ON NORM CLOSED IDEALS IN $L(\ell_p \oplus \ell_q)$.

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Abstract. It is well known that the only proper non-trivial norm-closed ideal in the algebra $L(X)$ for $X = \ell_p$ ($1 \leqslant p < \infty$) or $X = c_0$ is the ideal of compact operators. The next natural question is to describe all closed ideals of $L(\ell_p \oplus \ell_q)$ for $1 \leqslant p, q < \infty$, $p \neq q$, or, equivalently, the closed ideals in $L(\ell_p, \ell_q)$ for $p < q$. This paper shows that for $1 < p < 2 < q < \infty$ there are at least four distinct proper closed ideals in $L(\ell_p, \ell_q)$, including one that has not been studied before. The proofs use various methods from Banach space theory.

1. Introduction

This paper is concerned with the structure of norm closed ideals of the algebra $L(X)$ of all bounded linear operators on an infinite-dimensional Banach space X. The classical result of [Calk41] asserts that the only proper non-trivial closed ideal of $L(\ell_2)$ is the ideal of compact operators. The same was shown to be true for ℓ_p ($1 \leqslant p < \infty$) and c_0 in [GMF60]. It remains open if there are other Banach spaces with only one proper non-trivial closed ideal. The complete structure of closed ideals in $L(X)$ was recently described in [LLR04] for $X = \left(\bigoplus_{n=1}^{\infty} \ell_2^n\right)_{c_0}$ and in [LSZ06] for $X = \left(\bigoplus_{n=1}^{\infty} \ell_2^n\right)_{\ell_1}$. In the both cases, there are exactly two nested proper non-zero closed ideals. Apart from those mentioned above, there are no other separable Banach spaces X for which the structure of the closed ideals in $L(X)$ is completely known. The structure of the closed ideals of operators on non separable Hilbert spaces was independently obtained by Gramsch [Gram67] and Luft [Luft68]. Recently Daws [Daws06] extended their results to non separable ℓ_p-spaces, $1 \leqslant p < \infty$, and non separable c_0-spaces.

This motivates the study of the next natural special case $X = \ell_p \oplus \ell_q$ ($1 \leqslant p, q < \infty$, $p \neq q$), which is our main interest here. There were several results in this direction proved in the 1970's concerning various special ideals or special cases of p and q. We

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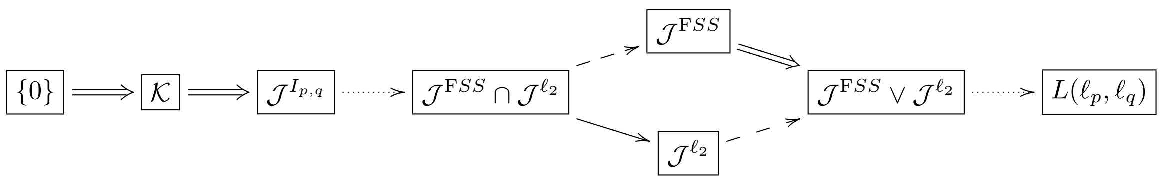
refer the reader to the book by Pietsch [Piet78, Chapter 5] for details. In particular, [Piet78, Theorem 5.3.2] asserts that $L(\ell_p \oplus \ell_q)$ (with, say, $p < q$) has exactly two proper maximal ideals (namely, the ideal of operators which factor through ℓ_p and the ideals of operators which factor through ℓ_q), and establishes a one-to-one correspondence between the non-maximal ideals in the algebra $L(\ell_p \oplus \ell_q)$ and the closed "ideals" in $L(\ell_p, \ell_q)$. Here an ***ideal*** in $L(\ell_p, \ell_q)$ means a linear subspace $\mathcal{J}$ of $L(\ell_p, \ell_q)$ such that $ATB \in \mathcal{J}$ whenever $A \in L(\ell_q)$, $T \in \mathcal{J}$, and $B \in L(\ell_p)$, and "closed" is always understood with respect to the operator norm topology. Consequently, the subject of the present paper is a study of the structure of closed ideals in $L(\ell_p, \ell_q)$ with $1 \leqslant p \leqslant q < \infty$.

In this paper we study a number of natural closed ideals in $L(\ell_p, \ell_q)$ and relations among them. In particular we show that if $1 < p < 2 < q < \infty$ then the following four closed ideals are proper and distinct: the ideal of all compact operators $\mathcal{K}$, the closed ideal $\mathcal{J}^{I_{p,q}}$ generated by the formal identity operator $I_{p,q} \colon \ell_p \to \ell_q$, the ideal of all finitely strictly singular operators $\mathcal{J}^{FSS}$, and the closure of the ideal of all ℓ_2-factorable operators $\mathcal{J}^{\ell_2}$ (see Section 2 for appropriate definitions). Although these ideals were identified earlier, they were not known to be distinct and proper except for special cases. The following diagram illustrates the relationship between these ideals.



Here arrows stand for inclusions. A solid arrow ($\Rightarrow$ or $\rightarrow$) between two ideals means that there are no other ideals sitting properly between the two, while a double arrow coming out of an ideal indicates the only immediate successor. A hyphenated arrow (−−>) indicates a proper inclusion, while a dotted one indicates that we do not know whether or not the inclusion is proper. In particular, the closed ideals in $L(\ell_p, \ell_q)$ are not totally ordered.

The paper is organized as follows. In Section 3 we study the ideal $\mathcal{J}^{I_{p,q}}$ for $1 \leqslant p < q < \infty$. In [Milm70], Milman proved that $\mathcal{J}^{I_{p,q}}$ is FSS, and, therefore, $\mathcal{J}^{I_{p,q}} \subseteq \mathcal{J}^{\mathrm{FSS}}$. Since $\mathcal{J}^{I_{p,q}}$ is not compact, $\mathcal{K}$ is properly contained in $\mathcal{J}^{I_{p,q}}$. We will show that every closed ideal that contains a non-compact operator necessarily contains $\mathcal{J}^{I_{p,q}}$, so that $\mathcal{J}^{I_{p,q}}$ is the least non-compact ideal. In Section 4 we consider the ideal $\mathcal{J}^{\ell_2}$ when $1 < p \leqslant 2 \leqslant q < \infty$. We find a specific non-FSS operator T in $\mathcal{J}^{\ell_2}$ such that the closed ideal $\mathcal{J}^T$ generated by T coincides with $\mathcal{J}^{\ell_2}$. This implies, in particular, that

$\mathcal{J}^{\mathrm{F}SS}$ is a proper ideal (a result proved in [Milm70] for $p = 2 \leqslant q$). Among results on other related ideals we also show that $\mathcal{J}^{\ell_2} \subseteq \mathcal{J}^{\ell_r}$ for all r between p and q, and we prove that every closed ideal of $L(\ell_p, \ell_q)$ which contains a non-FSS operator must also contain $\mathcal{J}^{\ell_2}$. In Section 5 we consider the "block Hadamard" operator U from ℓ_p to ℓ_q for $1 < p < 2 < q < \infty$. We show that $U \notin \mathcal{J}^{\ell_2}$, hence $\mathcal{J}^{\ell_2}$ is a proper ideal. Since, obviously, $I_{p,q} \in \mathcal{J}^{\ell_2}$, it follows that $\mathcal{J}^{I_{p,q}} \subsetneq \mathcal{J}^U$. We show in Section 6 that U is FSS, hence $\mathcal{J}^{I_{p,q}} \subsetneq \mathcal{J}^{\mathrm{F}SS}$.

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2. Notation and preliminaries

We use the standard notation from the Banach space theory as in [LT77, LT79, Tom89, DJT95] and we refer the reader to these books for unexplained notions. Given two Banach spaces X and Y, we write $L(X, Y)$ for the space of all continuous linear operators from X to Y, we write $L(X)$ for $L(X, X)$. A linear subspace $\mathcal{J}$ of $L(X, Y)$ is said to be an ***ideal*** if $ATB \in \mathcal{J}$ whenever $A \in L(Y)$, $T \in \mathcal{J}$, and $B \in L(X)$. By a ***closed ideal*** we mean an ideal closed in the operator norm topology. We denote by $\mathcal{K}$ the closed ideal of all compact operators.

Throughout this paper, p and q always satisfy $1 \leqslant p < q < \infty$. We denote by p' the conjugate of p, that is, $\frac{1}{p} + \frac{1}{p'} = 1$. It is well known (see, e.g., [CPY74]) that $\mathcal{K}$ is contained in every closed ideal of $L(\ell_p, \ell_q)$. If Z is a Banach space, we say that an operator $T \in L(X, Y)$ factors through Z if $T = AB$ where $A \in L(Z, Y)$ and $B \in L(X, Z)$; we denote by $\mathcal{J}^Z$ the closure of the set of all operators in $L(\ell_p, \ell_q)$ that factor through Z. It can be easily verified that if Z is isomorphic to $Z \oplus Z$ then $\mathcal{J}^Z$ is a subspace, hence an ideal. For $S \in L(\ell_p, \ell_q)$ we denote by $\mathcal{J}^S$ the closed ideal in $L(\ell_p, \ell_q)$ generated by S, that is, the smallest closed ideal containing S. It is easy to see that $\mathcal{J}^S$ consists of operators that can be approximated in norm by operators of the form $\sum_{i=1}^n A_i S B_i$, where $A_i \in L(\ell_q)$ and $B_i \in L(\ell_p)$ for $i = 1, \dots, n$. If A is an $n \times n$ scalar matrix, we write $\|A\|_{p,q}$ for the norm of A as an operator from ℓ_p^n to ℓ_q^n.

It is known that every operator in $L(\ell_p, \ell_q)$ is strictly singular, see, e.g., [LT77]. We call an operator $S\colon X \to Y$ ***finitely strictly singular*** or ***FSS*** if for every $\varepsilon > 0$ there exists $n \in \mathbb{N}$ such that $\inf\limits_{x \in E,\ \|x\|=1} \|Sx\| < \varepsilon$ for every n-dimensional subspace E of X. This class of operators already appeared in [Milm70] where its introduction has been credited to Mityagin and Pełczyński. It can be easily verified (see [Masc94]) that S is FSS if and only if every ultrapower of S is strictly singular. It follows immediately that the set of all FSS operators from X to Y is a closed ideal. Denote by $\mathcal{J}^{\mathrm{FSS}}$ the ideal of all FSS operators in $L(\ell_p,\ell_q)$.

We denote by (e_i) and (f_i) the standard bases of ℓ_p and ℓ_q respectively, and we denote their coordinate functionals by (e_i^*) and (f_i^*). If (x_n) is a sequence in a Banach space, we write $[x_n]$ for its closed linear span. A sequence (x_n) in a Banach space is ***semi-normalized*** if $\inf_n\|x_n\|>0$ and $\sup_n\|x_n\|<\infty$.

The following standard lemma follows immediately from Propositions 1.a.12 and 2.a.1 of [LT77].

