NORM CLOSED OPERATOR IDEALS IN LORENTZ SEQUENCE SPACES

ANNA KAMINSKA, ALEXEY I. POPOV, EUGENIU SPINU, ADI TCACIUC, AND VLADIMIR G. TROITSKY

Abstract. In this paper, we study the structure of closed algebraic ideals in the algebra of operators acting on a Lorentz sequence space.

1. Introduction

1.1. **Ideals.** This paper is concerned with the study of the structure of closed algebraic ideals in the algebra $L(X)$ of all bounded linear operatrors on a Banach space X.

Throughout the paper, by a ***subspace*** of a Banach space we mean a closed subspace; a vector subspace of X which is not necessarily closed will be referred to as ***linear subspace***. A (two-sided) ***ideal*** in $L(X)$ is a linear subspace J of $L(X)$ such that $ATB \in J$ whenever $T \in J$ and $A, B \in L(X)$. The ideal J is called ***proper*** if $J \neq L(X)$. The ideal J is ***non-trivial*** if J is proper and $J \neq \{0\}$.

The spaces for which the structure of closed ideals in $L(X)$ is well-understood are very few. It was shown in [7] that the only non-trivial closed ideal in the algebra $L(\ell_2)$ is the ideal of compact operators. This result was generalized in [13] to the spaces ℓ_p $(1 \leqslant p < \infty)$ and c_0. A space constructed recently in [5] is another space with this property. In [15] and [16], it was shown that the algebras $L\big((\oplus_{k=1}^{\infty}\ell_2^k)_{c_0}\big)$ and $L\big((\oplus_{k=1}^{\infty}\ell_2^k)_{\ell_1}\big)$ have exactly two non-trivial closed ideals. There are no other separable spaces for which the structure of closed ideals in $L(X)$ is completely known.

Partial results about the structure of closed ideals in $L(X)$ were obtained in [20, 5.3.9] for $X = L_p[0,1]$ $(1 < p < \infty,\ p \neq 2)$ and in [22] and [23] for $L(\ell_p \oplus \ell_q)$ $(1 \leqslant p, q < \infty)$. The purpose of this paper is to investigate the structure of ideals in $L(d_{w,p})$ where $d_{w,p}$ is a Lorentz sequence space (see the definition in Subsection 1.3).

For two closed ideals J_1 and J_2 in $L(X)$, we will denote by $J_1 \wedge J_2$ the largest closed ideal J in $L(X)$ such that $J \subseteq J_1$ and $J \subseteq J_2$ (that is, $J_1 \wedge J_2 = J_1 \cap J_2$), and we will

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denote by $J_1 \vee J_2$ the smallest closed ideal J in $L(X)$ such that $J_1 \subseteq J$ and $J_2 \subseteq J$. We say that J_2 is a ***successor*** of J_1 if $J_1 \subsetneq J_2$. If, in addition, no closed ideal J in $L(X)$ satisfies $J_1 \subsetneq J \subsetneq J_2$, then we call J_2 an ***immediate successor*** of J_1.

It is well-known that if X is a Banach space then every non-zero ideal in the algebra $L(X)$ must contain the ideal $\mathcal{F}(X)$ of all finite-rank operators on X. It follows that, at least in the presence of the approximation property (in particular, if X has a Schauder basis), every non-zero closed ideal in $L(X)$ contains the closed ideal $\mathcal{K}(X)$ of all compact operators.

Two ideals closely related to $\mathcal{K}(X)$ are the closed ideal $\mathcal{SS}(X)$ of strictly singular operators and the closed ideal $\mathcal{FSS}(X)$ of finitely strictly singular operators on X. Recall that an operator $T \in L(X)$ is called ***strictly singular*** if no restriction $T|_Z$ of T to an infinite-dimensional subspace Z of X is an isomorphism. An operator T is ***finitely strictly singular*** if for any $\varepsilon > 0$ there is $N \in \mathbb{N}$ such that any subspace Z of X with $\dim Z \geqslant N$ contains a vector $z \in Z$ satisfying $\|Tz\| < \varepsilon\|z\|$. It is not hard to show that $\mathcal{K}(X) \subseteq \mathcal{FSS}(X) \subseteq SS(X)$ (see [17, 19, 22, 4] for more information about these classes of operators).

If X is a Banach space and $T \in L(X)$ then the ideal in $L(X)$ generated by T is denoted by J_T. It is easy to see that $J_T = \big\{\sum_{i=1}^{n} A_i T B_i \ :\ A_i, B_i \in L(X)\big\}$. It follows that if $S \in L(X)$ factors through T, i.e., $S = ATB$ for some $A, B \in L(X)$ then $J_S \subseteq J_T$.

1.2. **Basic sequences.** The main tool in this paper is the notion of a basic sequence. In this subsection, we will fix some terminology and remind some classical facts about basic sequences. For a thorough introduction to this topic, we refer the reader to [9] or [12].

If (x_n) is a sequence in a Banach space X then its closed span will be denoted by $[x_n]$. We say that a basic sequence (x_n) ***dominates*** a basic sequence (y_n) and write $(x_n) \succeq (y_n)$ if the convergence of a series $\sum_{n=1}^{\infty} a_n x_n$ implies the convergence of the series $\sum_{n=1}^{\infty} a_n y_n$. We say that (x_n) is ***equivalent*** to (y_n) and write $(x_n) \sim (y_n)$ if $(x_n) \succeq (y_n)$ and $(y_n) \succeq (x_n)$.

Remark 1.1. It follows from the Closed Graph Theorem that $(x_n) \succeq (y_n)$ if and only if the linear map from $\operatorname{span}\{x_n\}$ to $\operatorname{span}\{y_n\}$ defined by the formula $T\colon x_n \mapsto y_n$ is bounded.

If (x_n) is a basis in a Banach space X, $z = \sum_{i=1}^{\infty} z_i x_i \in X$, and $A \subseteq \mathbb{N}$ then the vector $\sum_{i \in A} z_i x_i$ will be denoted by $z|_A$ (provided the series converges; this is always the case when the basis is unconditional). We will refer to $z|_A$ as the ***restriction of***

z ***to*** A. The restrictions $z|_{[n,\infty)\cap\mathbb{N}}$ and $z|_{(n,\infty)\cap\mathbb{N}}$, where $n \in \mathbb{N}$, will be abbreviated as $z|_{[n,\infty)}$ and $z|_{(n,\infty)}$, respectively. We say that a vector v is a ***restriction*** of z if there exists $A \subseteq \mathbb{N}$ such that $v = z|_A$. The vector $z = \sum_{i=1}^{\infty} z_i x_i$ will also be denoted by $z = (z_i)$. If $z = \sum_{i=1}^{\infty} z_i x_i$ then the ***support*** of z is the set $\operatorname{supp} z = \{i \in \mathbb{N} \colon z_i \neq 0\}$.

Every 1-unconditional basis (x_n) in a Banach space X defines a Banach lattice order on X by $\sum_{i=1}^{\infty} a_i x_i \geqslant 0$ if and only if $a_i \geqslant 0$ for all $i \in \mathbb{N}$ (see, e.g., [18, page 2]). For $x \in X$, we have $|x| = x \vee (-x)$. A Banach lattice is said to have ***order continuous norm*** if the condition $x_\alpha \downarrow 0$ implies $\|x_\alpha\| \to 0$. For an introduction to Banach lattices and standard terminology, we refer the reader to [1, §1.2].

If (x_n) is a basic sequence in a Banach space X, then a sequence (y_n) in $\operatorname{span}\{x_n\}$ is a ***block sequence*** of (x_n) if there is a strictly increasing sequence (p_n) in $\mathbb{N}$ and a sequence of scalars (a_i) such that $y_n = \sum_{i=p_n+1}^{p_{n+1}} a_i x_i$ for all $n \in \mathbb{N}$.

The following two facts are classical and will sometimes be used without any references. The first fact is known as the Principle of Small Perturbations (see, e.g., [12, Theorem 4.23]).

Theorem 1.2. *Let X be a Banach space, (x_n) a basic sequence in X, and (x_n^*) the correspondent biorthogonal functionals defined on $[x_n]$. If (y_n) is a sequence such that $\sum_{n=1}^{\infty} \|x_n^*\| \cdot \|x_n - y_n\| < 1$ then (y_n) is a basic sequence equivalent to (x_n). Moreover, if $[x_n]$ is complemented in X then so is $[y_n]$. If $[x_n] = X$ then $[y_n] = X$.*

The next fact, which is often called the Bessaga-Pełczyński selection principle, is a result of combining the "gliding hump" argument (see, e.g., [9, Lemma 5.1]) with the Principle of Small Perturbations.

Theorem 1.3. *Let X be a Banach space with a seminormalized basis (x_n) and let (x_n^*) be the correspondent biorthogonal functionals. Let (y_n) be a seminormalized sequence in X such that $x_n^*(y_k) \overset{k\to\infty}{\longrightarrow} 0$ for all $n \in \mathbb{N}$. Then (y_n) has a subsequence (y_{n_k}) which is basic and equivalent to a block sequence (u_k) of (x_n). Moreover, $y_{n_k} - u_k \to 0$, and u_k is a restriction of y_{n_k}.*

1.3. **Lorentz sequence spaces.** Let $1 \leqslant p < \infty$ and $w = (w_n)$ be a sequence in $\mathbb{R}$ such that $w_1 = 1$, $w_n \downarrow 0$, and $\sum_{i=1}^{\infty} w_i = \infty$. The Lorentz sequence space $d_{w,p}$ is a Banach space of all vectors $x \in c_0$ such that $\|x\|_{d_{w,p}} < \infty$, where

$$\|(x_n)\|_{d_{w,p}} = \Big(\sum_{n=1}^{\infty} w_n x_n^{*p} \Big)^{1/p}$$

is the norm in $d_{w,p}$. Here (x_n^*) is the ***non-increasing rearrangement*** of the sequence $(|x_n|)$. An overview of properties of Lorentz sequence spaces can be found in [17, Section 4.e].

The vectors (e_n) in $d_{w,p}$ defined by $e_n(i) = \delta_{ni}$ $(n, i \in \mathbb{N})$ form a 1-symmetric basis in $d_{w,p}$. In particular, (e_n) is 1-unconditional, hence $d_{w,p}$ is a Banach lattice. We call (e_n) the unit vector basis of $d_{w,p}$. The unit vector basis of ℓ_p will be denoted by (f_n) throughout the paper.

