itself a closed cone, and, for $x_1,x_2\in X_+$, $x_1+x_2\in E$ implies $x_1,x_2\in E$. One can easily see that a closed cone $E\subset X_+$ is a face of X_+ iff it is hereditary, that is, $x\in E$ whenever $0\leqslant x\leqslant y$ and $y\in E$.

Theorem 15. *Suppose that X is an ordered normed space with monotone norm and T is a positive operator on a X such that $\|T\|=1$ and $Te>e$ for some $e>0$. Then there exists a non-trivial closed face E of the positive cone of X which is invariant under T. Moreover, if X is a Banach lattice then $E-E$ is closed non-trivial ideal in X, invariant under T.*

Proof. Without loss of generality $\|e\|=1$. Let

$$E=\{x\geqslant 0\ :\ \lim_{\alpha\to 0^+}(\|e+\alpha x\|-1)/\alpha=0\}.$$

Note that if $x\in E$ and $0\leqslant y\leqslant x$ then $y\in E$. Note also that E is non-trivial as the positive vector $Te-e\in E$ because, for $\alpha\in(0,1)$,

$$1=\|e\|\leqslant\|e+\alpha(Te-e)\|\leqslant\|(1-\alpha)e+\alpha Te\|\leqslant(1-\alpha)\|e\|+\alpha\|Te\|=1.$$

Furthermore, E is T-invariant. Indeed, suppose $\alpha>0$ and $x\in E$. Then

$$\|e+\alpha Tx\|\leqslant\|Te+\alpha Tx\|\leqslant\|e+\alpha x\|.$$

Therefore,

$$\lim_{\alpha\to 0^+}(\|e+\alpha Tx\|-1)/\alpha\leqslant\lim_{\alpha\to 0^+}(\|e+\alpha x\|-1)/\alpha=0.$$

It is easy to see that E is a cone. Indeed, if $x,y\in E$, then $cx\in E$ for $c>0$, and

$$\|e+\alpha(x+y)/2\|-1\leqslant\tfrac{1}{2}\big((\|e+\alpha x\|-1)+(\|e+\alpha y\|-1)\big)=o(\alpha)$$

as α approaches 0. Thus, $x+y\in E$.

To show that E is closed, suppose x_i is a sequence of positive elements in E, converging to x in norm. We shall show that $x\in E$. Fix $\varepsilon>0$. It suffices to prove that, whenever $\alpha>0$ is sufficiently small, the inequality $\|e+\alpha x\|\leqslant 1+\varepsilon\alpha$ is satisfied. Find i for which $\|x-x_i\|<\varepsilon/2$. There exists α_0 such that $\|e+\alpha x_i\|\leqslant 1+\varepsilon\alpha/2$ whenever $0<\alpha<\alpha_0$. Thus, for $\alpha\in(0,\alpha_0)$,

$$\|e+\alpha x\|\leqslant\|e+\alpha x_i\|+\alpha\|x-x_i\|\leqslant(1+\varepsilon\alpha/2)+\varepsilon\alpha/2=1+\varepsilon\alpha.$$

Finally, $e \notin E$, hence E is a non-trivial face of the positive cone of X.

Next, suppose that X is a Banach lattice, and put $F = E - E$. Clearly, F is an order ideal, that is, F is a linear subspace such that $x \in F$ and $|y| \leqslant |x|$ imply $y \in F$. Show that F is closed. Suppose $z \in \overline{F}$, and (x_i), (y_i) are sequences in E such that $\lim_i \|z - (x_i - y_i)\| = 0$. Then $\lim_i \|z_+ - (x_i - y_i)_+\| = 0$. Let $a_i = (x_i - y_i)_+ \wedge z_+$. By the above, $\lim_i \|a_i - z_+\| = 0$. Note that

$$0 \leqslant a_i \leqslant (x_i - y_i)_+ \leqslant |x_i| + |y_i| \in E,$$

hence $a_i \in E$. But E is closed, thus $z_+ \in E$. Similarly, $z_- \in E$, and therefore, $z \in F$.