Lemma 2.1. *If $X=\ell_p$ ($1\leqslant p<\infty$) or c_0 and (x_n) is a semi-normalized sequence in X which converges to zero coordinate-wise (that is, for every i, $e_i^*(x_n)\to 0$ as $n\to\infty$), then there is a subsequence (x_{n_i}) equivalent to (e_i), and $[x_{n_i}]$ is complemented in X.*

Remark 2.2. Suppose that $1\leqslant p\leqslant q<\infty$ and $T\in L(\ell_p,\ell_q)$. We say that T is ***block-diagonal*** if $T=\bigoplus_{n=1}^\infty T_n$, where $T_n\colon \ell_p^{m_n}\to\ell_q^{m_n}$. Equivalently, there exists a strictly increasing sequence of integers (k_n) such that $T=\sum_{n=1}^\infty P_nTQ_n$, where Q_n and P_n are the canonical projections from ℓ_p and ℓ_q onto the finite-dimensional subspaces spanned by $e_{k_n+1},\dots,e_{k_{n+1}}$ and $f_{k_n+1},\dots,f_{k_{n+1}}$ respectively. Note that $m_n=k_{n+1}-k_n$ and T_n can be identified with P_nTQ_n. It can be easily verified that if $p\leqslant q$ then $\|T\|=\sup_n\|T_n\|$. Indeed, $\|T_n\|=\|P_nTQ_n\|\leqslant\|T\|$ as P_n and Q_n are contractions. On the other hand,

$$\|Tx\|=\Bigl(\sum_{n=1}^\infty\|P_nTQ_nx\|^q\Bigr)^{\frac1q}\leqslant\Bigl(\sup_n\|P_nTQ_n\|\Bigr)\Bigl(\sum_{n=1}^\infty\|Q_nx\|^q\Bigr)^{\frac1q}$$
$$\leqslant\bigl(\sup_n\|T_n\|\bigr)\Bigl(\sum_{n=1}^\infty\|Q_nx\|^p\Bigr)^{\frac1p}=\bigl(\sup_n\|T_n\|\bigr)\|x\|.$$

Remark 2.3. Suppose that $R\in L(\ell_p,\ell_q)$ for $1\leqslant p\leqslant q<\infty$, and T is a block-diagonal submatrix of R, that is, $T=\sum_{n=1}^\infty P_nRQ_n$, where (P_n) and (Q_n) are as in Remark 2.2. Then T can be written as a convex combination of operators of the form URV, where U and V are isometries. See Proposition 1.c.8 of [LT77] and Remark 1 following it for the construction.

3. The formal identity operator $I_{p,q}$

In this section we consider the formal identity operator $I_{p,q}\colon\ell_p\to\ell_q$ for $1\leqslant p<q<\infty$. Clearly, $I_{p,q}$ is not compact, so that $\mathcal{K}\subsetneq\mathcal{J}^{I_{p,q}}$. First, we show that $\mathcal{J}^{I_{p,q}}$ is contained in every closed ideal of $L(\ell_p,\ell_q)$ except $\mathcal{K}$. This result is probably known to specialists, but we provide a short proof for completeness.

Proposition 3.1. *Let $1 \leqslant p < q < \infty$. If $\mathcal{J}$ is any ideal in $L(\ell_p, \ell_q)$ containing a non-compact operator, then $I_{p,q} \in \mathcal{J}$.*

Proof. Assume that $\mathcal{J}$ contains a non-compact operator T. There exists a normalized sequence (x_n) in ℓ_p such that (Tx_n) has no convergent subsequences. By passing to subsequences and using a standard diagonalization argument, we can assume that (x_n) and (Tx_n) converge coordinate-wise. Let $y_n = x_n - x_{n-1}$, then (y_n) and (Ty_n) converge coordinate-wise to zero. Since (Tx_n) has no convergent subsequences, we can assume (by passing to a further subsequence if necessary) that (Ty_n) is semi-normalized. It follows that (y_n) is also semi-normalized. Using Lemma 2.1 twice, we can assume (by passing to a subsequence) that (y_n) is equivalent to (e_i), (Ty_n) is equivalent to (f_i), and $[Ty_n]$ is complemented in ℓ_q.

Let $B\colon \ell_p \to [y_n]$ be an isomorphism given by $Be_n = y_n$, and let $A\colon [Ty_n] \to \ell_q$ be an isomorphism given by $A(Ty_n) = f_n$. Since $[Ty_n]$ is complemented, A can be extended to an operator on all of ℓ_q. Thus we can view B and A as elements of $L(\ell_p)$ and $L(\ell_q)$ respectively. Observe that $ATBe_n = f_n$ for each n, hence $ATB = I_{p,q}$. It follows that $I_{p,q} \in \mathcal{J}$. □

Corollary 3.2. *If a closed ideal of $L(\ell_p, \ell_q)$ contains a non-compact operator, then it contains $\mathcal{J}^{I_{p,q}}$.*

The following result was proved in [Milm70]. For the reader's convenience we provide a short proof.

Proposition 3.3. *Suppose that $1 \leqslant p < q < \infty$. The formal identity operator $I_{p,q}$ is FSS.*

We will deduce this proposition from the following lemma, which appeared in [Milm70].

Lemma 3.4. *If E is an n-dimensional subspace of c_0 then there exists $x \in E$ such that x attains its sup-norm at at least n coordinates.*

Proof. The proof is by induction. The statement is trivial for $n = 1$. Suppose that it is true for n, take any subspace E of c_0 of dimension $n + 1$. By induction hypothesis, there exists $x \in E$ such that

$$\delta := \|x\|_\infty = |x_{i_1}| = \cdots = |x_{i_n}| \tag{1}$$

for a set of distinct indices $I = \{i_1, \ldots, i_n\}$. Suppose that $|x_i| < \delta$ for all $i \notin I$ (otherwise we are done). Let Y be the subspace of c_0 consisting of all the sequences that vanish at $i_1, \dots, i_n$. Since Y has co-dimension n, it follows that $Y \cap E \neq \{0\}$. Pick a non-zero $y \in Y \cap E$. We claim that for some $s > 0$ the sequence $x + sy$ attains its sup-norm at at least $n+1$ coordinates. Indeed, $|x_i + ty_i| = \delta$ for all $i \in I$ and $t \geqslant 0$. Consider the function

$$f(t) = \max_{j \notin I} |x_j + ty_j|.$$

Clearly, f is continuous, $f(0) < \delta$, and $\lim_{t \to +\infty} f(t) = +\infty$. It follows that $f(s) = \delta$ for some $s > 0$. Then $|x_i + sy_i| = \|x + sy\|_\infty = \delta$ for some $i \notin I$. □

Proof of Proposition 3.3. Given $\varepsilon > 0$, pick $n \in \mathbb{N}$ such that $n^{\frac{1}{q} - \frac{1}{p}} < \varepsilon$. Suppose that E is a subspace of ℓ_p with $\dim E = n$. By Lemma 3.4 there exists $x \in E$ and indices $i_1, \dots, i_n$ satisfying (1). Without loss of generality, $\|x\|_p = 1$. It follows that $1 = \|x\|_p^p \geqslant n\delta^p$, so that $\delta \leqslant n^{-\frac{1}{p}}$. Then

$$\|x\|_q^q \leqslant \|x\|_\infty^{q-p} \|x\|_p^p = \delta^{q-p} \leqslant n^{-\frac{1}{p}(q-p)},$$

so that $\|x\|_q \leqslant n^{\frac{1}{q} - \frac{1}{p}} < \varepsilon$. It follows that $I_{p,q}$ is FSS. □

Corollary 3.5. *Let $1 \leqslant p < q < \infty$. The ideal $\mathcal{K}$ is a proper subset of $\mathcal{J}^{\mathrm{F}SS}$.*

4. Operators factorable through ℓ_2

In this section we consider the ideal $\mathcal{J}^{\ell_2}$ for $1 < p < 2 < q$. Using Pełczyński's decomposition, we will construct an operator $T \colon \ell_p \to \ell_q$ such that $\mathcal{J}^{\ell_2} = \mathcal{J}^T$. That is, the closure of the ideal of all ℓ_2-factorable operators is exactly the closed ideal generated by T. Furthermore, we show that T fails to be FSS, hence the ideal $\mathcal{J}^{\mathrm{F}SS}$ is proper. It will be obvious from the definition of T that T factors through ℓ_r whenever $p \leqslant r \leqslant q$, so it follows that $\mathcal{J}^{\ell_2} \subseteq \mathcal{J}^{\ell_r}$. We also show that T factors through every non-FSS operator. It follows that any closed ideal containing a non-FSS operator necessarily contains $\mathcal{J}^{\ell_2}$.

To construct T, recall that it follows from Pełczyński's Decomposition Theorem that for every $1 < r < \infty$, ℓ_r is isomorphic to $\left(\bigoplus_{n=1}^\infty \ell_2^n\right)_r$, the ℓ_r-direct sum of ℓ_2^n's (see [LT77, p. 73]). Let $1 < p \leqslant q < \infty$, put $U \colon \ell_p \to \left(\bigoplus_{n=1}^\infty \ell_2^n\right)_p$ and $V \colon \left(\bigoplus_{n=1}^\infty \ell_2^n\right)_q \to \ell_q$ be two such isomorphisms. By $I_{2,p,q} \colon \left(\bigoplus_{n=1}^\infty \ell_2^n\right)_p \to \left(\bigoplus_{n=1}^\infty \ell_2^n\right)_q$ we denote the formal identity operator, that is, just the change of the norm on the direct sum. Then let $T = VI_{2,p,q}U$, that is,

$$T \colon \ell_p \xrightarrow{U} \left(\bigoplus_{n=1}^\infty \ell_2^n\right)_p \xrightarrow{I_{2,p,q}} \left(\bigoplus_{n=1}^\infty \ell_2^n\right)_q \xrightarrow{V} \ell_q. \tag{2}$$

We will call T a ***Pełczyński Decomposition*** operator.

Remark 4.1. Note that T is not unique, it is defined up to the isomorphisms U and V, so that we have actually constructed a class of operators. It is clear, however, that any two Pełczyński Decomposition operators factor through each other. Moreover, one can easily verify that if in the preceding construction we "skip" some of the blocks, that is, if we consider $\big(\bigoplus_{n=1}^{\infty} \ell_2^{k_n}\big)$ for some strictly increasing sequence of indices k_n then the resulting operator T' obviously factors through T. Conversely, T factors through T' because ℓ_2^n is a complemented subspace of $\ell_2^{k_n}$.

Furthermore, let $E_n = U^{-1}(\ell_2^n) \subset \ell_p$ be the pre-image of the n-th block of $(\bigoplus \ell_2^n)_p$. Similarly, put $F_n = V(\ell_2^n) \subset \ell_q$. Then $d(E_n, \ell_2^n) \leqslant \|U\| \cdot \|U^{-1}\|$ and $d(F_n, \ell_2^n) \leqslant \|V\| \cdot \|V^{-1}\|$, where $d(X, Y)$ stands for the Banach-Mazur distance between X and Y. Hence, (E_n) and (F_n) are sequences of uniformly Euclidean subspaces of ℓ_p and ℓ_q respectively. Note that $T(E_n) = F_n$, so that T fixes copies of ℓ_2^n for all $n \in \mathbb{N}$. This immediately implies the following result.

Proposition 4.2. *For $1 < p \leqslant q < \infty$, every Pełczyński Decomposition operator fails to be FSS.*

Corollary 4.3. *For $1 < p \leqslant q < \infty$, the ideal $\mathcal{J}^{\mathrm{FSS}}$ is proper.*

Our next goal is to show that if $1 < p \leqslant 2 \leqslant q < \infty$ then $\mathcal{J}^T = \mathcal{J}^{\ell_2}$. We will make use of the concept of ℓ_2-factorable norm γ_2. Recall that if $S \in L(X, Y)$ (X and Y Banach spaces) then $\gamma_2(S) = \inf \|S_1\| \|S_2\|$, where the infimum is taken over all factorizations $S = S_1 S_2$ where $S_2 \colon X \to \ell_2$ and $S_1 \colon \ell_2 \to Y$. It is known that γ_2 is a norm on the ideal of all ℓ_2-factorable operators, and $\gamma_2(ASB) \leqslant \|A\| \gamma_2(S) \|B\|$ whenever $X \xrightarrow{B} X \xrightarrow{S} Y \xrightarrow{A} Y$. See [Tom89, DJT95] for more information on γ_2.