Remark 1.4. It is proved in [3, Lemma 1] and [10, Lemma 15] that if (u_n) is a seminormalized block sequence of (e_n) in $d_{w,p}$, $u_n = \sum_{i=p_n+1}^{p_{n+1}} a_i e_i$, such that $a_i \to 0$, then there is a subsequence (u_{n_k}) such that $(u_{n_k}) \sim (f_n)$ and $[u_{n_k}]$ is complemented in $d_{w,p}$. Further, it was shown in [3, Corollary 3] that if (y_n) is a seminormalized block sequence of (e_n) then there is a seminormalized block sequence (u_n) of (y_n) such that $u_n = \sum_{i=p_n+1}^{p_{n+1}} a_i e_i$, with $a_i \to 0$. Therefore, every infinite dimensional subspace of $d_{w,p}$ contains a further subspace which is complemented in $d_{w,p}$ and isomorphic to ℓ_p ([10, Corollary 17]).

Remark 1.5. Remark 1.4 yields, in particular, that $d_{w,p}$ does not contain copies of c_0. Since the basis (e_n) of $d_{w,p}$ is unconditional, the space $d_{w,p}$ is weakly sequentially complete by [2, Theorem 4.60] (see also [17, Theorem 1.c.10]). Also, [2, Theorem 4.56] guarantees that $d_{w,p}$ has order continuous norm. In particular, if $x \in d_{w,p}$ then $\|x|_{[n,\infty)}\| \to 0$ as $n \to \infty$.

Remark 1.6. It was shown in [14] that if $p > 1$ then $d_{w,p}$ is reflexive. This can also be easily obtained from Remark 1.4 (cf. [17, Theorem 1.c.12]).

Remark 1.7. The unit vector basis (e_n) of $d_{w,p}$ is weakly null. Indeed, by Rosenthal's ℓ_1-theorem (see [21]; also [17, Theorem 2.e.5]), (e_n) is weakly Cauchy. Since it is symmetric, $(e_n) \sim (e_{2n} - e_{2n-1})$.

The next proposition will be used often in this paper.

Proposition 1.8 ([3, Proposition 5 and Corollary 2])**.** *If (u_n) is a seminormalized block sequence of (e_n) then $(f_n) \succeq (u_n)$. If (u_n) does not contain subsequences equivalent to (f_n) then also $(u_n) \succeq (e_n)$.*

The following lemma is standard.

Lemma 1.9. *Let (x_n) be a block sequence of (e_n), $x_n = \sum_{i=p_n+1}^{p_{n+1}} a_i e_i$. If (y_n) is a basic sequence such that $y_n = \sum_{i=p_n+1}^{p_{n+1}} b_i e_i$, where $|b_i| \leqslant |a_i|$ for all $i \in \mathbb{N}$, then (x_n) is basic and $(x_n) \succeq (y_n)$.*

Proof. Let

$$\gamma_i = \begin{cases} \frac{b_i}{a_i}, & \text{if } a_i \neq 0, \\ 0, & \text{if } a_i = 0. \end{cases}$$

Define an operator $T \in L(d_{w,p})$ by $T\left(\sum_{i=1}^{\infty} c_i e_i\right) = \sum_{i=1}^{\infty} c_i \gamma_i e_i$. Then T is, clearly, linear and, since the basis (e_n) is 1-unconditional, T is bounded with $\|T\| \leqslant 1$. In particular, $T|_{[x_n]}$ is bounded. Also, $T(x_n) = y_n$ for all $n \in \mathbb{N}$, hence $(x_n) \succeq (y_n)$. $\square$

1.4. **Outline of the results.** The purpose of the paper is to uncover the structure of ideals in $L(d_{w,p})$. We show that (some of) these ideals can be arranged into the following diagram.

$$\begin{array}{ccccc} & \nearrow & \mathcal{SS} & \Rightarrow & \\ \{0\} \Rightarrow \mathcal{K} \subsetneq \overline{J^j} \rightarrow \overline{J^{\ell_p}} \wedge \mathcal{SS} & & & & \overline{J^{\ell_p}} \vee \mathcal{SS} \rightarrow \mathcal{SS}_{d_{w,p}} \Rightarrow L(d_{w,p}) \\ & \Rightarrow & \overline{J^{\ell_p}} & \nearrow & \end{array}$$

(the notations will be defined throughout the paper). On this diagram, a single arrow between ideals, $J_1 \longrightarrow J_2$, means that $J_1 \subseteq J_2$. A double arrow between ideals, $J_1 \Longrightarrow J_2$, means that J_2 is the only immediate successor of J_1 (in particular, $J_1 \neq J_2$), whereas a dotted double arrow between ideals, $J_1 \Rightarrow J_2$, only shows that J_2 is an immediate successor for J_1 (in particular, J_1 may have other immediate successors).

While working with the diagram above, we obtain several important characterizations of some ideals in $L(d_{w,p})$. In particular, we show that $\mathcal{FSS}(d_{w,p}) = \mathcal{SS}(d_{w,p})$ (Theorem 3.5). We also characterize the ideal of weakly compact operators (Theorem 3.6) and Dunford-Pettis operators (Theorem 5.7) on $d_{w,p}$. We show in Theorem 4.7 that $\overline{J^j}$ is the only immediate successor of $\mathcal{K}$ under some assumption on the weights w. In the last section of the paper, we show that all strictly singular operators from ℓ_1 to $d_{w,1}$ can be approximated by operators factoring through the formal identity operator $j\colon \ell_1 \to d_{w,1}$ (see Section 4 for the definition). We also obtain a result on factoring positive operators from $\mathcal{SS}(d_{w,p})$ through the formal identity operator (Theorem 6.12).

2. Operators factorable through ℓ_p

Let X and Y be Banach spaces and $T \in L(X)$. We say that T ***factors through*** Y if there are two operators $A \in L(X, Y)$ and $B \in L(Y, X)$ such that $T = BA$.

The following two lemmas are standard. We present their proofs for the sake of completeness.

Lemma 2.1. *Let X and Y be Banach spaces and $T \in L(X,Y)$, $S \in L(Y,X)$ be such that $ST = \mathrm{id}_X$. Then T is an isomorphism and* $\operatorname{Range} T$ *is a complemented subspace of Y isomorphic to X.*

Proof. For all $x \in X$, we have $\|x\| = \|STx\| \leqslant \|S\|\|Tx\|$, so $\|Tx\| \geqslant \frac{1}{\|S\|}\|x\|$. This shows that T is an isomorphism. In particular, $\operatorname{Range} T$ is a closed subspace of Y isomorphic to X.

Put $P = TS \in L(Y)$. Then $P^2 = TSTS = T\mathrm{id}_X S = TS = P$, hence P is a projection. Clearly, $\operatorname{Range} P \subseteq \operatorname{Range} T$. Also, $PT = TST = T$, so $\operatorname{Range} T \subseteq \operatorname{Range} P$. Therefore $\operatorname{Range} P = \operatorname{Range} T$, and $\operatorname{Range} T$ is complemented. □

Lemma 2.2. *Let X and Y be Banach spaces such that Y is isomorphic to $Y \oplus Y$. Then the set $J = \{T \in L(X) : T$ factors through $Y\}$ is an ideal in $L(X)$.*

Proof. It is clear that J is closed under multiplication by operators in $L(X)$. In particular, J is closed under scalar multiplication. Let $A, B \in J$. Write $A = A_1A_2$ and $B = B_1B_2$, where $A_1, B_1 \in L(Y,X)$ and $A_2, B_2 \in L(X,Y)$. Then $A + B = UV$ where $V\colon x \in X \mapsto (A_2x, B_2x) \in Y \oplus Y$ and $U\colon (x,y) \in Y \oplus Y \mapsto A_1x + B_1y \in Y$. Clearly, UV factors through $Y \oplus Y \simeq Y$. Hence $A + B \in J$. □

We will denote the set of all operators in $L(d_{w,p})$ which factor through a Banach space Y by J^Y.

Theorem 2.3. *The sets J^{ℓ_p} and $\overline{J^{\ell_p}}$ are proper ideals in $L(d_{w,p})$.*

Proof. Since ℓ_p is isomorphic to $\ell_p \oplus \ell_p$, it follows from Lemma 2.2 that J^{ℓ_p} is an ideal in $L(d_{w,p})$. Let us show that $J^{\ell_p} \neq L(d_{w,p})$.

Assume that $J^{\ell_p} = L(d_{w,p})$, then the identity operator I on $d_{w,p}$ belongs to J. Write $I = ST$ where $T \in L(d_{w,p}, \ell_p)$ and $S \in L(\ell_p, d_{w,p})$. By Lemma 2.1, the range of T is complemented in ℓ_p and is isomorphic to $d_{w,p}$. This is a contradiction because all complemented infinite-dimensional subspaces of ℓ_p are isomorphic to ℓ_p (see, e.g., [17, Theorem 2.a.3]), while $d_{w,p}$ is not isomorphic to ℓ_p (see [6] for the case $p = 1$ and [14] for the case $1 < p < \infty$; see also [17, p. 176]).

Being the closure of a proper ideal, $\overline{J^{\ell_p}}$ is itself a proper ideal (see, e.g., [11, Corollary VII.2.4]). □

Proposition 2.4. *There exists a projection $P \in L(d_{w,p})$ such that* $\operatorname{Range} P$ *is isomorphic to ℓ_p. For every such P we have $J_P = J^{\ell_p}$.*

Proof. Such projections exist by Remark 1.4. Let $Y = \operatorname{Range} P$, $U\colon Y \to \ell_p$ be an isomorphism onto, and $i\colon Y \to d_{w,p}$ be the inclusion map. It is easy to see that $P = (iU^{-1})(UP)$, hence $P \in J^{\ell_p}$, so that $J_P \subseteq J^{\ell_p}$.