Finally we prove that F is non-trivial. More precisely, we show that $e \notin F$. Indeed, suppose there exist $x, y \in E$ such that $\|e - (x-y)\| \leqslant \frac{1}{3}$. Then $(x-y)_+ \leqslant |x| + |y| \in E$, so $(x-y)_+ \in E$. Pick $\alpha > 0$ for which $\|e + \alpha(x-y)_+\| \leqslant 1 + \alpha/3$. Then

$$\begin{aligned} 1 + \alpha = \|e + \alpha e\| &= \big\|e + \alpha(x-y)_+ + \alpha\big(e - (x-y)_+\big)\big\| \\ &\leqslant \big\|e + \alpha(x-y)_+\big\| + \alpha\big\|e - (x-y)_+\big\| \leqslant 1 + \alpha/3 + \alpha\big\|e - (x-y)\big\| = 1 + \tfrac{2\alpha}{3}, \end{aligned}$$

contradiction. □

Similar results hold for rearrangement invariant operator spaces, arising from von Neumann algebras. For the benefit of the reader, we give a brief introduction into this natural non-commutative generalization of Banach lattices.

Suppose N is a von Neumann algebra on a Hilbert space H, equipped with a faithful normal semifinite trace τ. Following [N74], we say that a closed, densely defined linear operator x on H is ***affiliated with*** N if $u^* x u = x$ for every unitary $u \in N'$ (the commutant of N). An operator x is called ***τ-measurable*** if for every $\varepsilon > 0$ there exists a (self-adjoint) projection $p \in N$ such that $p(H) \subset D(x)$ and $\tau(1 - p) < \varepsilon$ (1 is the identity in N). The set of all τ-measurable operators is denoted by $\tilde{N}$.

Following [FK86], we introduce for $x \in \tilde{N}$ the ***generalized eigenvalue function*** $\mu(\cdot, x) : [0, \infty) \to [0, \infty)$, defined by

$$\mu(t, x) = \inf\{s \geqslant 0 \;:\; \tau(\chi_{(s,\infty)}(|x|)) \leqslant t\}.$$

Equivalently (see [FK86]), we have

$$\mu(t, x) = \inf\{\|xp\| \;:\; p \in N \, a \, projection, \tau(1-p) \leqslant t\}.$$

Following [DDdP93], we call a linear manifold $G \subset \tilde{N}$, equipped with the norm $\|\cdot\|$, a ***(normed) rearrangement invariant operator space*** (***r.i.o.s.***, in short) if whenever $x \in G$, $y \in \tilde{N}$, and $\mu(t,y) \leqslant \mu(t,x)$ for every t, then $y \in G$ and $\|y\| \leqslant \|x\|$. E is called ***symmetric*** if, in addition, $\|y\| \leqslant \|x\|$ whenever

$$\int_0^a \mu(t,y)dt \leqslant \int_0^a \mu(t,x)dt$$

for every $a > 0$.

To underscore the connections between r.i.o.s. and Banach lattices, consider the commutative case of $N = L_\infty(I)$, where I is an interval $(0,a)$ $(a \in (0,\infty])$. By Proposition 2.a.8 of [LT79], any r.i.o.s. G which satisfies

$$L_1(I) \cap L_\infty(I) \subset G \subset L_1(I) + L_\infty(I) \tag{$*$}$$

is symmetric. We say that G has the ***Fatou property*** if, whenever $f \in G$, (f_n) is a sequence of non-negative elements of G, and $f_n(\omega) \nearrow f(\omega)$, then $\|f_n\| \to \|f\|$.

Suppose N is a von Neumann algebra with a normal faithful semifinite trace τ, and G is as in the previous paragraph (with $I = (0,\tau(1))$). Following [DDdP93], we define the space $G(N) = \{x \in \tilde{N} \mid \mu(\cdot,x) \in G\}$, equipped with the norm $\|x\|_{G(N)} = \|\mu(\cdot,x)\|_G$. If G satisfies (*), then $N \cap N_* \subset G(N) \subset N + N_*$. We identify $L_\infty(N)$ with N itself, and $L_1(N)$ with N_* (the predual of N). If, in addition, G has Fatou property, then $G(N)$ is norm closed (see Proposition 1.7 and Corollary 2.4 of [DDdP93]).