Lemma 4.4. *Suppose that $R \in L(\ell_p, \ell_q)$, $1 < p \leqslant q < \infty$, and $\varepsilon > 0$.*

(i) *There exist two block-diagonal operators $V, W \in L(\ell_p, \ell_q)$ such that $\|W\| \leqslant \|R\| + \varepsilon$, $\|V\| \leqslant 2\|R\| + 2\varepsilon$, and $\big\|R - (W + V)\big\| < \varepsilon$.*

(ii) *Suppose that, in addition, R is ℓ_2-factorable. Then V and W can be chosen to be ℓ_2-factorable, and $\gamma_2(W) \leqslant \gamma_2(R) + \varepsilon$, $\gamma_2(V) \leqslant 2\gamma_2(R) + 2\varepsilon$, and $\gamma_2\big(R - (W + V)\big) < \varepsilon$.*

Proof. Let $r_{i,j}$ stand for the (i, j)-th entry of the matrix of R, that is, $r_{i,j} = f_i^*(Re_j)$. For the purpose of this proof we introduce the following notation: for $\Omega \subset \mathbb{N} \times \mathbb{N}$, we define the matrix $R_\Omega = (\rho_{i,j})$ by

$$\rho_{i,j} = \begin{cases} r_{i,j} & \text{if } (i, j) \in \Omega, \\ 0 & \text{otherwise.} \end{cases}$$

We start by approximating R by a matrix S with finitely many entries in every row and every column. Namely, by truncating each row and each column of R sufficiently far we can find two strictly increasing sequences (M_j) and (N_i) of positive integers such that $\|R - R_\Gamma\| < \varepsilon$ where $\Gamma \subseteq \mathbb{N} \times \mathbb{N}$ is defined by

$$(i,j) \in \Gamma \text{ iff } i \leqslant M_j \text{ and } j \leqslant N_i.$$

Put $S = R_\Gamma$.

We will define two strictly increasing sequences (k_n) and (l_n) of positive integers, such that Γ is contained in the union of two block-diagonal sets $\Delta = \bigcup_{n=1}^\infty \Delta_n$ and $\Lambda = \bigcup_{n=1}^\infty \Lambda_n$ where

$$\begin{aligned} \Delta_n &= \big\{(i,j) \in \Gamma \mid k_{n-1} < i,j \leqslant k_n\big\} \text{ and} \\ \Lambda_n &= \big\{(i,j) \in \Gamma \mid l_{n-1} < i,j \leqslant l_n\big\}. \end{aligned}$$

We define the sequences (k_n) and (l_n) by an interlaced induction. Put $k_0 = 0$, $l_0 = 1$. For $n \geqslant 0$ we let

$$k_{n+1} = \max\{M_{l_n}, N_{l_n}\} \quad \text{and} \quad l_{n+1} = \max\{M_{k_{n+1}}, N_{k_{n+1}}\}.$$

Clearly, (k_n) and (l_n) are strictly increasing. Next, we show that $\Gamma \subseteq \Delta \cup \Lambda$. Let $(i,j) \in \Gamma$. There exists n such that $l_n < \max\{i,j\} \leqslant l_{n+1}$. If $l_n < \min\{i,j\}$, then $l_n < i,j \leqslant l_{n+1}$, so that $(i,j) \in \Lambda$. Suppose now that $\min\{i,j\} \leqslant l_n$. Then either i or j is less than or equal to l_n, while the other is greater than l_n. Say, $i \leqslant l_n$ and $j > l_n$. It follows that

$$i \leqslant l_n \leqslant N_{l_n} \leqslant k_{n+1} \quad \text{and} \quad j > l_n \geqslant N_{k_n} \geqslant k_n.$$

Therefore $j \leqslant N_i \leqslant N_{l_n} \leqslant k_{n+1}$. Also, $N_i \geqslant j > l_n \geqslant N_{k_n}$ yields $i > k_n$. Hence, $k_n < i,j \leqslant k_{n+1}$, so that $(i,j) \in \Delta$.

Set $W = S_\Delta$ and $V = S - W$. Then the non-zero entries of W and V are located in Δ and Λ, respectively, so that W and V are block-diagonal. By the definition of S we have $\big\|R - (W+V)\big\| < \varepsilon$. Since W is a block-diagonal part of S, Remark 2.3 yields that $\|W\| \leqslant \|S\| \leqslant \|R\| + \varepsilon$. Finally, it follows from $V = S - W$ that $\|V\| \leqslant 2\|R\| + 2\varepsilon$.

If R is ℓ_2-factorable, then we can choose S with finitely many entries in each row and column such that S is also ℓ_2-factorable and $\gamma_2(R-S) < \varepsilon$. Indeed, let $R = R_1 R_2$ be a factorization of R through ℓ_2. Approximate R_1 and R_2 in norm by S_1 and S_2 respectively, such that S_1 and S_2 have finitely many entries in every row and column. Put $S = S_1 S_2$, then S is as claimed. We use triangle inequality to show that $\gamma_2(R-S) < \varepsilon$ when $\|R_1 - S_1\|$ and $\|R_2 - S_2\|$ are sufficiently small.

Set $W = S_\Delta$ and $V = S - W$. It follows from Remark 2.3 that $\gamma_2(W) \leqslant \gamma_2(S) \leqslant \gamma_2(R) + \varepsilon$. Then then $\gamma_2(V) = \gamma_2(S - W) \leqslant 2\gamma_2(R) + 2\varepsilon$. In particular, W and V are ℓ_2-factorable. □

Remark 4.5. In a similar fashion one can show that every operator between two Banach spaces with shrinking unconditional bases can be approximated by a sum of two block-diagonal operators.

Remark 4.6. A slight modification of the proof Lemma 4.4(i) yields $\|W\| \leqslant \|R\|$. Indeed, choose M_j and N_i sufficiently large so that not only $\|R - R_\Gamma\| < \varepsilon$ but also $\|R - R_\Omega\| < \varepsilon$ for every $\Omega \subseteq \mathbb{N} \times \mathbb{N}$ such that $\Gamma \subseteq \Omega$. Then, after constructing Δ and Λ, put $W = R_\Delta$ and $V = R_{\Delta\cup\Lambda} - W$. Then the non-zero entries of W and V are located in Δ and Λ, respectively, so that W and V are block-diagonal. Since $\Delta \cup \Lambda \supseteq \Gamma$ then $\big\|R - (W + V)\big\| < \varepsilon$. Since W is a block-diagonal part of R, Remark 2.3 yields that $\|W\| \leqslant \|R\|$. Finally, $\|V\| \leqslant \|R_{\Delta\cup\Lambda}\| + \|W\| \leqslant 2\|R\| + \varepsilon$.

Theorem 4.7. *If $1 < p \leqslant 2 \leqslant q$ and T is a Pełczyński Decomposition operator, then $\mathcal{J}^T = \mathcal{J}^{\ell_2}$.*

Proof. Observe that $I_{2,p,q}$, being the formal identity from $\big(\bigoplus_{n=1}^\infty \ell_2^n\big)_p$ to $\big(\bigoplus_{n=1}^\infty \ell_2^n\big)_q$, factors through $\big(\bigoplus_{n=1}^\infty \ell_2^n\big)_2 = \ell_2$. It follows that T factors through ℓ_2 and, therefore, $\mathcal{J}^T \subseteq \mathcal{J}^{\ell_2}$.

We show that $\mathcal{J}^{\ell_2} \subseteq \mathcal{J}^T$. Clearly, it suffices to show that every ℓ_2-factorable operator belongs to $\mathcal{J}^T$. In view of Lemma 4.4(ii), it suffices to show this for block-diagonal operators. Let W be an ℓ_2-factorable block-diagonal operator. Then we can write $W = \bigoplus_{n=1}^\infty A_n B_n$, where $B_n\colon \ell_p^{k_n} \to \ell_2^{k_n}$ and $A_n\colon \ell_2^{k_n} \to \ell_q^{k_n}$ such that $\sup_n\|A_n\|$ and $\sup_n\|B_n\|$ are finite. By merging consecutive blocks if necessary, we can assume without loss of generality that (k_n) is strictly increasing. Observe that the operators

$$\begin{aligned} B &= \bigoplus_{n=1}^\infty B_n\colon \Big(\bigoplus_{n=1}^\infty \ell_p^{k_n}\Big)_p \to \Big(\bigoplus_{n=1}^\infty \ell_2^{k_n}\Big)_p \quad \text{and} \\ A &= \bigoplus_{n=1}^\infty A_n\colon \Big(\bigoplus_{n=1}^\infty \ell_2^{k_n}\Big)_q \to \Big(\bigoplus_{n=1}^\infty \ell_q^{k_n}\Big)_q \end{aligned}$$

are bounded, and $W = AI_0B$, where I_0 is the formal identity from $\big(\bigoplus_{n=1}^\infty \ell_2^{k_n}\big)_p$ to $\big(\bigoplus_{n=1}^\infty \ell_q^{k_n}\big)_q$. Thus, W factors through I_0. It follows from Remark 4.1 that I_0 factors through T. Hence, W factors through T. □

Remark 4.8. Actually, we proved that every operator in $\mathcal{J}^{\ell_2}$ can be approximated by sums of two T-factorable operators.

Remark 4.9. Suppose that $1 < p < r < q$. Then $I_{2,p,q}$ in (2) factors through $\left(\bigoplus_{n=1}^{\infty} \ell_2^n\right)_r$, which is isomorphic to ℓ_r. It follows that T factors through ℓ_r. Then Theorem 4.7 implies that $\mathcal{J}^{\ell_2} \subseteq \mathcal{J}^{\ell_r}$ when $p \leqslant 2 \leqslant q$.

Next, we show that if $p < 2 < q$ then $\mathcal{J}^{\ell_2}$ is the least closed ideal beyond $\mathcal{J}^{\mathrm{F}SS}$, that is, every closed ideal that contains a non-FSS operator also contains $\mathcal{J}^{\ell_2}$. For the proof we need the following well-known fact.

Theorem 4.10. *For every $1 < r < \infty$ there exists $K > 0$ such that for all $n \in \mathbb{N}$ there exists $N \in \mathbb{N}$ such that every N-dimensional subspace $F \subset \ell_r$ contains an n-dimensional subspace E which is K-complemented in ℓ_r and 2-isomorphic to ℓ_2^n.*

Remark 4.11. The theorem follows by simultaneous use of Dvoretzky's theorem both in a subspace $F \subset \ell_r$ and in its dual F^* (see e.g., [MS86]). This gives the result with $N = Cn^{r/2}$ and $K = C'\sqrt{\max\{r, r'\}}$, where $C, C' > 0$ are absolute constants. This theorem can be also viewed, for example, as a special case of results in [FT79].

We will also routinely use the following observation.

Remark 4.12. Suppose that (E_n) is a sequence of subspaces of a Banach space X which are uniformly Euclidean and uniformly complemented in X. That is, there exist a constant $C > 0$ and sequences (P_n) and (V_n) such that P_n is a projection from X onto E_n with $\|P_n\| < C$, and $V_n\colon E_n \to \ell_2^n$ is an isomorphism with $\|V_n\| \cdot \|V_n^{-1}\| \leqslant C$ for every n. Let G_n be a subspace of E_n ($n \in \mathbb{N}$). Then it is easy to see that the G_n's are uniformly Euclidean and uniformly complemented in X as well.

For $x \in \ell_r$ we write $\operatorname{supp} x = \{i \in \mathbb{N} \mid x_i \neq 0\}$. For $A \subseteq \ell_r$ put $\operatorname{supp} A = \cup_{x \in A} \operatorname{supp} x$.

Theorem 4.13. *Let $1 < p \leqslant 2 \leqslant q < \infty$. If $R \in L(\ell_p, \ell_q)$ is not FSS, then every Pełczyński Decomposition operator factors through R.*

Proof. Since R is not FSS, there exist a constant $C > 0$ and a sequence (E_n) of subspaces of ℓ_p such that $\dim E_n \to \infty$ as $n \to \infty$, and $R_{|E_n}$ is invertible with $\|(R_{|E_n})^{-1}\| \leqslant C$. We can assume, in addition, that $\operatorname{supp} E_n$ is finite by truncating all the vectors in a basis of E_n sufficiently far (and adjusting C if necessary). Let $F_n = R(E_n)$. Using Theorem 4.10 and Remark 4.12 we can easily obtain subspaces $E_n' \subset E_n$ and $F_n' \subset F_n$ which are C-Euclidean, C-complemented in ℓ_p and ℓ_q respectively, and such that $F_n' = R(E_n')$. By passing to a subsequence we may assume that $\dim E_n' = n$, and we relabel so obtained sequences by (E_n) and (F_n). Let $Q_n\colon \ell_q \to F_n$ be a projection with $\|Q_n\| \leqslant C$.