On the other hand, if $T \in J^{\ell_p}$ is arbitrary, $T = AB$ with $A \in L(\ell_p, d_{w,p})$, $B \in L(d_{w,p}, \ell_p)$, then one can write $T = (AUP)P(iU^{-1}B)$, so that $T \in J_P$. Thus $J^{\ell_p} \subseteq J_P$. □

Corollary 2.5. *The ideal $\overline{J^{\ell_p}}$ properly contains the ideal of compact operators $\mathcal{K}(d_{w,p})$.*

Proof. It was already mentioned in the introductory section that compact operators form the smallest closed ideal in $L(d_{w,p})$. Since a projection onto a subspace isomorphic to ℓ_p is not compact, it follows that $\mathcal{K}(d_{w,p}) \neq \overline{J^{\ell_p}}$. □

3. Strictly singular operators

In this section we will study properties of strictly singular operators in $L(d_{w,p})$. Since projections onto the subspaces of $d_{w,p}$ isomorphic to ℓ_p are clearly not strictly singular, it follows from Proposition 2.4 that $\mathcal{SS}(d_{w,p}) \neq J^{\ell_p}$. Moreover, $\mathcal{SS} \neq \overline{J^{\ell_p}} \vee \mathcal{SS}$ and $\overline{J^{\ell_p}} \wedge \mathcal{SS} \neq \overline{J^{\ell_p}}$. So, the ideals we discussed so far can be arranged as follows:

$$\begin{array}{ccccccccc} & & & & & \nearrow & \mathcal{SS} & \overset{\neq}{\searrow} & & & \\ \{0\} & \Longrightarrow & \mathcal{K} & \longrightarrow & \overline{J^{\ell_p}} \wedge \mathcal{SS} & & & & \overline{J^{\ell_p}} \vee \mathcal{SS} & \longrightarrow & L(d_{w,p}) \\ & & & & & \underset{\neq}{\searrow} & \overline{J^{\ell_p}} & \nearrow & & & \end{array}$$

The following theorem shows that there can be no other closed ideals between $\mathcal{SS}$ and $\overline{J^{\ell_p}} \vee \mathcal{SS}$ on this diagram.

Theorem 3.1. *Let $T \in L(d_{w,p})$. If $T \notin \mathcal{SS}(d_{w,p})$ then $J^{\ell_p} \subseteq J_T$.*

Proof. Let $T \notin \mathcal{SS}(d_{w,p})$. Then there exists an infinite-dimensional subspace Y of $d_{w,p}$ such that $T|_Y$ is an isomorphism. By Remark 1.4, passing to a subspace, we may assume that Y is complemented in $d_{w,p}$ and isomorphic to ℓ_p. Let (x_n) be a basis of Y equivalent to the unit vector basis of ℓ_p. Define $z_n = Tx_n$, then (z_n) is also equivalent to the unit vector basis of ℓ_p. By Remark 1.4, (z_n) has a subsequence (z_{n_k}) such that $[z_{n_k}]$ is complemented in $d_{w,p}$ and isomorphic to ℓ_p.

Denote $W = [x_{n_k}]$. Then W and $T(W)$ are both complemented subspaces of $d_{w,p}$ isomorphic to ℓ_p. Let P and Q be projections onto W and $T(W)$, respectively. Put $S = \big(T|_W\big)^{-1}$, $S \in L\big(T(W), d_{w,p}\big)$. Then it is easy to see that $P = (SQ)TP$. Since SQ and P are in $L(d_{w,p})$, we have $J_P \subseteq J_T$. By Proposition 2.4, $J^{\ell_p} \subseteq J_T$. □

Corollary 3.2. *$\overline{J^{\ell_p}} \bigvee \mathcal{SS}(d_{w,p})$ is the only immediate successor of $\mathcal{SS}(d_{w,p})$ and $\overline{J^{\ell_p}}$ is an immediate successor of $\overline{J^{\ell_p}} \wedge \mathcal{SS}(d_{w,p})$.*

Now we will investigate the ideal of finitely strictly singular operators on $d_{w,p}$. To prove the main statement (Theorem 3.5), we will need the following lemma due to Milman [19] (see also a thorough discussion in [22]). This lemma will be used more than once in the paper.

Lemma 3.3 ([19])**.** *If F is a k-dimensional subspace of c_0 then there exists a vector $x \in F$ such that x attains its sup-norm at at least k coordinates (that is, x^* starts with a constant block of length k).*

We will also use the following simple lemma.

Lemma 3.4. *Let $s_n = \sum_{i=1}^n w_i$ ($n \in \mathbb{N}$) where $w = (w_i)$ is the sequence of weights for $d_{w,p}$. If $x \in d_{w,p}$, $y = x^*$, and $N \in \mathbb{N}$ then $0 \leqslant y_N \leqslant \frac{\|x\|}{s_N^{1/p}}$.*

Proof. $\|x\|^p = \|y\|^p = \sum_{i=1}^\infty y_i^p w_i \geqslant y_N^p \sum_{i=1}^N w_i = y_N^p s_N$. ☐

Theorem 3.5. *Let X and Y be subspaces of $d_{w,p}$. Then $\mathcal{FSS}(X,Y) = \mathcal{SS}(X,Y)$. In particular, $\mathcal{FSS}(\ell_p, d_{w,p}) = \mathcal{SS}(\ell_p, d_{w,p})$ and $\mathcal{FSS}(d_{w,p}) = \mathcal{SS}(d_{w,p})$.*

Proof. Let $T \in L(X,Y)$. Suppose that T is not finitely strictly singular. We will show that it is not strictly singular. Since T is not finitely strictly singular, there exists a constant $c > 0$ and a sequence F_n of subspaces of X with $\dim F_n \geqslant n$ such that for each n and for all $x \in F_n$ we have $\|Tx\| \geqslant c\|x\|$.

Fix a sequence (ε_k) in $\mathbb{R}$ such that $1 > \varepsilon_k \downarrow 0$. We will inductively construct a sequence (x_k) in X and two strictly increasing sequences $(n_k), (m_k)$ in $\mathbb{N}$ such that:

(i) (x_k) and (Tx_k) are seminormalized; we will denote Tx_k by u_k;
(ii) for all $k \in \mathbb{N}$, $\operatorname{supp} x_k \subseteq [n_k, \infty)$ and $\operatorname{supp} u_k \subseteq [m_k, \infty)$;
(iii) if $k \geqslant 2$ then $\|x_{k-1}|_{[n_k,\infty)}\| < \varepsilon_k$, $\|u_{k-1}|_{[m_k,\infty)}\| < \varepsilon_k$, and all the coordinates of u_{k-1} where the sup-norm is attained are less than m_k;
(iv) for each $k \in \mathbb{N}$, the vector u_k^* begins with a constant block of length at least k.

That is, (x_n) and (u_n) are two almost disjoint sequences and u_n's have long "flat" sections.

Take x_1 to be any nonzero vector in F_1 and put $n_1 = m_1 = 1$. Suppose we have already constructed $x_1, \dots, x_{k-1}$, $n_1, \dots, n_{k-1}$, and $m_1, \dots, m_{k-1}$ such that the conditions (i)–(iv) are satisfied. Choose $n_k \in \mathbb{N}$ and $m_k \in \mathbb{N}$ such that $n_k > n_{k-1}$, $m_k > m_{k-1}$ and the condition (iii) is satisfied.

Consider the space

$$V=\big\{y=(y_i)\in F_{n_k+m_k+k}\ :\ y_i=0 \text{ for } i<n_k\big\}\subseteq F_{n_k+m_k+k}.$$

It follows from $\dim F_{n_k+m_k+k}\geqslant n_k+m_k+k$ that $\dim V\geqslant m_k+k$. Since $V\subseteq F_{n_k+m_k+k}$, $\|Ty\|\geqslant c\|y\|$ for all $y\in V$. In particular, $\dim(TV)\geqslant m_k+k$. Define

$$Z=\big\{z=(z_i)\in TV\ :\ z_i=0 \text{ for } i<m_k\big\}.$$

It follows that $\dim Z\geqslant k$.

Clearly, $\operatorname{supp} y\subseteq[n_k,\infty)$ for all $y\in V$ and $\operatorname{supp} z\subseteq[m_k,\infty)$ for all $z\in Z$. By Lemma 3.3, we can choose $u_k\in Z$ such that u_k is normalized and u_k^* starts with a constant block of length k. Put $x_k=(T|_V)^{-1}(u_k)\in Y$. Since $x_k\in V$ and $\|u_k\|=1$, it follows that $\frac{1}{\|T\|}\leqslant\|x_k\|\leqslant\frac{1}{c}$, so the conditions (i)–(iv) are satisfied for (x_k).

For each $k\in\mathbb{N}$, let $x_k'=x_k|_{[n_k,n_{k+1})}$ and $u_k'=u_k|_{[m_k,m_{k+1})}$. Passing to tails of sequences, if necessary, we may assume that both (x_k') and (u_k') are seminormalized block sequences of (e_n).

Since the non-increasing rearrangement of each u_k' starts with a constant block of length k by (iii), the coefficients in u_k' converge to zero by Lemma 3.4. Therefore, passing to a subsequence, we may assume by Remark 1.4 that (u_k') is equivalent to the unit vector basis (f_n) of ℓ_p. Using Theorem 1.2 and passing to a further subsequence, we may also assume that $(x_k)\sim(x_k')$ and $(u_k)\sim(u_k')$.

By Proposition 1.8, the sequence (x_k') is dominated by (f_n). Notice that the condition $u_k=Tx_k$ implies $(x_k)\succeq(u_k)$. Therefore, we get the following chain of dominations and equivalences of basic sequences:

$$(f_n)\succeq(x_k')\sim(x_k)\succeq(u_k)\sim(u_k')\sim(f_n).$$

It follows that all the dominations in this chain are, actually, equivalences. In particular, $(x_k)\sim(u_k)$. Thus, T is an isomorphism on the space $[x_k]$, hence T is not strictly singular. □

Recall that an operator T on a Banach space X is weakly compact if the image of the unit ball of X under T is relatively weakly compact. Alternatively, T is weakly compact if and only if for every bounded sequence (x_n) in X there exists a subsequence (x_{n_k}) of (x_n) such that (Tx_{n_k}) is weakly convergent.

If $1<p<\infty$ then $d_{w,p}$ is reflexive, and, hence, every operator in $L(d_{w,p})$ is weakly compact. In case $p=1$ we have the following.

Theorem 3.6. *Let $T\in L(d_{w,1})$. Then T is weakly compact if and only if T is strictly singular.*

Proof. Suppose that T is strictly singular. We will show that T is weakly compact.

Let (x_n) be a bounded sequence in X. By Rosenthal's ℓ_1-theorem, there is a subsequence (x_{n_k}) of (x_n) such that (x_{n_k}) is either equivalent to the unit vector basis (f_n) of ℓ_1 or is weakly Cauchy. In the latter case, (Tx_{n_k}) is also weakly Cauchy. If $(x_{n_k}) \sim (f_n)$ then, since T is strictly singular, (Tx_{n_k}) cannot have subsequences equivalent to (f_n). Hence, using Rosenthal's theorem one more time and passing to a further subsequence, we may assume that, again, (Tx_{n_k}) is weakly Cauchy. Since $d_{w,1}$ is weakly sequentially complete, the sequence (Tx_{n_k}) is weakly convergent. It follows that T is weakly compact.