If $G \subset N + N_*$ is a r.i.o.s., we denote by G_+ the set of positive elements in G, i.e. $G \cap \tilde{N}_+$. Then every self-adjoint element in G can be represented as a difference of two positive ones (see [DDdP89] and [DDdP93]). Moreover, every element $x \in G$ can be written as $x = x_1 - x_2 + i(x_3 - x_4)$, with $x_j \in G_+$. Finally, the trace τ extends naturally to $(N + N_*)_+$ by setting $\tau(x) = \int_0^\infty \mu(t,x)dt$ for $x \geqslant 0$.

Theorem 16. *Suppose N is a von Neumann algebra with a faithful normal semifinite trace τ, G is a norm closed symmetric rearrangement invariant subspace of $\tilde{N}$ satisfying $N \cap N_* \subset G \subset N + N_*$, and $T : G \to G$ is a positive contraction such that $Te > e$ for some positive $e \in G$. Then T has an invariant non-trivial face E of the positive cone of G Moreover, $\overline{E - E}$ is a non-trivial closed subspace of G, invariant under T.*

To prove the theorem, we need to collect some facts related to conditional expectations on von Neumann algebras. Suppose N is a von Neumann algebra equipped with a normal faithful semifinite trace τ, and M is a Neumann subalgebra of N s.t. the restriction of τ to M is semifinite. Then (see Proposition V.2.36 of [T79]) there exists a positive contractive projection Φ from N onto M s.t. $\Phi(abc) = a\Phi(b)c$ and $\tau(\Phi(ab)) = \tau(a\Phi(b))$ whenever $a, c \in N_*$ and $b \in N$. Moreover, it follows from the proof that, for any $x \in N \cap N_*$, $\Phi(x) \in M \cap M_*$ and $\|\Phi(x)\|_{M_*} \leqslant \|x\|_{N_*}$. Since $N \cap N_*$ (or $M \cap M_*$) is dense in N_* (respectively, M_*), Φ can be extended to a contraction from N_* to M_*. Thus, Φ can be thought of as an operator from $N + N_*$ to $M + M_*$ respectively, which maps N to M and N_* to M_* contractively.

Lemma 17. *Suppose N, M and τ are as above, and G is a symmetric r.i.o.s. with $N \cap N_* \subset G \subset N + N_*$. Then Φ maps G into $G \cap \tilde{M}$, and $\|\Phi(x)\|_G \leqslant \|x\|_G$ for any $x \in G$.*

Proof. As noted above, Φ acts contractively from N to M and from N_* to M_*. For $x \in G$ we have, by Theorem 4.7 of [DDdP93], $\int_0^a \mu(t, \Phi(x))dt \leqslant \int_0^a \mu(t, x)dt$ for any $a > 0$. Thus, $\Phi(x) \in G$, and $\|\Phi(x)\| \leqslant \|x\|$. □

Proof of Theorem 16. Suppose $e \in G_+$, $\|e\| = 1$, and $T : G \to G$ is a positive operator s.t. $Te > e$. Let

$$E = \{x \geqslant 0 \ : \ \lim_{\alpha \to 0^+} (\|e + \alpha x\| - 1)/\alpha = 0\}.$$

As in the proof of Theorem 15, we can show that E is a closed non-trivial face of G_+ (E is non-empty, and $e \notin E$). Moreover, E is invariant under T. Therefore the closed linear span of E is invariant under T. It remains to show that e does not belong to the closed linear span of E. It suffices to show that, whenever $x_1, x_2 \in E$, we have $\|e + x_1 - x_2\| \geqslant 1/6$.