We are going to define sequences $(\widehat{E}_n)$, $(\widehat{F}_n)$ and $(\widehat{Q}_n)$ which satisfy all the properties described in the previous paragraph and, in addition, there exists a strictly increasing sequence (m_n) in $\mathbb{N}$ such that the following four conditions are satisfied

(i) $m_{n-1} < \min\operatorname{supp}\widehat{E}_n$ and $m_{n-1} < \min\operatorname{supp}\widehat{F}_n$;
(ii) $\widehat{Q}_n y = 0$ whenever $\max\operatorname{supp} y \leqslant m_{n-1}$;
(iii) $m_n \geqslant \max\operatorname{supp}\widehat{E}_n$;
(iv) $\|\widehat{Q}_n y\| \leqslant 2^{-n}\|y\|$ whenever $\min\operatorname{supp} y > m_n$.

We construct the sequences inductively. Let $m_0 = 0$, and suppose that we already constructed $\widehat{E}_i$, $\widehat{F}_i$, $\widehat{Q}_i$, and m_i for all $i < n$. Let G and G' be the subspaces of ℓ_p and ℓ_q respectively, consisting of all vectors whose first m_{n-1} coordinates are zero. Put $k = 2m_{n-1} + n$. It follows from $\dim F_k = k$ and $\operatorname{codim} G' = m_{n-1}$ that $m_{n-1} + n \leqslant \dim F_k \cap G' = \dim R^{-1}\big(F_k \cap G'\big)$, because $R_{|E_k}$ is an isomorphism. Since $\operatorname{codim} G = m_{n-1}$ we have $G \cap R^{-1}\big(F_k \cap G'\big) \geqslant n$. Let $\widehat{E}_n$ be an n-dimensional subspace of $G \cap R^{-1}\big(F_k \cap G'\big)$, and $\widehat{F}_n = R(\widehat{E}_n)$. Then $\widehat{E}_n \subseteq G$ and $\widehat{F}_n \subseteq G'$, hence (i) is satisfied. Clearly, $\widehat{F}_n$ is $\widehat{C}$-complemented in ℓ_q, where $\widehat{C} = C^2$. Then there exists a projection $Q' \colon \ell_q \to \widehat{F}_n$ such that $\|Q'\| \leqslant \widehat{C}$. Let $\widehat{Q}_n = Q'P$, where P is the basis projection of ℓ_q onto $[f_i]_{i \geqslant m_{n-1}}$. Then $\widehat{Q}_n$ is again a projection from ℓ_q onto $\widehat{F}_n$, $\|\widehat{Q}_n\| \leqslant \widehat{C}$, and (ii) is satisfied. Since $\operatorname{rank}\widehat{Q}_n = n$, we can write $\widehat{Q}_n = \sum_{j=1}^n z_j \otimes z_j^*$, where $z_1, \dots, z_j \in \ell_p$ and $z_1^*, \dots, z_j^* \in \ell_q^*$. Then we can find $r \in \mathbb{N}$ sufficiently large, such that if $\|y\| \leqslant 1$ and $\min\operatorname{supp} y > r$ then $|z_j^*(y)|$ is sufficiently small for all $j = 1, \dots, n$, so that $\|\widehat{Q}y\| \leqslant 2^{-n}$. Let $m_n = \max\{r, s\}$, where $s = \max\operatorname{supp}\widehat{E}_n$, then (iii) and (iv) are satisfied.

For convenience, we relabel $\widehat{E}_n$, $\widehat{F}_n$, $\widehat{Q}_n$, and $\widehat{C}$ as E_n, F_n, Q_n, and C again. For every n suppose that V_n is a C-isomorphism of ℓ_2^n onto E_n with $\|V_n\| = 1$ and $\|V_n^{-1}\| \leqslant C$. Put

$$V = \bigoplus_{n=1}^{\infty} V_n \colon \Big(\bigoplus_{n=1}^{\infty} \ell_2^n\Big)_p \to \Big(\bigoplus_{n=1}^{\infty} E_n\Big)_p.$$

Since E_n's are disjointly supported, we can consider $\big(\bigoplus_{n=1}^{\infty} E_n\big)_p$ as a subspace of ℓ_p. It follows that V is a C-isomorphism between $\big(\bigoplus_{n=1}^{\infty} \ell_2^n\big)_p$ and a subspace of ℓ_p. Define

$$W \colon \ell_q \to \Big(\bigoplus_{n=1}^{\infty} \ell_2^n\Big)_q \text{ via } W \colon x \mapsto \Big(V_n^{-1}\big(R_{|E_n}\big)^{-1} Q_n x\Big)_{n=1}^{\infty}.$$

We claim that W is bounded. Indeed, pick $x \in \ell_q$. Then

$$\|Wx\| = \Big(\sum_{n=1}^{\infty} \big\|V_n^{-1}\big(R_{|E_n}\big)^{-1} Q_n x\big\|_2^q\Big)^{\frac{1}{q}} \leqslant C^2 \Big(\sum_{n=1}^{\infty} \|Q_n x\|^q\Big)^{\frac{1}{q}}. \tag{3}$$

Let P_k be the basis projection from ℓ_q onto $[f_i]_{i=m_{k-1}+1}^{m_k}$. Then $x=\sum_{k=1}^\infty P_k x$. It follows from (ii) that $Q_nP_kx=0$ whenever $k<n$. Furthermore, (iv) yields $\big\|Q_n\big(\sum_{k>n}P_kx\big)\big\|\leqslant 2^{-n}\|x\|$. Also, $\|Q_nP_nx\|\leqslant C\|P_nx\|$. Therefore, $\|Q_nx\|\leqslant C\|P_nx\|+2^{-n}\|x\|$. Using Cauchy-Schwartz inequality, we get

$$\Big(\sum_{n=1}^\infty\|Q_nx\|^q\Big)^{\frac{1}{q}}\leqslant\Big(\sum_{n=1}^\infty\big(C\|P_nx\|\big)^q\Big)^{\frac{1}{q}}+\Big(\sum_{n=1}^\infty\big(2^{-n}\|x\|\big)^q\Big)^{\frac{1}{q}}\leqslant(C+1)\|x\|.$$

Together with (3) this yields that W is bounded.

Finally, it is easy to see that $WRV=I_{2,p,q}$, it follows easily that every Pełczyński Decomposition operator factors through R. ☐

Corollary 4.14. *Let $1<p\leqslant 2\leqslant q<\infty$. If $R\in L(\ell_p,\ell_q)$ is not FSS, then $\mathcal{J}^{\ell_2}\subseteq\mathcal{J}^R$.*

5. Operators not factorable through ℓ_2

We employ the following known theorem (see [DJT95, Theorem 9.13] or [Tom89, Theorem 27.1]) to deduce conditions for an operator in $L(\ell_p,\ell_q)$ to factor through ℓ_r.

Theorem 5.1. *Let $1\leqslant r<\infty$, let $U\colon X\to Y$ be a bounded linear operator between Banach spaces X and Y, and let $C\geqslant 0$. The following are equivalent:*

(i) *There exists a subspace L of $L_r(\mu)$, μ a measure, and a factorization $U=VW$, where $V\colon L\to Y$ and $W\colon X\to L$ are bounded linear operators with $\|V\|\cdot\|W\|\leqslant C$.*

(ii) *Whenever finite sequences $(x_i)_{i=1}^n$ and $(z_i)_{i=1}^m$ in X satisfy*

$$\sum_{i=1}^m\big|\langle x^*,z_i\rangle\big|^r\leqslant\sum_{i=1}^n\big|\langle x^*,x_i\rangle\big|^r \text{ for all } x^*\in X^*, \text{ then } \sum_{i=1}^m\|Uz_i\|^r\leqslant C^r\sum_{i=1}^n\|x_i\|^r.$$

Let us use Theorem 5.1 to state a criterion for an operator $U:\ell_p^m\to\ell_q^m$ not to factor as $U=AB$ with $\|B\|_{p,r}\cdot\|A\|_{r,q}\leqslant C$.

Corollary 5.2. *Let $m\in\mathbb{N}$, $C>1$, and $r>1$, and assume that U is an invertible m by m matrix. Let $\delta=\|U^{-1}\|_{r',r'}$. Then $\|B\|_{p,r}\cdot\|A\|_{r,q}\geqslant\delta^{-1}$ for any factorization $U=AB$. Moreover, if $\widetilde{U}$ is another m by m matrix with*

$$\|\widetilde{U}-U\|_{p,q}\leqslant\big(2\max_{1\leqslant i\leqslant m}\|U^{-1}e_i\|_p\big)^{-1}, \tag{4}$$

then it follows that for any factorization $\widetilde{U}=AB$ we have $\|B\|_{p,r}\cdot\|A\|_{r,q}\geqslant(2\delta)^{-1}$.

Proof. For $i = 1, \ldots, m$ let $x_i = e_i$ and $z_i = \delta^{-1}U^{-1}e_i$ and observe that for any $x^* \in \mathbb{R}^m$:

$$\Big(\sum_{i=1}^m \big|\langle x^*, z_i\rangle\big|^r\Big)^{1/r} = \delta^{-1}\Big(\sum_{i=1}^m \big|\langle (U^{-1})^* x^*, e_i\rangle\big|^r\Big)^{1/r} = \delta^{-1}\|(U^{-1})^* x^*\|_r$$

$$\leqslant \delta^{-1}\|U^{-1}\|_{r',r'}\|x^*\|_r = \Big(\sum_{i=1}^m \big|\langle x^*, x_i\rangle\big|^r\Big)^{1/r},$$

which implies that the hypothesis of (ii) in Theorem 5.1 is satisfied. Secondly it follows that

$$\sum_{i=1}^m \|Uz_i\|_q^r = \delta^{-r} m = \delta^{-r}\sum_{i=1}^m \|x_i\|_p^r, \tag{5}$$

which means that the conclusion of (ii) in Theorem 5.1 is not satisfied for any $C < \delta^{-1}$. It follows that condition (i) in Theorem 5.1 fails whenever $C < \delta^{-1}$.

Now assume that $\widetilde{U}$ is another m by m matrix satisfying (4), then it follows for $i = 1, \ldots, m$ that

$$\begin{aligned}
\|\widetilde{U}(z_i)\|_q &\geqslant \|U(z_i)\|_q - \|(U - \widetilde{U})(z_i)\|_q \\
&\geqslant \tfrac{1}{2}\|U(z_i)\|_q + \Big(\tfrac{1}{2}\|U(z_i)\|_q - \|U - \widetilde{U}\|_{p,q}\|z_i\|_p\Big) \\
&= \tfrac{1}{2}\|U(z_i)\|_q + \Big(\tfrac{1}{2\delta} - \|U - \widetilde{U}\|_{p,q}\delta^{-1}\|U^{-1}e_i\|_p\Big) \geqslant \tfrac{1}{2}\|U(z_i)\|_q,
\end{aligned}$$

which implies, together with (5), that for $\widetilde{U}$ the conclusion of (ii) in Theorem 5.1 is not satisfied for any $C < \delta^{-1}/2$, hence (i) fails in this case. □

We will now define an operator which will be crucial for the rest of the paper, and we start with the following notations. Let H_n be the n-th Hadamard matrix. That is, $H_1 = (1)$, $H_{n+1} = \left(\begin{smallmatrix} H_n & H_n \\ H_n & -H_n \end{smallmatrix}\right)$ for every $n \geqslant 1$. Then H_n is an $N \times N$ matrix where $N = 2^n$. We use the identifications $\ell_p = \big(\bigoplus_{n=1}^\infty X_n\big)_p$ and $\ell_q = \big(\bigoplus_{n=1}^\infty Y_n\big)_q$, where $X_n = \ell_p^{2^n}$ and $Y_n = \ell_q^{2^n}$ are block subspaces of ℓ_p and ℓ_q respectively. We consider H_n as an operator from X_n to Y_n. Put

$$U_n = N^{-\frac{1}{\min\{p',q\}}} H_n \text{ where } N = 2^n, \text{ and let } U = \bigoplus_{n=1}^\infty U_n \colon \ell_p \to \ell_q. \tag{6}$$

Remark 5.3. Observe that $N^{-\frac{1}{2}}H_n$ is a unitary matrix on ℓ_2^N. In particular, it is an isometry on ℓ_2^N, hence $\|H_n\|_{2,2} = N^{\frac{1}{2}}$, and $H_n^2 = NI$. One can easily verify that $\|H_n\|_{1,\infty} = 1$ and $\|H_n\|_{1,1} = \|H\|_{\infty,\infty} = N$.