Conversely, let J be the closed ideal of weakly compact operators in $L(d_{w,1})$. By the first part of the proof, J is a successor of $\mathcal{SS}(d_{w,1})$. Suppose that $J \neq \mathcal{SS}(d_{w,1})$. By Theorem 3.1, $J^{\ell_1} \subseteq J$. This, however, is a contradiction since a projection onto a copy of ℓ_1 (which belongs to J^{ℓ_1} by Proposition 2.4) is not weakly compact. □

4. Operators factorable through the formal identity

The operator $j\colon \ell_p \to d_{w,p}$ defined by $j(e_n) = f_n$ is called ***the formal identity operator from*** ℓ_p ***to*** $d_{w,p}$. It follows immediately from the definition of the norm in $d_{w,p}$ that $\|j\| = 1$.

We will denote by the symbol J^j the set of all operators $T \in L(d_{w,p})$ which can be factored as $T = AjB$ where $A \in L(d_{w,p})$ and $B \in L(d_{w,p}, \ell_p)$.

Proposition 4.1. *J^j is an ideal in $L(d_{w,p})$.*

Proof. It is clear from the definition that the set J^j is closed under both right and left multiplication by operators from $L(d_{w,p})$. We have to show that if T_1 and T_2 are in J^j then $T_1 + T_2$ is in J^j, as well.

Write $T_1 = A_1 j B_1$, $T_2 = A_2 j B_2$ with $A_1, A_2 \in L(d_{w,p})$ and $B_1, B_2 \in L(d_{w,p}, \ell_p)$. Let $A \in L(d_{w,p} \oplus d_{w,p}, d_{w,p})$ and $B \in L(d_{w,p}, \ell_p \oplus \ell_p)$ be defined by

$$A(x_1, x_2) = A_1 x_1 + A_2 x_2 \quad \text{and} \quad Bx = (B_1 x, B_2 x).$$

Define also $U\colon \ell_p \to \ell_p \oplus \ell_p$ and $V\colon d_{w,p} \to d_{w,p} \oplus d_{w,p}$ by

$$U\big((x_n)\big) = \big((x_{2n-1}), (x_{2n})\big), \quad \text{and} \quad V\big((x_n)\big) = \big((x_{2n-1}), (x_{2n})\big).$$

Since the bases of ℓ_p and $d_{w,p}$ are both unconditional, U and V are bounded.

Now observe that for each $x = (x_n) \in d_{w,p}$ we can write

$$AVjU^{-1}Bx = AVjU^{-1}(B_1 x, B_2 x) =$$
$$A(jB_1 x, jB_2 x) = A_1 j B_1 x + A_2 j B_2 x = T_1 x + T_2 x.$$

This shows that $T_1+T_2 = AVjU^{-1}B$ with $AV \in L(d_{w,p})$ and $U^{-1}B \in L(d_{w,p},\ell_p)$, hence $T_1 + T_2 \in J^j$. $\square$

As we already mentioned before, the space $d_{w,p}$ contains many complemented copies of ℓ_p. Consider the operator $jUP \in L(d_{w,p})$ where P is a projection onto any subspace Y isomorphic to ℓ_p and $U\colon Y \to \ell_p$ is an isomorphism onto. It turns out that the ideal generated by any such operator does not depend on the choice of Y and, in fact, coincides with J^j.

Proposition 4.2. *Let Y be a complemented subspace of $d_{w,p}$ isomorphic to ℓ_p, $P \in L(d_{w,p})$ be a projection with range Y, and $U\colon Y \to \ell_p$ be an isomorphism onto. If $T = jUP$ then $J_T = J^j$.*

Proof. Clearly, $J_T \subseteq J^j$. Let $S \in J^j$. Then $S = AjB$ where $A \in L(d_{w,p})$ and $B \in L(d_{w,p},\ell_p)$. It follows that

$$S = AjB = Aj(UPU^{-1})B = AT(U^{-1}B) \in J_T.$$

$\square$

The next goal is to show that the ideal $\overline{J^j}$ "sits" between $\mathcal{K}(X)$ and $\mathcal{SS}(X) \wedge \overline{J^{\ell_p}}$.

Theorem 4.3. *The formal identity operator $j\colon \ell_p \to d_{w,p}$ is finitely strictly singular.*

Proof. Let $\varepsilon > 0$ be arbitrary. Take $n \in \mathbb{N}$ such that $\frac{1}{n}\sum_{i=1}^n w_i < \varepsilon$; such n exists by $w_n \to 0$. Since (w_n) is also a decreasing sequence, it follows that $w_i < \varepsilon$ for all $i \geqslant n$.

Let $Y \subseteq \ell_p$ be a subspace with $\dim Y \geqslant n$. By Lemma 3.3, there exists a vector $x \in Y$ such that $\|x\|_{\ell_p} = 1$ and x attains its sup-norm at at least n coordinates. Denote $\delta = \|x\|_{\sup} > 0$. Then $\|x\|_{\ell_p} \geqslant n^{1/p}\delta$, so $\delta \leqslant n^{-1/p}$.

Observe that the non-increasing rearrangement x^* of x satisfies the condition that $x^*_i = \delta$ for all $1 \leqslant 1 \leqslant n$. Therefore

$$\|jx\|^p_{d_{w,p}} = \sum_{i=1}^{\infty} x_i^{*p} w_i \leqslant \delta^p \sum_{i=1}^{n} w_i + \varepsilon \sum_{i=n+1}^{\infty} x_i^{*p} \leqslant \delta^p n\varepsilon + \varepsilon\|x\|^p_{\ell_p} \leqslant 2\varepsilon.$$

Hence $\|jx\|_{d_{w,p}} \leqslant (2\varepsilon)^{1/p}$. $\square$

Corollary 4.4. *The following inclusions hold: $\mathcal{K}(d_{w,p}) \subsetneq \overline{J^j}$ and $J^j \subseteq \mathcal{SS}(d_{w,p}) \wedge J^{\ell_p}$.*

Proof. Let Y, P, and U be as in Proposition 4.2. Then $jUP \in J^j$. If $x_n = U^{-1}f_n \in d_{w,p}$ then (x_n) is seminormalized and $jUPx_n = e_n$. Hence the sequence $(jUPx_n)$ has no convergent subsequences, so that jUP is not compact.

The inclusion $J^j \subseteq \mathcal{SS}(d_{w,p}) \wedge J^{\ell_p}$ is obvious since j is strictly singular. $\square$

Conjecture 4.5. *The ideal $\overline{J^j}$ is the only immediate successor of $\mathcal{K}(d_{w,p})$.*

In [3] and [10] (see also [17]), conditions on the weights $w = (w_n)$ are given under which $d_{w,p}$ has exactly two non-equivalent symmetric basic sequences. We will show that the conjecture holds true in this case.

Lemma 4.6. *If $T \in \mathcal{SS}(d_{w,p})\backslash\mathcal{K}(d_{w,p})$ then there exists a seminormalized basic sequence (x_n) in $d_{w,p}$ such that $(f_n) \succeq (x_n)$ and (Tx_n) is weakly null and seminormalized.*

Proof. Let (z_n) be a bounded sequence in $d_{w,p}$ such that (Tz_n) has no convergent subsequences. Then (z_n) has no convergent subsequences either. Applying Rosenthal's ℓ_1-theorem and passing to a subsequence, we may assume that (z_n) is either equivalent to the unit vector basis of ℓ_1 or is weakly Cauchy.

Case: (z_n) is equivalent to the unit vector basis of ℓ_1. Since a reflexive space cannot contain a copy of ℓ_1, we conclude that $p = 1$, so $(z_n) \sim (f_n)$. Again, by Rosenthal's theorem, (Tz_n) has a subsequence which is either equivalent to (f_n) or is weakly Cauchy. If $(Tz_{n_k}) \sim (f_n)$ then T is an isomorphism on the space $[z_{n_k}]$, contrary to the assumption that $T \in \mathcal{SS}(d_{w,p})$. Therefore, (Tz_{n_k}) is weakly Cauchy. Put $x_k = z_{n_{2k}} - z_{n_{2k-1}}$. Then (x_k) is basic and (Tx_k) is weakly null. Passing to a further subsequence of (z_{n_k}) we may assume that (Tx_k) is seminormalized. Also, (x_k) is still equivalent to (f_n), hence is dominated by (f_n).

Case: (z_n) is weakly Cauchy. Clearly, (Tz_n) is also weakly Cauchy. Consider the sequence (u_n) in $d_{w,p}$ defined by $u_n = z_{2n} - z_{2n-1}$. Then both (u_n) and (Tu_n) are weakly null. Passing to a subsequence of (z_n), we may assume that (Tu_n) and, hence, (u_n) are seminormalized. Applying Theorem 1.3, we get a subsequence (u_{n_k}) of (u_n) which is basic and equivalent to a block sequence (v_n) of (e_n). Denote $x_k = u_{n_k}$. By Proposition 1.8, (f_n) dominates (v_n) and, hence, (x_k). $\square$

Theorem 4.7. *If $d_{w,p}$ has exactly two non-equivalent symmetric basic sequences, then $\overline{J^j}$ is the only immediate successor of $\mathcal{K}(d_{w,p})$.*

Proof. Let T be a non-compact operator on $d_{w,p}$. It suffices to show that $J^j \subseteq J_T$. We may assume that T is strictly singular because, otherwise, we have $J^j \subseteq J^{\ell_p} \subseteq J_T$ by Theorem 3.1.

Let (x_n) be a sequence as in Lemma 4.6. Passing to a subsequence and using Theorem 1.3, we may assume that (Tx_n) is basic and equivalent to a block sequence (h_n) of (e_n) such that $Tx_n - h_n \to 0$. We claim that (h_n) has no subsequences equivalent to (f_n). Indeed, otherwise, for such a subsequence (h_{n_k}) of (h_n), we would have

$(f_n) \sim (f_{n_k}) \succeq (x_{n_k}) \succeq (Tx_{n_k}) \sim (h_{n_k}) \sim (f_n)$, so $(x_{n_k}) \sim (Tx_{n_k})$, contrary to $T \in \mathcal{SS}(d_{w,p})$. By [10, Theorem 19], (h_n) has a subsequence which spans a complemented subspace in $d_{w,p}$ and is equivalent to (e_n). Therefore, by Theorem 1.2, we may assume (by passing to a further subsequence) that $(Tx_n) \sim (e_n)$ and $[Tx_n]$ is complemented in $d_{w,p}$.