First suppose that either $\tau(1) < \infty$, or $\lim_{t \to \infty} \mu(t, e) = 0$. Then there exists a commutative von Neumann algebra M s.t. $e \in \tilde{M}$ and the restriction of τ to M is semifinite. Indeed, if $\tau(1) < \infty$, we can consider the von Neumann algebra generated by projections $\chi_{(a,\infty)}(e)$, where $a > 0$. If $\lim_{t \to \infty} \mu(t, e) = 0$, observe that $\tau(\chi_{(a,\infty)}(e)) < \infty$ for any $a > 0$, and let $p = \sup_{a>0} \chi_{(a,\infty)}(e)$. Use Zorn's lemma to find mutually orthogonal projections $(p_i) \in N$ s.t. $\tau(p_i) < \infty$ and $\sum_i p_i = 1 - p$. Then let M be the von Neumann algebra generated by projections $\chi_{(a,\infty)}(e)$ and p_i. Clearly M satisfies our conditions.

Let Φ be the conditional expectation from N onto M. By Lemma 17, Φ acts as a contraction from G to $G_1 = G \cap \tilde{M}$. Then G_1 can be regarded as a Banach lattice. Let

$$E_1 = \{x \in G_1 \ : \ x \geqslant 0, \lim_{\alpha \to 0^+} (\|e + \alpha x\| - 1)/\alpha = 0\}.$$

As in the proof of Theorem 15, $\|e + x - y\| \geqslant 1/3$ whenever $x, y \in E_1$. However, $\Phi(E) \subset E_1$, and therefore

$$\|e + x - y\| \geqslant \|e + \Phi(x) - \Phi(y)\| \geqslant \frac{1}{3}$$

whenever $x, y \in E$.

The case of $a = \lim_{t\to\infty} \mu(t, e) > 0$ is more complicated. Note that $\|a1\|_G \leqslant \|e\| = 1$, hence $\|x\|_G \leqslant \|x\|_N \|1\|_G \leqslant \|x\|_N/a$ for any $x \in N$. Let $k = \lceil 6/a \rceil$, $p_i = \chi_{[ia/k,(i+1)a/k)}(e)$ for $0 \leqslant i \leqslant k-1$, $p_k = \chi_{[(k-1)a/k,a]}(e)$, and $e_1 = \chi_{(a,\infty)}(e)e + \sum_{i=1}^k \frac{i}{k} a p_i$. Then $e \geqslant e_1$, $e - e_1 \in N$, and

$$\|e - e_1\|_G \leqslant \|e - e_1\|_N/a \leqslant 1/6.$$

By definition, $\mu(t, e) = \mu(t, e_1)$ for any t. Moreover, a projection p_i can be represented as $p_i = \sum_j q_{ij}$, where projections q_{ij} are mutually orthogonal and $\tau(q_{ij}) < \infty$. Note also that $\tau(\chi_{(b,\infty)}(e)) < \infty$ whenever $b > a$, and $\chi_{(a,\infty)}(e) = \sup_{b>a} \chi_{(b,\infty)}(e)$. Consider the (commutative) von Neumann algebra M, generated by projections q_{ij} and $\chi_{(b,\infty)}(e)$ $(b > a)$. Then $e_1 \in G_1 = G \cap \tilde{M}$. Let

$$E_1 = \{x \in G_1 \,:\, x \geqslant 0, \lim_{\alpha\to 0^+} (\|e_1 + \alpha x\| - 1)/\alpha = 0\}.$$

As above, we show that $\|e_1 + x - y\| \geqslant 1/3$ if $x, y \in E_1$. However, $\Phi(e) \geqslant e_1$ (since Φ is positive), and therefore, $\|e_1 + \Phi(x)\| \leqslant \|e + x\|$ for any $x \in G$. Thus, $\Phi(E) \in E_1$ and, for any $x, y \in E$, we have

$$\|e + x - y\| \geqslant \|\Phi(e) + \Phi(x) - \Phi(y)\| \geqslant \|e_1 + \Phi(x) - \Phi(y)\| - \|e - e_1\| \geqslant \frac{1}{3} - \frac{1}{6}.$$

The proof is complete. □

Thanks are due to Y.A. Abramovich and C.D. Aliprantis for their interest in the work and useful suggestions. The authors would like to thank the organizers of the workshop in Linear Analysis in College Station in 2001 where much of the work on this paper was done. The authors also thank the Department of Mathematics of the University of Texas at Austin for its hospitality and support.

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