Theorem 5.4. *If $1 < p \leqslant 2 \leqslant q < \infty$, then the operator U defined by (6) has the following properties.*

(i) $\|U\|_{p,q} = 1$.
(ii) *U is not compact.*
(iii) *If $p' \neq q$ then U is FSS.*
(iv) *Let $p \leqslant r \leqslant q$. Then U factors through ℓ_r when $p \leqslant r \leqslant q'$ or $p' \leqslant r \leqslant q$; otherwise $U \notin \mathcal{J}^{\ell_r}$.*
(v) *In particular, if $p \neq q$ then $U \notin \mathcal{J}^{\ell_2}$.*

Remark 5.5. In Section 6 we treat (iii) in the much harder case when $p' = q$ and show that in this case U is still FSS.

Proof. Using the Riesz-Thorin interpolation theorem (e.g., [BL76, LT79]) for H_n acting as an operator in $L(\ell_1, \ell_\infty)$ and as an operator in $L(\ell_2, \ell_2)$, and using Remark 5.3, we obtain $\|H_n\|_{r,r'} \leqslant N^{\frac{1}{r'}}$ whenever $1 \leqslant r \leqslant 2$. Similarly, interpolating between $\|H\|_{1,1}$ and $\|H_n\|_{2,2}$, and between $\|H_n\|_{2,2}$ and $\|H\|_{\infty,\infty}$, we obtain $\|H\|_{r,r} \leqslant N^{\frac{1}{\min\{r,r'\}}}$ whenever $1 \leqslant r \leqslant \infty$.

Define $U_n^{(r)} = N^{-\frac{1}{r'}} H_n$ and $U^{(r)} = \bigoplus_{n=1}^{\infty} U_n^{(r)}$, then $\|U_n^{(r)}\|_{r,r'} \leqslant 1$ for every n, hence $\|U^{(r)}\|_{r,r'} \leqslant 1$. Considering U as an operator in $L(\ell_p, \ell_q)$, we can write

$$U = \begin{cases} \ell_p \xrightarrow{U^{(p)}} \ell_q & \text{when } p' = q, \\ \ell_p \xrightarrow{U^{(p)}} \ell_{p'} \xrightarrow{I_{p',q}} \ell_q & \text{when } p' < q, \text{ and} \\ \ell_p \xrightarrow{I_{p,q'}} \ell_{q'} \xrightarrow{U^{(q')}} \ell_q & \text{when } p < q'. \end{cases} \tag{7}$$

It follows immediately that $\|U\|_{p,q} \leqslant 1$. Since $\mathcal{J}^{\mathrm{FSS}}$ is an ideal, (iii) follows from Proposition 3.3. It also follows from (7) that U factors through ℓ_r if $p \leqslant r \leqslant q'$ or $p' \leqslant r \leqslant q$,

Consider first the case $p' \leqslant q$. Then $U_n = N^{-\frac{1}{p'}} H_n$. Let $h_{n,i} = H_n e_i$, the i-th column of the n-th Hadamard matrix. It follows from $H_n^2 = NI$ that $U_n h_{n,i} = N^{-\frac{1}{p'}} H_n^2 e_i = N^{\frac{1}{p}} e_i$. Thus, $\|U_n h_{n,i}\|_q = N^{\frac{1}{p}} = \|h_{n,i}\|_p$, so that $\|U_n\|_{p,q} = 1$. Hence, U is not compact, and $\|U\|_{p,q} = 1$ by Remark 2.2.

Next, suppose that $p < r < p' \leqslant q$. We use Corollary 5.2 to show that $U \notin \mathcal{J}^{\ell_r}$ in this case. Indeed, assume to the contrary that $U \in \mathcal{J}^{\ell_r}$. Then there exists $\widetilde{U}$ such that $\|U - \widetilde{U}\| < \frac{1}{2}$ and $\widetilde{U}$ factors through ℓ_r. Let C be the ℓ_r-factorization constant of $\widetilde{U}$. Since $p < \min\{r, r'\}$ one can choose n so that $C < \frac{1}{2} N^{\frac{1}{p} - \frac{1}{\min\{r,r'\}}}$, where $N = 2^n$. Let $\widetilde{U}_n$ be the $N \times N$ submatrix of $\widetilde{U}$ corresponding to the n-th block of U, that is, $\widetilde{U}_n = Q_n \widetilde{U} P_n$, where P_n (respectively, Q_n) is the canonical projection from ℓ_p (respectively, ℓ_q) onto the span of $e_{N+1}, \dots, e_{2N}$. Then the ℓ_r-factorization constant of

$\widetilde{U}_n$ is at most C. It follows from $\|U_n^{-1}e_i\|_p = \|N^{-\frac{1}{p}}h_{n,i}\|_p = 1$ that

$$\big\|U_n - \widetilde{U}_n\big\| \leqslant \big\|U - \widetilde{U}\big\| < \tfrac{1}{2} = \big(2\max_{1\leqslant i\leqslant N}\big\|U_n^{-1}e_i\big\|_p\big)^{-1}.$$

Let $\delta = \|U_n^{-1}\|_{r',r'}$. It follows from $H_n^2 = NI$ and $U_n = N^{-\frac{1}{p'}}H_n$ that $U_n^{-1} = N^{-\frac{1}{p}}H$, so that

$$\delta = N^{-\frac{1}{p'}}\|H_n\|_{r',r'} \leqslant N^{-\frac{1}{p}+\frac{1}{\min\{r,r'\}}}.$$

Corollary 5.2 yields that the ℓ_r-factorization constant of $\widetilde{U}_n$ is at least $(2\delta)^{-1} \geqslant \frac{1}{2}N^{\frac{1}{p}-\frac{1}{\min\{r,r'\}}} > C$, which is a contradiction.

The case $p < q'$ can be reduced to the previous case by duality. Indeed, it follows from (7) that $U^* = I_{q,p'}U^{(q')}\colon \ell_{q'} \to \ell_{p'}$. It follows that if $p \leqslant r \leqslant q'$ then $I_{q,p'}$ and, therefore, U^* factors through $\ell_{r'}$. Hence, U factors through ℓ_r. Furthermore, since H_n is symmetric for every n, it follows that U_n^* coincides with U_n as a matrix and $\|U_n^*\|_{q',p'} = 1$. Applying the previous argument, we observe that U^* is non-compact and $\|U^*\|_{q',p'} = 1$, hence the same is true for U. Furthermore, if $q' < r < q$, then $U^* \notin \mathcal{J}^{\ell_{r'}}$ so that $U \notin \mathcal{J}^{\ell_r}$.

Finally, (v) follows immediately from (iv). □

Remark 5.6. If $p < r < r' < q$ then the operator $\widetilde{U}$ defined as

$$\ell_p \xrightarrow{I_{p,r}} \ell_r \xrightarrow{U^{(r)}} \ell_{r'} \xrightarrow{I_{r',q}} \ell_q$$

is compact. Indeed, as a matrix

$$\widetilde{U}_n = U_n^{(r)} = N^{-\frac{1}{r'}}H_n = N^{\frac{1}{\min\{p',q\}}-\frac{1}{r'}}U_n.$$

It follows from $\|U_n\|_{p,q} = 1$ and $r' < \min\{p',q\}$ that $\|\widetilde{U}_n\|_{p,q} = N^{\frac{1}{\min\{p',q\}}-\frac{1}{r'}} \to 0$ as $n \to 0$.

Remark 5.7. It follows from Theorem 5.4(iv) that $\mathcal{J}^{\ell_r}$ is proper when $\max\{p,q'\} < r < \min\{p',q\}$. In particular, $\mathcal{J}^{\ell_2}$ is proper. It follows from Remark 4.9 and Theorem 5.4(iv) that $\mathcal{J}^{\ell_2} \subsetneq \mathcal{J}^{\ell_r}$ whenever $p < r < q'$ or $p' < r < q$. We do not know, however, whether $\mathcal{J}^{\ell_r}$ is proper in this case.

Next, we show that if U' is another "U-like" operator then U and U' factor through each other.

Again, we view $\ell_p = \big(\bigoplus_{n=1}^\infty X_n\big)_p$ and $\ell_q = \big(\bigoplus_{n=1}^\infty Y_n\big)_q$, where $X_n = \ell_p^{2^n}$ and $Y_n = \ell_q^{2^n}$. Denote the basis vectors in X_n and Y_n by $e_1^{(n)},\dots,e_{2^n}^{(n)}$ and $f_1^{(n)},\dots,f_{2^n}^{(n)}$, respectively. We can view H_n and U_n as operators from X_n to Y_n.

Theorem 5.8. *Suppose that* (n_i) *is an increasing sequence, and let* $\widetilde{U} = \bigoplus_{i=1}^{\infty} U_{n_i}$*, viewed as an operator from* $\ell_p = \big(\bigoplus_{i=1}^{\infty} X_{n_i}\big)_p$ *to* $\ell_q = \big(\bigoplus_{i=1}^{\infty} Y_{n_i}\big)_q$*. Then* U *and* $\widetilde{U}$ *factor through each other.*

Proof. Consider the following diagram

$$\ell_p = \big(\textstyle\bigoplus_{i=1}^{\infty} X_{n_i}\big)_p \overset{\imath}{\hookrightarrow} \big(\bigoplus_{n=1}^{\infty} X_n\big)_p \xrightarrow{U} \big(\bigoplus_{n=1}^{\infty} Y_n\big)_q \xrightarrow{R} \big(\bigoplus_{i=1}^{\infty} Y_{n_i}\big)_q = \ell_q,$$

where $\imath$ is the canonical embedding, and R is the canonical projection. We can view $\imath$ and R as operators on ℓ_p and ℓ_q respectively. Thus, we get $\widetilde{U} = RU\imath$.

Next, we prove that U factors through $\widetilde{U}$. First, we show that whenever $n < m$ then there exists operators $C\colon X_n \to X_m$ and $D\colon Y_m \to Y_n$ such that $U_n = DU_mC$ and $\|C\|_{p,p} \leqslant 1$ and $\|D\|_{q,q} \leqslant 1$.

First, we consider the case $q \leqslant p'$. Define $C_n\colon X_n \to X_{n+1}$ via $C_n e_i^{(n)} = e_i^{(n+1)}$ as $i = 1, \dots, 2^n$. Clearly, C_n is an isometry.

Let Z_n be the subspace of Y_{n+1} consisting of all the vectors whose first half coordinates are equal to the last half coordinates respectively, that is, $Z_n = \operatorname{span}\{f_i^{(n+1)} + f_{i+2^n}^{(n+1)} \mid i = 1, \dots, 2^n\}$. Let P_n be the "averaging" projection from Y_{n+1} onto Z_n given by

$$P_n\Big(\sum_{i=1}^{2^{n+1}} \alpha_i f_i^{(n+1)}\Big) = \sum_{i=1}^{2^n} \frac{\alpha_i + \alpha_{i+2^n}}{2}\big(f_i^{(n+1)} + f_{i+2^n}^{(n+1)}\big).$$

Then $\|P_n\| = 1$.

Define $B_n\colon Z_n \to Y_n$ via $B_n(f_i^{(n+1)} + f_{i+2^n}^{(n+1)}) = 2^{\frac{1}{q}} f_i^{(n)}$, then B_n is an isometry. Hence, $D_n = B_nP_n\colon Y_{n+1} \to Y_n$ is of norm one.