We have proved that there exists a sequence (x_n) in $d_{w,p}$ such that $[Tx_n]$ is complemented in $d_{w,p}$ and

$$(f_n) \succeq (x_n) \succeq (Tx_n) \sim (e_n).$$

Let $A \in L(\ell_p, d_{w,p})$ and $B \in L([Tx_n], d_{w,p})$ be defined by $Af_n = x_n$ and $B(Tx_n) = e_n$. Let $Q \in L(d_{w,p})$ be a projection onto $[Tx_n]$. Then for all $n \in \mathbb{N}$, we obtain: $BQTAf_n = BQTx_n = BTx_n = e_n$. It follows that $BQTA = j$, so that $J^j \subseteq J_T$. □

In order to prove Conjecture 4.5 without additional conditions on w, it suffices to show that if $T \in \overline{J^j} \setminus \mathcal{K}(d_{w,p})$ then $J^j \subseteq \overline{J_T}$. We will prove a weaker statement: if $T \in J^j \setminus \mathcal{K}(d_{w,p})$ then $J^j \subseteq J_T$.

Recall (see [3, p.148]) that if $x = (a_n) \in d_{w,p}$ then a block sequence (y_n) of (e_n) is called a ***block of type I generated by*** x if it is of the form $y_n = \sum_{i=p_n+1}^{p_{n+1}} a_{i-p_n} e_i$ for all n. A set $A \subseteq d_{w,p}$ will be said to be ***almost lengthwise bounded*** if for each $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that $\|x^*|_{[N,\infty)}\| < \varepsilon$ for all $x \in A$. We will usually use it in the case when $A = \{x_n\}$ for some sequence (x_n) in $d_{w,p}$. We need the following result, which is a slight extension of [3, Theorem 3]. We include the proof for completeness.

Theorem 4.8. *Let (x_n) be a seminormalized block sequence of (e_n) in $d_{w,p}$.*

(i) *If (x_n) is not almost lengthwise bounded then there exists a subsequence (x_{n_k}) such that $(x_{n_k}) \sim (f_n)$.*

(ii) *If (x_n) is almost lengthwise bounded, then there exists a subsequence (x_{n_k}) equivalent to a block of type I generated by a vector $u = \sum_{i=1}^{\infty} b_i e_i \in d_{w,p}$ with $b_i \downarrow 0$. Moreover, if the sequence (x_n) is bounded in ℓ_p then u is in ℓ_p.*[1]

Proof. (i) Without loss of generality, $\|x_n\| \leqslant 1$ for all $n \in \mathbb{N}$. By the assumption, there exists $\varepsilon > 0$ with the property that for each $k \in \mathbb{N}$, there is $n_k \in \mathbb{N}$ such that $\|x^*_{n_k}|_{(k,\infty)}\| \geqslant \varepsilon$. Let u_k be a restriction of x_{n_k} such that $u^*_k = x^*_{n_k}|_{[1,k]}$ and $v_k = x_{n_k} - u_k$.

Clearly, each nonzero entry of u_k is greater than or equal to the greatest entry of v_k. By Lemma 3.4, the k-th coordinate of u^*_k is less than or equal to $\frac{1}{s_k^{1/p}}$ where $s_k = \sum_{i=1}^{k} w_i$.

[1]As a sequence space, ℓ_p is a subset of $d_{w,p}$. That is, we can identify ℓ_p with Range j. More precisely, we claim here that if $(j^{-1}x_n)$ is bounded in ℓ_p then u is in Range j. Being a block sequence of (e_n), (x_n) is contained in Range j.

It follows that (v_k) is a block sequence of (e_n) such that $\varepsilon \leqslant \|v_k\| \leqslant 1$ and absolute values of the entries of v_k are all at most $\frac{1}{s_k^{1/p}}$. Since $\lim_k s_k = +\infty$ by the definition of $d_{w,p}$, passing to a subsequence and using Remark 1.4 we may assume that (v_k) is equivalent to (f_n). By Proposition 1.8, (f_n) dominates (x_{n_k}). Using also Lemma 1.9, we obtain the following diagram:

$$(f_n) \succeq (x_{n_k}) \succeq (v_k) \sim (f_n).$$

Hence (x_{n_k}) is equivalent to (f_n).

(ii) Suppose that $x_n = \sum_{i=p_n+1}^{p_{n+1}} a_i e_i$. Clearly, the sequence (a_i) is bounded. Without loss of generality, $a_{p_n+1} \geqslant \dots \geqslant a_{p_{n+1}} \geqslant 0$ for each n. Put $y_n = x_n^*$. Using a standard diagonalization argument and passing to a subsequence, we may assume that (y_n) converges coordinate-wise; put $b_i = \lim\limits_{n\to\infty} y_{n,i}$. It is easy to see that $b_i \geqslant b_{i+1}$ for all i. Put $u = (b_i)$.

Case: the sequence $(p_{n+1} - p_n)$ *is bounded.* Passing to a subsequence, we may assume that $N := p_{n_k+1} - p_{n_k}$ is a constant. Note that $\operatorname{supp} u \subseteq [1, N]$ and $\operatorname{supp} y_{n_k} \subseteq [1, N]$ for all k. Put $u_k = \sum_{i=p_{n_k}+1}^{p_{n_k+1}} b_{i-p_{n_k}} e_i$, then $u = u_k^*$ and (u_k) as a block of type I generated by u. By compactness, $\|x_{n_k} - u_k\| = \|y_{n_k} - u\| \to 0$. Therefore, passing to a further subsequence, we have $(x_{n_k}) \sim (u_k)$. Being a vector with finite support, u belongs to ℓ_p.

Case: the sequence $(p_{n+1} - p_n)$ *is unbounded.* We will construct the required subsequence (x_{n_k}) and a sequence (N_k) inductively. Put $n_1 = N_1 = 1$ and let $k > 1$. Suppose that $n_1, \dots, n_{k-1}$ and $N_1, \dots, N_{k-1}$ have already been selected. Since (x_n) is almost lengthwise bounded, we can find $N_k > N_{k-1}$ such that $\|y_n|_{(N_k,\infty)}\| < \frac{1}{k}$ for all n. Put $v_k = u|_{[1,N_k]}$. Using coordinate-wise convergence, we can find $n_k > n_{k-1}$ such that $\|y_{n_k}|_{[1,N_k]} - v_k\|_{\ell_p} < \frac{1}{k}$ and $p_{n_k} + N_k \leqslant p_{n_k+1}$. Put $u_k = \sum_{i=p_{n_k}+1}^{p_{n_k}+N_k} b_{i-p_{n_k}} e_i$. Then $u_k^* = v_k$, so that

$$\|x_{n_k}|_{(p_{n_k}, p_{n_k}+N_k]} - u_k\|_{\ell_p} = \|y_{n_k}|_{[1,N_k]} - v_k\|_{\ell_p} < \tfrac{1}{k} \tag{1}$$

and

$$\|x_{n_k}|_{(p_{n_k}+N_k, p_{n_k+1}]}\| = \|y_{n_k}|_{(N_k,\infty)}\| < \tfrac{1}{k}.$$

It follows that $\|x_{n_k} - u_k\| \to 0$. Passing to a subsequence, we get $(x_{n_k}) \sim (u_k)$.

Next, we show that $u \in d_{w,p}$. Since $\|\cdot\| \leqslant \|\cdot\|_{\ell_p}$, it follows from (1) that

$$\|v_k\| = \|u_k\| \leqslant \|x_{n_k}|_{(p_{n_k}, p_{n_k}+N_k]}\| + \tfrac{1}{k} \leqslant \|x_{n_k}\| + \tfrac{1}{k}.$$

Since (x_n) is bounded, so is (v_k). Since $\operatorname{supp} v_k = N_k \to \infty$, we have $u \in d_{w,p}$. For the "moreover" part, we argue in a similar way. By (1), we have

$$\|v_k\|_{\ell_p} \leqslant \|u_k\|_{\ell_p} \leqslant \|x_{n_k}|_{(p_{n_k}, p_{n_k}+N_k]}\|_{\ell_p} + \tfrac{1}{k} \leqslant \|x_{n_k}\|_{\ell_p} + \tfrac{1}{k}.$$

Therefore, if (x_n) is bounded in ℓ_p then so is (v_k), hence $u \in \ell_p$. □

Lemma 4.9. *Suppose that (u_n) is a block of type I in $d_{w,p}$ generated by some $u = \sum_{i=1}^{\infty} b_i e_i$. If $b_i \downarrow 0$ and $u \in \ell_p$ then (u_n) has a subsequence equivalent to (e_n)*

Proof. By Corollary 4 of [3], we may assume that the basic sequence (u_n) is symmetric. It suffices to show that $[u_n]$ is isomorphic to $d_{w,p}$ because all symmetric bases in $d_{w,p}$ are equivalent; see e.g., Theorem 4 of [3]. Without loss of generality, $\|u\| = 1$. Lemma 4 of [3] asserts that $[u_n]$ is isomorphic to $d_{w,p}$ iff $(s_n^{(u)}) \sim (s_n)$, where $s_n = \sum_{i=1}^{n} w_i$, $s_n^{(u)} = \sum_{i=1}^{\infty} b_i^p (s_{ni} - s_{n(i-1)})$, and $(\alpha_n) \sim (\beta_n)$ means that there exist positive constants A and B such that $A\alpha_n \leqslant \beta_n \leqslant B\alpha_n$ for all n. Let's verify that this condition is, indeed, satisfied. On one hand, taking only the first term in the definition of $s_n^{(u)}$, we get $s_n^{(u)} \geqslant b_1^p s_n$. On the other hand, it follows from $w_i \downarrow$ that $s_{ni} - s_{n(i-1)} \leqslant s_n$ for every i, hence $s_n^{(u)} \leqslant \sum_{i=1}^{\infty} b_i^p s_n = \|u\|_{\ell_p}^p s_n$. □

Lemma 4.10. *Let (x_n) be a block sequence of (f_n) in ℓ_p such that the sequences (x_n) and (jx_n) are seminormalized in ℓ_p and $d_{w,p}$, respectively. Then there exists a subsequence (x_{n_k}) such that $(jx_{n_k}) \sim (e_n)$.*

Proof. Clearly, $(x_n) \sim (f_n)$. It follows that $(jx_n) \not\sim (f_n)$ because, otherwise, j would be an isomorphism on $[x_n]$, which is impossible because j is strictly singular by Theorem 4.3. Applying Theorem 4.8 to (jx_n) and passing to a subsequence, we may assume that $(jx_n) \sim (u_n)$, where (u_n) is a block of type I generated by some $u = \sum_{i=1}^{\infty} b_i e_i$ such that $b_i \downarrow 0$ and $u \in \ell_p$. Applying Lemma 4.9 and passing to a subsequence, we get $(u_n) \sim (e_n)$. □

Theorem 4.11. *If $T \in J^j \setminus \mathcal{K}(d_{w,p})$ then $J^j \subseteq J_T$.*

Proof. Write $T = AjB$ where $B\colon d_{w,p} \to \ell_p$ and $A\colon d_{w,p} \to d_{w,p}$. Let (x_n) be as in Lemma 4.6. The sequence (Bx_n) is bounded, hence we may assume by passing to a subsequence that it converges coordinate-wise. Since (Tx_n) is weakly null and seminormalized, it has no convergent subsequences. It follows that, after passing to a subsequence of (x_n), we may assume that (Tz_n) is seminormalized, where $z_n = x_{2n} - x_{2n-1}$. In particular, (z_n), (Bz_n), and (jBz_n) are seminormalized. Also, (Bz_n) converges to zero coordinate-wise. Using Theorem 1.3 and passing to a further subsequence, we may assume that (Bz_n) is equivalent to a block sequence (u_n) of (f_n) and $Bz_n - u_n \to 0$. It follows from $(f_n) \succeq (x_n)$ that $(f_n) \succeq (z_n) \succeq (Bz_n) \sim (u_n) \sim (f_n)$. In particular, $(z_n) \sim (f_n)$.