Fix $1 \leqslant i \leqslant 2^n$. Since $C_n e_i^{(n)} = e_i^{(n+1)}$, $H_{n+1}C_n e_i^{(n)}$ is the i-th column of H_{n+1}. Since $i \leqslant 2^n$ it follows from the construction of H_n's that the i-th column of H_{n+1} is exactly the i-th column of H_n repeated twice. In particular, $H_{n+1}C_n e_i^{(n)} \in Z_n$ and, therefore, $H_{n+1}C_n e_i^{(n)} = P_nH_{n+1}C_n e_i^{(n)}$. Finally,

$$B_nP_nH_{n+1}C_n e_i^{(n)} = 2^{\frac{1}{q}}(\text{the } i\text{-th column of } H_n) = 2^{\frac{1}{q}} H_n e_i^{(n)}.$$

Consequently, $D_nH_{n+1}C_n = 2^{\frac{1}{q}}H_n$. It follows from $H_n = 2^{\frac{n}{q}}U_n$ that $D_nU_{n+1}C_n = U_n$. Iterating this $m - n$ times, we get $DU_mC = U_n$ where $C\colon X_n \to X_m$ is an isometry, $D\colon Y_m \to Y_n$ is of norm one.

If $q \geqslant p'$, then we consider the adjoint operators. Note that $U_n^* = U_n$ as matrices. Applying the previous argument we find matrices C and D such that $U_n^* = DU_m^*C$ with $\|C\|_{q',q'} \leqslant 1$ and $\|D\|_{p',p'} \leqslant 1$. Then $U_n = C^*U_mD^*$ is a required factorization in the case $q \geqslant p'$.

It follows that for every i we have

$$\widetilde{D}_i U_{n_i} \widetilde{C}_i = U_i \tag{8}$$

for some contractions $\widetilde{C}_i \colon X_i \to X_{n_i}$ and $\widetilde{D}_i \colon X_{n_i} \to X_i$. Let

$$\widetilde{C} = \left(\oplus_{i=1}^{\infty} \widetilde{C}_i\right) \colon \left(\oplus_{i=1}^{\infty} X_i\right)_p \to \left(\oplus_{i=1}^{\infty} X_{n_i}\right)_p$$

and

$$\widetilde{D} = \left(\oplus_{i=1}^{\infty} \widetilde{D}_i\right) \colon \left(\oplus_{i=1}^{\infty} X_{n_i}\right)_q \to \left(\oplus_{i=1}^{\infty} X_i\right)_q.$$

Then $\widetilde{C} \colon \ell_p \to \ell_p$ and $\widetilde{D} \colon \ell_q \to \ell_q$ are bounded, and by (8) we have $\widetilde{D}\widetilde{U}\widetilde{C} = U$. □

It follows that any two operators of type $\widetilde{U}$ generated by different sequences factor through each other.

6. The operator U is FSS

Again, let U be the operator defined by (6). Theorem 5.4(iii) states that U is FSS when $p \neq q'$. We will show in this section that U is still FSS when $1 < p = q'$. The argument requires some preparation.

Recall that the n-th s-number of an operator $T \in L(H)$ on a Hilbert space H is defined via $s_n(T) = \inf\{\|T - R\| \mid \operatorname{rank} R < n\}$. For $1 \leqslant r < \infty$, the Schatten norm $\|T\|_{S_r}$ of T equals the ℓ_r norm of the sequence of the s-numbers. We say that T belongs to Schatten class S_r if $\|T\|_{S_r} < \infty$. We denote by S_∞ the set of all compact operators equipped with the operator norm.

Lemma 6.1. *If $T \in L(H)$ such that $\|T\|_{S_q} = 1$ and $\inf_{x \in F, \|x\|=1} \|Tx\| \geqslant \varepsilon$ for a subspace F of H, then $\dim F \leqslant \varepsilon^{-q}$.*

Proof. Suppose that $\dim F = k$. For every operator S of rank $k - 1$ there exists $x \in F$ such that $\|x\| = 1$ and $Sx = 0$. It follows that $\|T - S\| \geqslant \|Tx\| \geqslant \varepsilon$, so that $s_1 \geqslant \ldots \geqslant s_k \geqslant \varepsilon$. Therefore, $1 = \|T\|_{S_q}^q \geqslant k\varepsilon^q$. Hence $k \leqslant \varepsilon^{-q}$. □

We will also utilize the following result of Maurey, [Maur74, Corollary 11, p. 21].

Theorem 6.2. *Let (Ω, μ) be a measure space, Y a Banach space, $0 < u \leqslant v < \infty$, $\frac{1}{u} = \frac{1}{v} + \frac{1}{r}$, T a bounded operator from a closed subspace E of $L_v(\mu)$ to Y, and $C > 0$. Then the following are equivalent.*

(i) *There exists a closed subspace F of $L_u(\mu)$ such that T factors as $T = V M_g$, where $V \colon F \to Y$ with $\|V\| \leqslant C$, and $M_g \colon L_v(\mu) \to L_u(\mu)$ is a multiplication operator defined by $M_g f = gf$ for every $f \in L_v(\mu)$, with $g \in L_r(\mu)$ and $\|g\|_r \leqslant 1$.*

(ii) *For any* $x_1,\dots,x_n$ *in* E,

$$\Big(\sum_{i=1}^n \|Tx_i\|^u\Big)^{\frac{1}{u}} \leqslant C\Big[\int\Big(\sum_{i=1}^n |x_i|^u\Big)^{\frac{v}{u}}\,d\mu\Big]^{\frac{1}{v}}.$$

In what follows, K_G will denote the so-called Grothendieck constant, a fundamental constant in the Banach space theory, see [DJT95, Tom89, LT77] for details.

Corollary 6.3. *Let* (Ω,μ) *be a measure space. Suppose that* $q=p'$ *and* $\frac{1}{p}=\frac{1}{2}+\frac{1}{r}$.

(i) *If* $T\colon L_q(\mu)\to\ell_2^k$ *then* T *can be factored through a multiplication operator from* $L_q(\mu)$ *to* $L_2(\mu)$, *that is,* $T=S\,M_g$, *where* $S\colon L_2(\mu)\to\ell_2^k$ *with* $\|S\|\leqslant K_G\|T\|$ *and* $\|g\|_r=1$.

(ii) *If* $T\colon \ell_2^k\to L_p(\mu)$ *then* T *can be factored through a multiplication operator from* $L_2(\mu)$ *to* $L_p(\mu)$, *that is,* $T=M_h\,S$, *where* $S\colon \ell_2^k\to L_2(\mu)$ *with* $\|S\|\leqslant K_G\|T\|$ *and* $\|h\|_r\leqslant 1$.

Proof. Suppose that $T\colon L_q(\mu)\to\ell_2^k$. We verify that condition (ii) of Theorem 6.2 holds for $u=2$, $v=q=p'$, and $r>1$ such that $\frac{1}{p}=\frac{1}{2}+\frac{1}{r}$ (which is equivalent to $\frac{1}{2}=\frac{1}{v}+\frac{1}{r}$). Let $f_1,\dots,f_n\in L_q$. Then

$$\sum_{i=1}^n\big\|Tf_i\big\|^2=\sum_{i=1}^n\sum_{j=1}^k\big|(Tf_i)_j\big|^2=\sum_{j=1}^k\sum_{i=1}^n\big|(Tf_i)_j\big|^2=\Big\|\Big(\sum_{i=1}^n|Tf_i|^2\Big)^{\frac{1}{2}}\Big\|_{\ell_2}^2,$$

where the last expression is the norm of the sequence $\Big(\big(\sum_{i=1}^n|(Tf_i)_j|^2\big)^{\frac{1}{2}}\Big)_{j=1}^n$. It follows from [LT79, Theorem 1.f.14] that

$$\Big\|\Big(\sum_{i=1}^n|Tf_i|^2\Big)^{\frac{1}{2}}\Big\|_{\ell_2}\leqslant K_G\|T\|\Big\|\Big(\sum_{i=1}^n|f_i|^2\Big)^{\frac{1}{2}}\Big\|_{L_q}=K_G\|T\|\Big[\int\Big(\sum_{i=1}^n|f_i|^2\Big)^{\frac{q}{2}}\,d\mu\Big]^{\frac{1}{q}}.$$

Now (i) follows from Theorem 6.2. To prove (ii), apply (i) to T^*. □

For $N\in\mathbb{N}$ and $1\leqslant p\leqslant\infty$, by L_p^N we denote the space $L_p(\mu)$ where μ is the uniform probability measure on $\Omega=\{1,\dots,N\}$. Thus, $L_p^N=\big(\mathbb{R}^N,\|\cdot\|_{L_p^N}\big)$ where, for $\bar{x}=(x_i)\in\mathbb{R}^N$, $\|\bar{x}\|_{L_p^N}=\big(\frac{1}{N}\sum_{i=1}^N|x_i|^p\big)^{\frac{1}{p}}$ for $p<\infty$ and $\|\bar{x}\|_{L_\infty^N}=\max_{1\leqslant i\leqslant N}|x_i|$. Clearly, $\|\cdot\|_{L_p^N}$ is a scalar multiple of $\|\cdot\|_{\ell_p^N}$.

The following easy lemma is well-known to specialists. We state it exactly in the form required later and we provide a short proof.

Lemma 6.4. *Consider a product of three operators*

$$S\colon L_2^N\xrightarrow{M_\psi}L_1^N\xrightarrow{T}\ell_\infty^N\xrightarrow{D}\ell_2^N$$

where $D = \operatorname{diag}(d_j)_{j=1}^N$, i.e., the diagonal operator with diagonal (d_j). Then the Hilbert-Schmidt norm of S satisfies $\|S\|_{\mathrm{HS}} \leqslant \|\psi\|_{L_2^N} \|T\| \big\|(d_j)\big\|_{\ell_2^N}$.

Proof. Observe that, in the notation of function spaces on (Ω, μ),

$$S\colon f \mapsto \psi f \mapsto \big(\langle g_n, \psi f\rangle\big)_{n=1}^N \mapsto \big(d_n \langle g_n, \psi f\rangle\big)_{n=1}^N,$$

for $f \in L_2^N$, and for some sequence $(g_n)_{n=1}^N$ in L_∞^N, so that $\|T\| = \sup_n \|g_n\|_{L_\infty^N}$. (Here $\langle \cdot, \cdot \rangle$ denotes the inner product with respect to μ.) Let $(f_i)_{i=1}^N$ be an orthonormal basis of L_2^N, then

$$\begin{aligned}\|S\|_{\mathrm{HS}}^2 = \sum_{i=1}^N \|Sf_i\|_{\ell_2^N}^2 = \sum_{i=1}^N \sum_{n=1}^N d_n^2 \langle g_n, \psi f_i\rangle^2 = \sum_{n=1}^N d_n^2 \sum_{i=1}^N \langle \psi g_n, f_i\rangle^2 \\ = \sum_{n=1}^N d_n^2 \cdot \|\psi g_n\|_{L_2^N}^2 \leqslant \|\psi\|_{L_2^N}^2 \cdot \big(\sup_n \|g_n\|_{L_\infty^N}^2\big) \cdot \big\|(d_j)\big\|_{\ell_2^N}^2.\end{aligned}$$

□

Theorem 6.5 ([Pis04])**.** *Suppose that $T\colon L_p^N \to \ell_q^N$ for some $1 \leqslant p < 2$ and $q = p'$. Let E be a k-dimensional subspace of L_p^N, and C_1, C_2, and C_3 be positive constants such that*

(i) *$\|T\|_{L_2^N, \ell_2^N} \leqslant 1$ and $\|T\|_{L_1^N, \ell_\infty^N} \leqslant 1$;*
(ii) *E is C_1-isomorphic to ℓ_2^k;*
(iii) *$F = T(E)$ is C_2-complemented in ℓ_q^N; and*
(iv) *$T_{|E}$ is invertible and $\big\|(T_{|E})^{-1}\big\| \leqslant C_3$.*

Then $k \leqslant \big(C_1^3 C_2 C_3^2 K_G^2\big)^q$.

Proof. Suppose that T, E, and F satisfy the hypotheses for some C_1, C_2, and C_2. Let r be such that $\frac{1}{p} = \frac{1}{2} + \frac{1}{r}$. There exists an isomorphism $V\colon \ell_2^k \to E$ such that $\|V\| \leqslant 1$ and $\|V^{-1}\| \leqslant C_1$. By Corollary 6.3(ii) V factors through L_2^N. Namely, $V = M_g S$ such that $S\colon \ell_2^k \to L_2^N$ with $\|S\| \leqslant C_1 K_G$ and $\|g\|_r \leqslant 1$. Let $J\colon E \to L_p^N$ be the canonical inclusion map.

$$\begin{array}{ccccccc} L_2^N & \xrightarrow{M_g} & L_p^N & \xrightarrow{T} & \ell_q^N & \xrightarrow[\text{diagonal}]{D} & \ell_2^N \\ S\uparrow & & J\uparrow \text{incl.} & & \text{proj.}\downarrow Q & & \downarrow R \\ \ell_2^k & \xrightarrow[C_1-\text{isom.}]{V} & E & \xrightarrow{T_{|E}} & F & \xrightarrow[C_1C_3-\text{isom}]{W} & \ell_2^k \end{array}$$

Let Q be a projection from ℓ_q^N onto F with $\|Q\| \leqslant C_2$. It follows from (i) that $\|T\|_{L_p^N, \ell_q^N} \leqslant 1$. Then F is C_1C_3-isomorphic to ℓ_2^k. Let $W\colon F \to \ell_2^k$ be an isomorphism such that $\|W\| \leqslant 1$ and $\|W^{-1}\| \leqslant C_1 C_3$. Corollary 6.3(i) implies that WQ factors through ℓ_2^N, that is, $WQ = RD$ where $R\colon \ell_2^N \to \ell_2^k$ with $\|R\| \leqslant K_G\|WQ\| \leqslant C_2K_G$, and D is a multiplication (or diagonal) operator $D = \operatorname{diag}(d_j)_{j=1}^N$ with $\big\|(d_j)\big\|_{\ell_r^N} \leqslant 1$.

We are going to show that $\|DTM_g\|_{S_q^N} \leqslant 1$, using the classical complex interpolation argument (see, e.g., [BL76]). For the convenience of the reader not familiar with the subject, we provide the details. Let $Z = \{z \in \mathbb{C} \mid 0 \leqslant \operatorname{Re} z \leqslant 1\}$, and define a function F from Z to the unit ball $B(L_2^N, \ell_2^N)$ of $L(L_2^N, \ell_2^N)$ as follows:

$$F(z) = |D|^{(1-z)\frac{r}{2}} \operatorname{sign} DTM_{|g|^{(1-z)\frac{r}{2}} \operatorname{sign} g}. \tag{9}$$

Here, as usually, $|D| = \operatorname{diag}(|d_j|)$ and $\operatorname{sign} D = \operatorname{diag}(\operatorname{sign} d_j)$. Observe that F is analytic in the interior of Z as a function from Z to $\mathbb{C}^N \times \mathbb{C}^N$. Furthermore, F is continuous and bounded on Z. A direct calculation shows that if $\frac{1}{r} = \frac{1-\theta}{2}$ then $F(\theta) = DTM_g$.

If $\operatorname{Re} z = 1$, it follows from (9) that $F(1+it) = A_tTB_t$, where $A_t = |D|^{-\frac{itr}{2}} \operatorname{sign} D$ and $B_t = M_{|g|^{-\frac{itr}{2}} \operatorname{sign} g}$. Notice that A_t and B_t viewed as operators from ℓ_2^N to ℓ_2^N and from L_2^N to L_2^N respectively are contractions. It follows that

$$\big\|F(z)\big\|_{L_2^N,\ell_2^N} \leqslant \|T\|_{L_2^N,\ell_2^N} \leqslant 1 \text{ whenever } \operatorname{Re} z = 1. \tag{10}$$

If $\operatorname{Re} z = 0$ then we can write

$$F(it) = A_t|D|^{\frac{r}{2}}TM_{|g|^{\frac{r}{2}}}B_t.$$

It can be easily verified that $\big\||g|^{\frac{r}{2}}\big\|_{L_2^N} \leqslant 1$ and $\big\|\big(|d_i|^{\frac{r}{2}}\big)\big\|_{\ell_2^N} \leqslant 1$. Since $\|T\|_{L_1^N,\ell_\infty^N} \leqslant 1$, it follows by Lemma 6.4 that

$$\big\|F(z)\big\|_{\mathrm{HS}} \leqslant 1 \text{ whenever } \operatorname{Re} z = 0. \tag{11}$$

Put $S_q^N = S_q(L_2^N, \ell_2^N)$. It is known (see, e.g. [GK65, Theorem 13.1]) that the Schatten classes interpolate like L_p-spaces. Since

$$\tfrac{1}{\infty}(1-\theta) + \tfrac{1}{2}\theta = \tfrac{1}{2} - \tfrac{1}{r} = \tfrac{1}{q},$$

it follows that $(S_\infty^N, S_2^N)_\theta = S_q^N$.

On the other hand, by definition of a complex interpolation space,

$$B_{(S_\infty^N,S_2^N)_\theta} = \Big\{f(\theta) \mid f\colon Z \to B(L_2^N, \ell_2^N) \text{ analytic}, \\ \big\|f_{|\{\operatorname{Re} z=0\}}\big\|_{S_2} \leqslant 1 \text{ and } \big\|f_{|\{\operatorname{Re} z=1\}}\big\|_{S_\infty} \leqslant 1\Big\}.$$

Since $\|\cdot\|_{S_2} = \|\cdot\|_{\mathrm{HS}}$ and $\|\cdot\|_{S_\infty} = \|\cdot\|_{L_2^N,\ell_2^N}$, it follows from (10) and (11) that $DTM_g = F(\theta) \in B_{(S_\infty^N,S_2^N)_\theta}$ and, thus, $\|DTM_g\|_{S_q^N} \leqslant 1$. It follows that

$$\|WTV\|_{S_q} = \big\|RDTM_gS\big\|_{S_q} \leqslant \|R\|\big\|DTM_g\big\|_{S_q}\|S\| \leqslant C_1C_2K_G^2.$$

Note that $\|(WTV)^{-1}\| \leqslant C_1^2 C_3^2$. It follows from Lemma 6.1 that

$$k \leqslant \Big(\frac{1}{C_1^2 C_3^2}\frac{1}{C_1 C_2 K_G^2}\Big)^{-q} = \big(C_1^3 C_2 C_3^2 K_G^2\big)^q.$$

This concludes the proof. □

We also need the following lemma, which generalizes Lemma 3.4. Assume that X is a Banach space with an FDD $(X_n)_{n=1}^\infty$, see [LT77] for the definition of FDD. Let P_n be the canonical projection from X onto X_n, and assume that X satisfies the following condition, which means that X is far from a c_0-sum of the X_n's:

(12) *for any $\delta > 0$ there is a $k = k(\delta)$ in $\mathbb{N}$ so that whenever $x \in S_X$, then* $\operatorname{card}\{n \in \mathbb{N} : \|P_n x\| \geqslant \delta\} < k$.

Suppose that for every $n \in \mathbb{N}$ we are given a seminorm q_n on X_n such that $q_n(x) \leqslant \|x\|$, where $q_n(x)$ stands for $q_n(P_n x)$ whenever $x \in X$.

Lemma 6.6. *Suppose that X, (X_n), and (q_n) are as in the preceding paragraph and $0 < r \leqslant 1$. Then there exists $\varepsilon > 0$ such that for every $l \in \mathbb{N}$ there exists $L \in \mathbb{N}$ such that for every L-dimensional subspace G of X such that* $\max_{n\in\mathbb{N}} q_n(x) \geqslant r\|x\|$ *for all $x \in G$ there exists an l-dimensional subspace $F \subseteq G$ and an index n_0 such that $q_{n_0}(x) \geqslant \varepsilon\|x\|$ for all $x \in F$.*

To prove Lemma 6.6 we need the following stabilization result, see, e.g., [MS86, p.6].

Theorem 6.7. *For every $n \in \mathbb{N}$, $\varepsilon > 0$ and $c > 0$ there is an $N = N(n, \varepsilon, c) \in \mathbb{N}$ so that for any N-dimsensional space E, and any Lipschitz map $f : S_E \to \mathbb{R}$ whose Lipschitz constant does not exceed c, there is an n-dimensional subspace F of E so that*

$$\max\{f(x) : x \in S_F\} - \min\{f(x) : x \in S_F\} \leqslant \varepsilon.$$

Proof of Lemma 6.6. Let $k(\cdot)$ be the function defined in (12). Put

$$m = k\big(\tfrac{r^2}{4}\big), \quad \delta = \tfrac{r}{4m}, \quad \text{and} \quad s = k(\delta).$$

It suffices to show that for $l' \in \mathbb{N}$ there exists L so that if G is a subspace of X of dimension L and $\max_{n\in\mathbb{N}} q_n(x) \geqslant r\|x\|$ for all $x \in G$ then G has an l'-dimensional subspace F' and a set $I \subset \mathbb{N}$ with $\operatorname{card} I = s$ such that $\max_{n\in I} q_n(x) \geqslant \delta\|x\|$ for all $x \in F'$.

Indeed, once we prove this formally weaker claim, we can take a number l' large enough, so that we can apply Theorem 6.7 s times to deduce that F' has an l-dimensional subspace F, which has the property that for all $n \in I$

$$\max_{x\in S_F} q_n(x) - \min_{x\in S_F} q_n(x) \leqslant \tfrac{\delta}{2}.$$

Now pick any $y \in S_F$, then $q_{n_0}(y) = \max_{n\in I} q_n(y) \geqslant \delta$ for some $n_0 \in I$. Then for every $x \in S_F$ we have

$$q_{n_0}(x) \geqslant \min_{z\in S_F} q_{n_0}(z) \geqslant \max_{z\in S_F} q_{n_0}(z) - \tfrac{\delta}{2} \geqslant q_{n_0}(y) - \tfrac{\delta}{2} \geqslant \tfrac{\delta}{2},$$

so that the statement of our Lemma is satisfied for $\varepsilon = \frac{\delta}{2}$.

Let $l' \in \mathbb{N}$ and define numbers $L_0, L_1, \dots, L_m$ as follows. Put $L_0 = l'$, and, assuming that $L_0, L_1, \dots, L_n$, $n < m$, have already been defined, we use Theorem 6.7 to choose L_{n+1} large enough so that for every L_{n+1}-dimensional subspace G of X and every Lipschitz-1 map $f\colon S_G \to \mathbb{R}$ there is an L_n-dimensional subspace $G' \subseteq G$ such that

$$\max_{x\in G'} f(x) - \min_{x\in G'} f(x) \leqslant \delta.$$

Let $L = L_m$. Assume that out claim is false. This would mean that there exists a subspace G of X of $\dim G = L$ such that

$$\max_{n\in\mathbb{N}} q_n(x) \geqslant r\|x\| \text{ for all } x \in G, \text{ and} \tag{13}$$

$$\text{for each } I \subset \mathbb{N} \text{ of } \operatorname{card} I = s \text{ and each subspace } F' \subseteq G \text{ of } \dim F' = l' \text{ there exists } x \in S_{F'} \text{ such that } \max_{n\in I} q_n(x) \leqslant \delta. \tag{14}$$

Choose an arbitrary vector $x_1 \in S_G$ and a subset $I_1 \subset \mathbb{N}$ of $\operatorname{card} I_1 = s$ so that $\min_{n\in I_1} q_n(x_1) \geqslant \max_{n\in\mathbb{N}\setminus I_1} q_n(x_1)$. It follows from (13) that there exists an index n_1 such that $q_{n_1}(x_1) \geqslant r$; we can assume that $n_1 \in I_1$. On the other hand, the definition of s implies that $q_n(x_1) \leqslant \delta$ whenever $n \notin I_1$. It follows from the definition of L_m that there exists a subspace G_{m-1} of G of dimension L_{m-1} so that

$$\max_{x\in S_{G_{m-1}}} \max_{n\in I_1} q_n(x) \leqslant \min_{x\in S_{G_{m-1}}} \max_{n\in I_1} q_n(x) + \delta \leqslant 2\delta, \tag{15}$$

where the last inequality follows from (14).