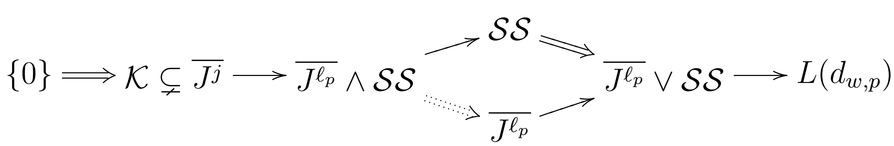
Since $Bz_n - u_n \to 0$ and (jBz_n) is seminormalized, we may assume that the sequence (ju_n) is seminormalized. By Lemma 4.10, passing to a further subsequence, we may assume that (ju_n) and, hence, (jBz_n) are equivalent to (e_n).

Passing to a subsequence and using Theorem 1.3, we may assume that (Tz_n) is equivalent to a block sequence (v_n) of (e_n) such that $Tz_n - v_n \to 0$. Since $T \in \mathcal{SS}(d_{w,p})$, no subsequence of (Tz_n) and, therefore, of (v_n) is equivalent to (f_n). By Proposition 1.8, $(v_n) \succeq (e_n)$. It follows from $(jBz_n) \sim (e_n)$ that $(e_n) \succeq (Tz_n)$, hence $(Tz_n) \sim (e_n) \sim (v_n)$.

Write $v_n = \sum_{i=p_n+1}^{p_{n+1}} a_n e_n$. By Remark 1.4, $a_n \not\to 0$. Hence, passing to a subsequence and using [10, Remark 9], we may assume that $[v_n]$ is complemented. By Theorem 1.3, we may assume that $[Tz_n]$ is complemented. Let $P \in L(d_{w,p})$ be a projection onto $[Tz_n]$ and $U \in L(\ell_p, d_{w,p})$ and $V \in L([Tz_n], d_{w,p})$ be defined by $Uf_n = z_n$ and $VTz_n = e_n$. Then we can write $j = VPTU$. Therefore $J^j \subseteq J_T$. □

5. $d_{w,p}$-STRICTLY SINGULAR OPERATORS

The ideals in $L(d_{w,p})$ we have obtained so far can be arranged into the following diagram.



(see the Introduction for the notations). In this section, we will characterize the greatest ideal in the algebra $L(d_{w,p})$, that is, a proper ideal in $L(d_{w,p})$ that contains all other proper ideals in $L(d_{w,p})$.

If X and Y are two Banach spaces, then an operator $T \in L(X)$ is called ***Y-strictly singular*** if for any subspace Z of X isomorphic to Y, the restriction $T|_Z$ is not an isomorphism. The set of all Y-strictly singular operators in $L(d_{w,p})$ will be denoted by $\mathcal{SS}_Y$.

According to this notation, the symbol $\mathcal{SS}_{d_{w,p}}$ stands for the set of all $d_{w,p}$-strictly singular operators in $L(d_{w,p})$ (not to be confused with $\mathcal{SS}(d_{w,p})$).

Lemma 5.1. *Suppose that $T \in \mathcal{SS}_{d_{w,p}}$ and (x_n) is a basic sequence in $d_{w,p}$ equivalent to the unit vector basis (e_n). Then $Tx_n \to 0$.*

Proof. Suppose, by way of contradiction, that $Tx_n \not\to 0$. Then there is a subsequence (x_{n_k}) such that (Tx_{n_k}) is seminormalized. Since (x_n) is weakly null (Remark 1.7), we may assume by using Theorem 1.3 and passing to a further subsequence that (Tx_{n_k}) is a basic sequence equivalent to a block sequence (z_k) of (e_n).

By Proposition 1.8, either (z_k) has a subsequence equivalent to (f_n) or $(z_k) \succeq (e_n)$. Since (Tx_{n_k}) cannot have subsequences equivalent to (f_n) (this would contradict boundedness of T), the former is impossible. Therefore $(z_k) \succeq (e_n)$. We obtain the following diagram:

$$(e_n) \sim (x_{n_k}) \succeq (Tx_{n_k}) \sim (z_k) \succeq (e_n).$$

Therefore $T|_{[x_{n_k}]}$ is an isomorphism. This contradicts T being in $\mathcal{SS}_{d_{w,p}}$. □

Corollary 5.2. *Let $T \in \mathcal{SS}_{d_{w,p}}$. If $Y \subseteq d_{w,p}$ is a subspace isomorphic to $d_{w,p}$ then there is a subspace $Z \subseteq Y$ such that Z is isomorphic to $d_{w,p}$ and $T|_Z$ is compact.*

Proof. Let (x_n) be a basis of Y equivalent to (e_n). By Lemma 5.1, $Tx_n \to 0$. There is a subsequence (x_{n_k}) of (x_n) such that $\sum_{k=1}^{\infty} \frac{\|Tx_{n_k}\|}{\|x_{n_k}\|}$ is convergent. Let $Z = [x_{n_k}]$. It follows that Z is isomorphic to $d_{w,p}$ and $T|_Z$ is compact (see, e.g., [8, Lemma 5.4.10]). □

Theorem 5.3. *The set $\mathcal{SS}_{d_{w,p}}$ of all $d_{w,p}$-strictly singular operators in $L(d_{w,p})$ is the greatest proper ideal in the algebra $L(d_{w,p})$. In particular, $\mathcal{SS}_{d_{w,p}}$ is closed.*

Proof. First, let us show that $\mathcal{SS}_{d_{w,p}}$ is an ideal. Let $T \in \mathcal{SS}_{d_{w,p}}$. If $A \in L(d_{w,p})$ then, trivially, $AT \in \mathcal{SS}_{d_{w,p}}$. If $TA \notin \mathcal{SS}_{d_{w,p}}$ then there exists a subspace Y of $d_{w,p}$ such that Y and $TA(Y)$ are both isomorphic to $d_{w,p}$. Then $A|_Y$ is bounded below, hence $A(Y)$ is isomorphic to $d_{w,p}$. It follows that T is an isomorphism on a copy of $d_{w,p}$, contrary to $T \in \mathcal{SS}_{d_{w,p}}$. So, $\mathcal{SS}_{d_{w,p}}$ is closed under two-sided multiplication by bounded operators.

Let $T, S \in \mathcal{SS}_{d_{w,p}}$. We will show that $T + S \in \mathcal{SS}_{d_{w,p}}$. Let Y be a subspace of $d_{w,p}$ isomorphic to $d_{w,p}$. By Corolary 5.2, there exists a subspace Z of Y such that Z is isomorphic to $d_{w,p}$ and $T|_Z$ is compact. Applying Corolary 5.2 again, we can find a subspace V of Z such that V is isomorphic to $d_{w,p}$ and $S|_V$ is compact. Therefore $(T + S)|_V$ is compact, so that $(T + S)|_Y$ is not an isomorphism. So, $\mathcal{SS}_{d_{w,p}}$ is an ideal.

Clearly, the identity operator I does not belong to $\mathcal{SS}_{d_{w,p}}$, so $\mathcal{SS}_{d_{w,p}}$ is proper. Let us show that $\mathcal{SS}_{d_{w,p}}$ is the greatest ideal in $L(d_{w,p})$.

Let $T \notin \mathcal{SS}_{d_{w,p}}$. Then there exists a subspace Y of $d_{w,p}$ such that Y and $T(Y)$ are isomorphic to $d_{w,p}$. By [10, Corollary 12], there exists a complemented (in $d_{w,p}$) subspace Z of $T(Y)$ such that Z is isomorphic to $d_{w,p}$. Let $P \in L(d_{w,p})$ be a projection onto Z. Put $H = T^{-1}(Z)$. It follows that H is isomorphic to $d_{w,p}$. Let $U\colon d_{w,p} \to H$ and $V\colon Z \to d_{w,p}$ be surjective isomorphisms. Then $S \in L(d_{w,p})$ defined by $S = (VP)TU$ is an invertible operator. Clearly $S \in J_T$, hence $J_T = L(X)$.

The fact that $\mathcal{SS}_{d_{w,p}}$ is closed follows from [11, Corollary VII.2.4]. $\square$

The next theorem provides a convenient characterization of $d_{w,p}$-strictly singular operators.

Lemma 5.4. *Let $T \in L(d_{w,p})$ be such that $Te_n \to 0$. Suppose that (x_n) is a bounded block sequence of (e_n) in $d_{w,p}$ such that (x_n) is almost lengthwise bounded. Then $Tx_n \to 0$.*

Proof. Write $x_n = \sum_{i=p_n+1}^{p_{n+1}} a_i e_i$. Since (x_n) is bounded, there is $C > 0$ such that $|a_i| \leqslant C$ for all i and $n \in \mathbb{N}$. Let $\varepsilon > 0$. Find $N \in \mathbb{N}$ such that $\|x_n^*|_{[N,\infty)}\| < \varepsilon$ for all $n \in \mathbb{N}$. Let u_n be a restriction of x_n such that $u_n^* = x_n^*|_{[1,N)}$ and $v_n = x_n - u_n$. It is clear that $\|v_n\| = \|x_n^*|_{[N,\infty)}\| < \varepsilon$. Also, $\|Tu_n\| \leqslant NC \cdot \max_{p_n+1 \leqslant i \leqslant p_{n+1}} \|Te_i\|$.