Next, pick an $x_2 \in S_{G_{m-1}}$ and $I_2 \subset \mathbb{N} \setminus I_1$ so that $\operatorname{card} I_2 = s$ and $\min_{n\in I_2} q_n(x_2) \geqslant \max_{n\notin I_1\cup I_2} q_n(x_2)$. Again, it follows from (13) that there exists an index n_2 such that $q_{n_2}(x_2) \geqslant r$; we can assume that $n_2 \in I_1 \cup I_2$. By (15), $q_n(x_2) \leqslant 2\delta < r$ for each $n \in I_1$, so that $n_2 \in I_2$. Again, $q_n(x_2) \leqslant \delta$ whenever $n \notin I_1 \cup I_2$. We can choose a subspace G_{m-2} of G_{m-1} of dimension L_{m-2} so that

$$\max_{x\in S_{G_{m-2}}} \max_{n\in I_2} q_n(x) \leqslant 2\delta.$$

Proceeding this way, we obtain a sequence of vectors $x_1, \dots, x_m$ and disjoint sets $I_1, \dots, I_m$ of cardinality s, and indices $n_1, \dots, n_m$ such that for each $i = 1, \dots, m$ we have $n_i \in I_i$ and $q_{n_i}(x_i) \geqslant r$. Also,

$$q_n(x_i) \leqslant \begin{cases} 2\delta & \text{if } n \in I_1 \cup \dots \cup I_{i-1}, \text{ and} \\ \delta & \text{if } n \notin I_1 \cup \dots \cup I_i, \end{cases}$$

hence $q_n(x_i) \leqslant 2\delta$ whenever $n \notin I_i$. If $n \in I_i$ then $q_n(x_i) \leqslant \|x_i\| = 1$.

Put $x = \sum_{i=1}^m x_i$, then for every $n \in \mathbb{N}$ we have $q_n(x) \leqslant 1 + m \cdot 2\delta \leqslant 2$. On the other hand,

$$r \leqslant q_{n_i}(x_i) \leqslant q_{n_i}(x) + q_{n_i}(x - x_i) \leqslant q_{n_i}(x) + 2m\delta,$$

so that $q_{n_i}(x) \geqslant r - 2m\delta = \frac{r}{2}$ for each $i = 1, \dots, m$. It follows from the definition of m that there can be at most $m-1$ indices n such that $q_n(x) \geqslant \frac{r^2}{4}\|x\|$, hence $\frac{r^2}{4}\|x\| > \frac{r}{2}$. It follows that $\|x\| > \frac{2}{r}$, so that $q_n(x) \leqslant 2 < r\|x\|$ for every $n \in \mathbb{N}$, which is a contradiction. □

Now we are ready to prove that U is FSS.

Theorem 6.8. *The operator U constructed in (6) is FSS for all $1 < p \leqslant 2 \leqslant q < \infty$, unless $p = q = 2$.*

Proof. In view of Theorem 5.4(iii) we may assume that $q = p'$. Recall that $U = \bigoplus_{n=1}^\infty U_n$ is composed of blocks $U_n\colon X_n \to Y_n$, where $X_n = \ell_p^{2^n}$ and $Y_n = \ell_q^{2^n}$. For each n, let $P_n\colon \ell_p \to X_n$ be the canonical projection. For $x \in \ell_p$ put $q_n(x) = \big\|U_n P_n x\big\|$. By Theorem 5.4(i) we have $q_n(x) \leqslant \|x\|$.

Assume that U is not FSS. Then there exists a constant C such that there are subspaces G of ℓ_p of arbitrarily large dimension such that the restriction of U to G is a C-isomorphism. Let $x \in S_G$, write $x = \sum_{n=1}^\infty x_n$ where $x_n \in X_n$, then $\|Ux\| \geqslant \frac{1}{C}$. On the other hand,

$$\|Ux\|^q = \sum_{n=1}^\infty \|U_n x_n\|^q \leqslant \max_{n\in\mathbb{N}} \|U_n x_n\|^{q-p} \sum_{n=1}^\infty \|U_n x_n\|^p \leqslant \max_{n\in\mathbb{N}} q_n(x)^{q-p}.$$

Hence, $\max\limits_{n\in\mathbb{N}} q_n(x) \geqslant C^{\frac{q}{p-q}}$.

It follows from Lemma 6.6 that there exists $\varepsilon > 0$ such that for every k and for every $G \subseteq \ell_p$ of sufficiently large dimension there exists a subspace F of G and an index n such that $\dim F = k$ and $q_n(x) \geqslant \varepsilon$ for all $x \in S_F$. This implies that the restriction of $U_n P_n$ to F is a $\frac{1}{\varepsilon}$-isomorphism. Put $E = P_n(F)$, then E is a k-dimensional subspace of X_n, and U_n is a $\frac{1}{\varepsilon}$-isomorphism on E. In view of Theorem 4.10 we may assume that E is 2-isomorphic to ℓ_2^k and $U_n(E)$ is K-complemented in $\ell_q^{2^n}$.

Let V_n be the canonical isometry between L_p^N and $X_n = \ell_p^N$, where $N = 2^n$. It follows that $\|V_n x\|_{\ell_r^N} = N^{\frac{1}{r} - \frac{1}{p}} \|x\|_{L_r^N}$ for every $x \in L_p^N$ and every $r \in [p, q]$. It follows from the definition of U_n and Remark 5.3 that

$$\|U_nV_n\|_{L_2^N,\ell_2^N} = N^{\frac{1}{2}-\frac{1}{p}}\|U_n\|_{\ell_2^N,\ell_2^N} = N^{\frac{1}{2}-\frac{1}{p}-\frac{1}{q}}\|H_n\|_{\ell_2^N,\ell_2^N} = 1$$

and

$$\|U_nV_n\|_{L_1^N,\ell_\infty^N} = N^{1-\frac{1}{p}}\|U_n\|_{\ell_1^N,\ell_\infty^N} = N^{1-\frac{1}{p}-\frac{1}{q}}\|H_n\|_{\ell_1^N,\ell_\infty^N} = 1.$$

Now applying Theorem 6.5 to U_nV_n and $V_n^{-1}(E)$ we obtain a contradiction with the fact that $k = \dim E$ was chosen arbitrarily. □

Remark 6.9. If $p = q = 2$ then U is an isometry, hence not FSS. Consider the case when $p = 1$ and $q = \infty$. The preceding proof does not work, since now we cannot use Theorem 4.10. Actually, U is not FSS in this case. Indeed, we now have $U_n = H_n$. It is easy to see that among the columns of H_n one finds all the Rademacher vectors (of length $N = 2^n$). Since the span of these vectors in ℓ_∞^N is isometrically isomorphic to ℓ_1^n, it follows that the restriction of H_n to the appropriate subspace of ℓ_1^N preserves a copy of ℓ_1^n.

Question. Are there any other closed ideals in $L(\ell_p, \ell_q)$? In view of the diagram at the beginning of our paper this question can be subdivided in the following subquestions:

(i) Is $\mathcal{J}^{I_{p,q}}$ equal to $\mathcal{J}^{\mathrm{F}SS} \cap \mathcal{J}^{\ell_2}$? If not, is $\mathcal{J}^{\mathrm{F}SS} \cap \mathcal{J}^{\ell_2}$ an immediate successor of $\mathcal{J}^{I_{p,q}}$?

(ii) Is $\mathcal{J}^{\mathrm{F}SS}$ an immediate successor of $\mathcal{J}^{\mathrm{F}SS} \cap \mathcal{J}^{\ell_2}$? More generally, are there any immediate successors of $\mathcal{J}^{\mathrm{F}SS} \cap \mathcal{J}^{\ell_2}$, other than $\mathcal{J}^{\ell_2}$?

(iii) Is $\mathcal{J}^{\mathrm{F}SS} \vee \mathcal{J}^{\ell_2}$ an immediate successor of $\mathcal{J}^{\ell_2}$?

(iv) Is $\mathcal{J}^{\mathrm{F}SS} \vee \mathcal{J}^{\ell_2}$ equal to $L(\ell_p, \ell_q)$?

Question. Suppose again that U is the operator defined in (6). Since U is FSS, we have $\mathcal{J}^U \subseteq \mathcal{J}^{\mathrm{F}SS}$. Does $\mathcal{J}^U$ equal $\mathcal{J}^{\mathrm{F}SS}$?

References

[BL76] Bergh, Jöran and Löfström, Jörgen. *Interpolation spaces. An introduction.* Grundlehren der Mathematischen Wissenschaften, No. 223. Springer-Verlag, Berlin-New York, 1976.

[Calk41] J.W. Calkin, Two-sided ideals and congruences in the ring of bounded operators in Hilbert space, *Ann. of Math.* 42(2):839–873, 1941.

[CPY74] S.R. Caradus, W.E. Pfaffenberger, and B. Yood. *Calkin algebras and algebras of operators on Banach spaces.* Marcel Dekker, Inc., New York, 1974. Lecture Notes in Pure and Applied Mathematics, Vol. 9.

[Daws06] M. Daws, Closed ideals in the Banach algebra of operators on classical non-separable spaces, Math.Proc.Camb.Phil. Soc. 140: 317–332, 2006.

[DJT95] J. Diestel, H. Jarchow, and A. Tonge. *Absolutely summing operators*, volume 43 of *Cambridge Studies in Advanced Mathematics.* Cambridge University Press, Cambridge, 1995.

[FT79] T. Figiel and N. Tomczak-Jaegermann. Projections onto Hilbertian subspaces of Banach spaces. *Israel J. Math.*, 33(2):155–171, 1979.

[GMF60] I.C. Gohberg, A.S. Markus, I.A. Feldman, Normally solvable operators and ideals associated with them, (Russian) *Bul. Akad. Štiince RSS Moldoven*, 76(10):51–70, 1960. English translation: *American. Math. Soc. Translat.* 61:63–84, 1967.

[GK65] I.C. Gohberg and M.G. Kreĭn, *Introduction to the theory of linear non-selfadjoint operators in Hilbert space*, (Russian), Nauka, Moscow, 1965. English translation: Amer. Math. Soc., Providence, R.I., 1969

[Gram67] B. Gramsch, Eine Idealstruktur Banachscher Operatoralgebren, J. Reine Angew. Math 225:97–115, 1967.

[LLR04] N.J. Laustsen, R.J. Loy, and C.J. Read, The lattice of closed ideals in the Banach algebra of operators on certain Banach spaces, J. Funct. Anal. 214(1):106–131, 2004.

[LSZ06] N.J. Laustsen, Th. Schlumprecht, and A. Zsak, The lattice of closed ideals in the Banach algebra of operators on a certain dual Banach space, J. of Operator Theory, 56 (2): 391-402, 2006.

[LT77] J. Lindenstrauss and L. Tzafriri, *Classical Banach spaces. I.* Springer-Verlag, Berlin, 1977.

[LT79] J. Lindenstrauss and L. Tzafriri, *Classical Banach spaces. II.* Springer-Verlag, Berlin, 1979.

[Luft68] E. Luft, The two-sided closed ideals of the algebra of bounded linear operators of a Hilbertspace. Czechoslovak Math. J. 18: 595–605, 1968.

[Masc94] V. Mascioni. A restriction-extension property for operators on Banach spaces. *Rocky Mountain J. Math.*, 24(4):1497–1507, 1994.

[Maur74] B. Maurey. *Théorèmes de factorisation pour les opérateurs linéaires à valeurs dans les espaces L^p*. Astérisque, 11, Soc. Math. France, Paris, 1974.

[MS86] V.D. Milman and G. Schechtman. *Asymptotic Theory of Finite Dimensional Normed Spaces*, LNM 1200, Springer-Verlag, New York, 1986.

[Milm70] V.D. Milman. Operators of class C_0 and C_0^*. (Russian), *Teor. Funkciĭ Funkcional. Anal. i Priložen.*, 10:15–26, 1970.

[Piet78] A. Pietsch. *Operator ideals*, volume 16 of *Mathematische Monographien [Mathematical Monographs]*. VEB Deutscher Verlag der Wissenschaften, Berlin, 1978.

[Pis04] G. Pisier. Private communications.

[Tom89] N. Tomczak-Jaegermann, *Banach-Mazur distances and finite-dimensional operator ideals*, Pitman Monographs and Surveys in Pure and Applied Mathematics, (38), Longman Scientific & Technical, Harlow, 1989.

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