Pick $M \in \mathbb{N}$ such that $\|Te_k\| < \frac{\varepsilon}{N}$ for all $k \geqslant M$. Then

$$\|Tx_n\| \leqslant \|Tu_n\| + \|Tv_n\| \leqslant NC\frac{\varepsilon}{N} + \varepsilon\|T\| = \varepsilon\bigl(C + \|T\|\bigr)$$

for all n such that $p_n > M$. It follows that $Tx_n \to 0$. $\square$

Theorem 5.5. *An operator $T \in L(d_{w,p})$ is $d_{w,p}$-strictly singular if and only if $Te_n \to 0$.*

Proof. Suppose that $Te_n \to 0$ but $T \notin \mathcal{SS}_{d_{w,p}}$. Then there exists a subspace Y of $d_{w,p}$ such that Y is isomorphic to $d_{w,p}$ and $T|_Y$ is an isomorphism. Let (x_n) be a basis of Y equivalent to (e_n). By Remark 1.7, $x_n \xrightarrow{w} 0$. Using Theorem 1.3 and passing to a subsequence, we may assume that (x_n) is equivalent to a block sequence (z_n) of (e_n) such that $x_n - z_n \to 0$. Since (z_n) is equivalent to (e_n), it is almost lengthwise bounded by Theorem 4.8. By Lemma 5.4, $Tz_n \to 0$. Since $x_n - z_n \to 0$, we obtain $Tx_n \to 0$. This is a contradiction since (x_n) is seminormalized and $T|_{[x_n]}$ is an isomorphism.

The converse implication follows from Lemma 5.1. $\square$

Remark 5.6. In Theorem 5.3 we showed, in particular, that $\mathcal{SS}_{d_{w,p}}$ is closed under addition. Alternatively, we could have deduced this from Theorem 5.5.

Recall that an operator T on a Banach space X is called ***Dunford-Pettis*** if for any sequence (x_n) in X, $x_n \xrightarrow{w} 0$ implies $Tx_n \to 0$. If $1 < p < \infty$ then the class of Dunford-Pettis operators on $d_{w,p}$ coincides with $\mathcal{K}(d_{w,p})$ because $d_{w,p}$ is reflexive. For the case $p = 1$ we have the following result.

Theorem 5.7. *Let $T \in L(d_{w,1})$. Then T is $d_{w,1}$-strictly singular if and only if T is Dunford-Pettis.*

Proof. If T is Dunford-Pettis then then T is $d_{w,1}$-strictly singular by Theorem 5.5 because (e_n) is weakly null.

Conversely, suppose that T is $d_{w,1}$-strictly singular. Let (x_n) be a weakly null sequence. Suppose that (Tx_n) does not converge to zero. Then, passing to a subsequence, we may assume that (x_n) is a seminormalized weakly null basic sequence equivalent to a block sequence (u_n) of (e_n) such that $x_n - u_n \to 0$. Clearly, (u_n) is weakly null. In particular, (u_n) has no subsequences equivalent to (f_n). By Theorem 4.8, (u_n) is almost lengthwise bounded. Hence, by Lemma 5.4, $Tu_n \to 0$. It follows that $Tx_n \to 0$, contrary to the choice of (x_n). □

6. Strictly singular operators between ℓ_p and $d_{w,p}$.

We do not know whether the ideals $\overline{J^j}$, $\mathcal{SS} \wedge \overline{J^{\ell_p}}$, and $\mathcal{SS}$ are distinct. In this section, we discuss some connections between these ideals.

Conjecture 6.1. $\overline{J^j} = \mathcal{SS} \wedge \overline{J_{\ell_p}}$. *In particular, every strictly singular operator in* $L(d_{w,p})$ *which factors through* ℓ_p *can be approximated by operators that factor through* j.

The following statement is a refinement of Lemma 1.9. Recall that $d_{w,p}$ is a Banach lattice with respect to the coordinate-wise order.

Lemma 6.2. *Suppose that* (x_n) *and* (y_n) *are seminormalized sequences in* $d_{w,p}$ *such that* $|x_n| \geqslant |y_n|$ *for all* $n \in \mathbb{N}$ *and* $x_n \to 0$ *coordinate-wise. Then there exists an increasing sequence* (n_k) *in* $\mathbb{N}$ *such that* (x_{n_k}) *and* (y_{n_k}) *are basic and* $(x_{n_k}) \succeq (y_{n_k})$.

Proof. Clearly, $y_n \to 0$ coordinate-wise. By Theorem 1.3, we can find a sequence (n_k) and two block sequences (u_k) and (v_k) of (e_n) such that (x_{n_k}) and (y_{n_k}) are basic, $(x_{n_k}) \sim (u_k)$, $(y_{n_k}) \sim (v_k)$, $x_{n_k} - u_k \to 0$, $y_{n_k} - v_k \to 0$, and for each $k \in \mathbb{N}$, the vector u_k (v_k, respectively) is a restriction of (x_{n_k}) (of (y_{n_k}), respectively).

For each $k \in \mathbb{N}$, define $h_k \in d_{w,p}$ by putting its i-th coordinate to be equal to $h_k(i) = \operatorname{sign}\big(v_k(i)\big) \cdot \big(|u_k(i)| \wedge |v_k(i)|\big)$. Then (h_k) is a block sequence of (e_n) such that $|h_k| \leqslant |u_k|$. A straightforward verification shows that $|h_k - v_k| \leqslant |u_k - x_{n_k}|$. It follows that $h_k - v_k \to 0$. By Theorem 1.2, passing to a subsequence, we may assume that (h_k) is basic and $(h_k) \sim (v_k)$. By Lemma 1.9, $(u_k) \succeq (h_k)$. Hence $(x_{n_k}) \succeq (y_{n_k})$. □

The next lemma is a version of Theorem 4.8 for the case (x_n) is an arbitrary bounded sequence.

Lemma 6.3. *If the bounded sequence (x_n) in $d_{w,p}$ is not almost lengthwise bounded, then there is a subsequence (x_{n_k}) such that $(x_{n_{2k}} - x_{n_{2k-1}})$ is equivalent to the unit vector basis (f_n) of ℓ_p.*

Proof. We can assume without loss of generality that no subsequence of (x_n) is equivalent to the unit vector basis of ℓ_1. Indeed, if (x_{n_k}) is equivalent to the unit vector basis of ℓ_1 then $p = 1$. It follows that (x_{n_k}) is equivalent to (f_n) and hence $(x_{n_{2k}} - x_{n_{2k-1}})$ is equivalent to (f_n), as well.

Without loss of generality, $\sup_n \|x_n\| = 1$. Since (x_n) is not almost lengthwise bounded, there exists $c > 0$ such that

$$\forall N \in \mathbb{N} \quad \exists n \in \mathbb{N} \quad \|x_n^*|_{[N,\infty)}\| > c. \tag{2}$$

Let $\frac{c}{4} > \varepsilon_k \downarrow 0$. We will inductively construct increasing sequences (n_k) and (N_k) in $\mathbb{N}$ and a sequence (y_k) in $d_{w,p}$ such that the following conditions are satisfied for each k:

(i) $\|x_{n_k}|_{[N_{k+1},\infty)}\| < \varepsilon_k$;
(ii) y_k is supported on $[N_k, N_{k+1})$;
(iii) y_k is a restriction of x_{n_k};
(iv) $\|y_k\| > \frac{c}{2}$;
(v) $\|y_k\|_\infty \leqslant s_{N_k}^{-1/p}$ where s_N is as in Lemma 3.4.

For $k = 1$, we put $N_1 = 1$, and define n_1 to be the first number n such that $\|x_n\| > c$; such an n exists by (2). Pick $N_2 \in \mathbb{N}$ such that $\|x_{n_1}|_{[N_2,\infty)}\| < \varepsilon_1$. Put $y_1 = x_{n_1}|_{[N_1,N_2)}$. It follows that $1 \geqslant \|y_1\| > c - \varepsilon_1 > \frac{c}{2}$, and the coordinates of y_1 are all at most 1 $(= s_1^{-1/p})$, hence all the conditions (i)–(v) are satisfied for $k = 1$.

Suppose that appropriate sequences $(n_i)_{i=1}^k$, $(N_i)_{i=1}^{k+1}$, and $(y_i)_{i=1}^k$ have been constructed. Use (2) to find n_{k+1} such that $\|x^*_{n_{k+1}}|_{[2N_{k+1},\infty)}\| > c$. Let z be the vector obtained from $x_{n_{k+1}}$ by replacing its N_{k+1} largest (in absolute value) entries with zeros. Then $\|z|_{[N_{k+1},\infty)}\| \geqslant \|z^*|_{[N_{k+1},\infty)}\| = \|x^*_{n_{k+1}}|_{[2N_{k+1},\infty)}\| > c$. By Lemma 3.4, $\|z\|_\infty \leqslant s_{N_{k+1}}^{-1/p}$. Choose N_{k+2} such that $\|x_{n_{k+1}}|_{[N_{k+2},\infty)}\| < \varepsilon_{k+1}$. It follows that $\|z|_{[N_{k+2},\infty)}\| < \varepsilon_{k+1}$. Put $y_{k+1} = z|_{[N_{k+1},N_{k+2})}$. Then $\|y_{k+1}\| \geqslant c - \varepsilon_{k+1} > \frac{c}{2}$, and the inductive construction is complete.

The sequence (y_k) constructed above is a seminormalized block sequence of (e_n) such that the coordinates of (y_k) converge to zero by condition (v). Using Remark 1.4 and passing to a subsequence, we may assume that (y_k) is equivalent to the unit vector basis (f_n) of ℓ_p.

Since (x_n) contains no subsequences equivalent to the unit vector basis of ℓ_1, using the Rosenthal's ℓ_1-theorem and passing to a further subsequence, we may assume that

(x_{n_k}) is weakly Cauchy. For all $m > k \in \mathbb{N}$, we have: $\|x_{n_k}|_{[N_m,\infty)}\| \leqslant \|x_{n_k}|_{[N_{k+1},\infty)}\| \leqslant \varepsilon_k$. Therefore $\|x_{n_m} - x_{n_k}\| \geqslant \|(x_{n_m} - x_{n_k})|_{[N_m,\infty)}\| \geqslant \|x_{n_m}|_{[N_m,\infty)}\| - \varepsilon_k \geqslant \|y_m\| - \varepsilon_k \geqslant \frac{c}{2} - \varepsilon_k > \frac{c}{4}$. It follows that the sequence (u_k) defined by $u_k = x_{n_{2k}} - x_{n_{2k-1}}$ is seminormalized and weakly null. Passing to a subsequence of (x_{n_k}), we may assume that (u_k) is equivalent to a block sequence of (e_n). By Proposition 1.8, $(f_n) \succeq (u_k)$.

Let $v_k = x_{n_{2k}} - \big(x_{n_{2k-1}}|_{[1,N_{2k})}\big)$. Then $\|u_k - v_k\| = \|x_{n_{2k-1}}|_{[N_{2k},\infty)}\| < \varepsilon_{2k-1} \to 0$. By Theorem 1.2, passing to a subsequence of (x_{n_k}), we may assume that (v_k) is basic and $(v_k) \sim (u_k)$. Also, (v_k) is weakly null. Note that $|y_{2k}| \leqslant |v_k|$ for all $k \in \mathbb{N}$, since $\operatorname{supp} y_{2k} \subseteq [N_{2k}, N_{2k+1})$, so that y_{2k} is a restriction of v_k. By Lemma 6.2, passing to a subsequence, we may assume that $(v_k) \succeq (y_{2k})$. Therefore we obtain the following diagram:

$$(f_k) \succeq (u_k) \sim (v_k) \succeq (y_{2k}) \sim (f_{2k}) \sim (f_n).$$

It follows that (u_k) is equivalent to (f_k). □

Corollary 6.4. *If $T \in \mathcal{SS}(\ell_p, d_{w,p})$ then the sequence (Tf_n) is almost lengthwise bounded.*

Proof. Suppose that (Tf_n) is not almost lengthwise bounded. By Lemma 6.3, there is a subsequence (f_{n_k}) such that $(Tf_{n_{2k}} - Tf_{n_{2k-1}})$ is equivalent to (f_n). It follows that $T|_{[f_{n_{2k}} - f_{n_{2k-1}}]}$ is an isomorphism. □

Remark 6.5. If we view T as an infinite matrix, the vectors (Tf_n) represent its columns.

Theorem 6.6. *If $T \in L(\ell_1, d_{w,1})$ is such that the sequence (Tf_n) is almost lengthwise bounded, then for any $\varepsilon > 0$ there exists $S \in L(\ell_1)$ such that $\|T - jS\| < \varepsilon$, where $j \in L(\ell_1, d_{w,1})$ is the formal identity operator.*

Proof. Let $\varepsilon > 0$ be fixed. Find $N \in \mathbb{N}$ such that $\|(Tf_n)^*|_{[N,\infty)}\| < \varepsilon$ for all n. Let $z_n \in d_{w,1}$ be the vector obtained from Tf_n by keeping its largest N coordinates and replacing the rest of the coordinates with zeros.

Define $S\colon \ell_1 \to d_{w,1}$ by $Sf_n = z_n$. Note that $\|T - S\| = \sup_n\|(T - S)f_n\| = \sup_n\|Tf_n - z_n\| \leqslant \varepsilon$; in particular, S is bounded. Let $F = \operatorname{span}\{e_1, \dots, e_N\}$. Since $\dim F < \infty$, there exists $C > 0$ such that

$$\frac{1}{C}\|x\|_{\ell_1} \leqslant \|x\|_{d_{w,1}} \leqslant C\|x\|_{\ell_1}$$

for all $x \in F$. Observe that for each $n \in \mathbb{N}$, the non-increasing rearrangement $(Sf_n)^*$ is in F. Therefore, for all $n \in \mathbb{N}$, we have

$$\|Sf_n\|_{\ell_1} = \|(Sf_n)^*\|_{\ell_1} \leqslant C\|(Sf_n)^*\|_{d_{w,1}} = C\|Sf_n\|_{d_{w,1}} \leqslant C\|S\|.$$

It follows that the operator $\widetilde{S}\colon \ell_1 \to \ell_1$ defined by $\widetilde{S}f_n = Sf_n$ belongs to $L(\ell_1)$. Obviously, $S = j\widetilde{S}$. So, $\|T - j\widetilde{S}\| < \varepsilon$. □

The next corollary follows immediately from Theorem 6.6 and Corollary 6.4. This corollary can be considered as a support for Conjecture 6.1.

Corollary 6.7. *$\mathcal{SS}(\ell_1, d_{w,1})$ is contained in the closure of $\{jS : S \in L(\ell_1, d_{w,1})\}$.*

Question. Does Corollary 6.7 remain valid for $p > 1$?

The following fact is standard, we include its proof for convenience of the reader.

Proposition 6.8. *If X is a Banach space then $\mathcal{SS}(X, \ell_1) = \mathcal{K}(X, \ell_1)$.*

Proof. Let $T \notin \mathcal{K}(X, \ell_1)$. Pick a bounded sequence (x_n) in X such that (Tx_n) has no convergent subsequences. By Schur's theorem, (Tx_n) and, therefore, (x_n) have no weakly Cauchy subsequences. Applying Rosenthal's ℓ_1-theorem twice, we find a subsequence (x_{n_k}) such that (x_{n_k}) and (Tx_{n_k}) are both equivalent to the unit vector basis of ℓ_1. It follows that T is not strictly singular. □

Proposition 6.9. *For all $p \in [1, \infty)$, $\mathcal{SS}(d_{w,p}, \ell_p) = \mathcal{K}(d_{w,p}, \ell_p)$.*

Proof. By Proposition 6.8, we only have to consider the case $p > 1$. Let $T \notin \mathcal{K}(X, \ell_p)$. Pick a bounded sequence (x_n) in X such that (Tx_n) has no convergent subsequences. Since $d_{w,p}$ contains no copies of ℓ_1, by Rosenthal's ℓ_1-theorem we may assume that (x_n) is weakly Cauchy. Passing to a further subsequence, we may assume that the sequence (Ty_n), where $y_n = x_{2n} - x_{2n-1}$, is seminormalized. It follows that (y_n) is also seminormalized. Also, (y_n) and, therefore, (Ty_n) are weakly null. Passing to a subsequence of (x_n), we may assume that (y_n) and (Ty_n) are both basic, equivalent to block sequences of (e_n) and (f_n), respectively. By [3, Proposition 5] and [17, Proposition 2.a.1], $(f_n) \succeq (y_n)$ and $(f_n) \sim (Ty_n)$. So, we obtain the diagram

$$(f_n) \succeq (y_n) \succeq (Ty_n) \sim (f_n).$$

Hence $[y_n]$ is isomorphic to $[Ty_n]$, so that T is not strictly singular. □

The following lemma is standard.

Lemma 6.10. *Let X be a Banach space. Every seminormalized basic sequence in X is dominated by the unit vector basis of ℓ_1.*

Lemma 6.11. *Let* (x_n) *and* (y_n) *be two sequences in a Banach space* X *such that* (x_n) *is equivalent to the unit vector basis of* ℓ_1 *and* (y_n) *is convergent. Then the sequence* (z_n) *defined by* $z_n = x_n + y_n$ *has a subsequence equivalent to the unit vector basis of* ℓ_1.

Proof. Observe that (z_n) cannot have weakly Cauchy subsequences since (x_n) does not have such subsequences. Since (z_n) is bounded, the result follows from Rosenthal's ℓ_1-theorem. □

Recall that an operator A between two Banach lattices X and Y is called ***positive*** if $x \geqslant 0$ entails $Tx \geqslant 0$.

Conjecture 6.1 asserts, in particular, that if $T \in \mathcal{SS}(d_{w,p})$ and $T = AB$ for some $A\colon d_{w,p} \to \ell_p$ and $B\colon \ell_p \to d_{w,p}$ then $T \in \overline{\mathcal{J}^j}$. In the next theorem, we prove this under the additional assumptions that $p = 1$ and both A and B are positive.

Theorem 6.12. *Let* $T \in \mathcal{SS}(d_{w,1})$ *be such that* $T = AB$*, where* $A \in L(\ell_1, d_{w,1})$*,* $B \in L(d_{w,1}, \ell_1)$*, and both* A *and* B *are positive. Then* $T \in \overline{\mathcal{J}^j}$.

Proof. Define a sequence (A_N) of operators in $L(\ell_1, d_{w,1})$ by the following procedure. For each $n \in \mathbb{N}$, let $A_N f_n$ be obtained from Af_n by keeping the largest N coordinates and replacing the rest of the coordinates with zeros. Since $Af_n \geqslant 0$ for all $n \in \mathbb{N}$, this defines a positive operator $\ell_1 \to d_{w,1}$. Also, $\|A_N f_n\| \leqslant \|Af_n\| \leqslant \|A\|$ for all $n \in \mathbb{N}$, hence $\|A_N\| \leqslant \|A\|$.

Define $A'_N = A - A_N$. It is clear that $0 \leqslant A'_N f_n \leqslant Af_n$ for all $n \in \mathbb{N}$, hence $A'_N \geqslant 0$ and $\|A'_N\| \leqslant \|A\|$. We claim that $A'_N \to 0$ in the strong operator topology (SOT). Indeed, since $A'_N f_n$ is obtained from Af_n by removing the largest N coordinates, the elements of the matrix of A'_N are all smaller than $\frac{\|A\|}{s_N}$ by Lemma 3.4. In particular, if $0 \leqslant x \in \ell_1$, then $A'_N x \downarrow 0$; it follows that $\|A'_N x\| \to 0$ because $d_{w,1}$ has order continuous norm (see Remark 1.5). If $x \in \ell_1$ is arbitrary then $\|A'_N x\| \leqslant \big\|A'_N |x|\,\big\| \to 0$.

We will show that $\|A'_N B\| \to 0$ as $N \to \infty$, so that $\|AB - A_N B\| \to 0$ as $N \to \infty$. Since $(A_N f_n)_{n=1}^\infty$ is almost lengthwise bounded (in fact, the vectors in the sequence $(A_N f_n)_{n=1}^\infty$ all have at most N nonzero entries), the theorem will follow from Theorem 6.6.

Assume, by way of contradiction, that there are $c > 0$ and a sequence (N_k) in $\mathbb{N}$ such that $\|A'_{N_k} B\| > c$. Then there exists a normalized positive sequence (x_k) in $d_{w,p}$ such that $\|A'_{N_k} B x_k\| > c$. By Rosenthal's ℓ_1-theorem, we may assume that (x_k) is either weakly Cauchy or equivalent to (f_n).

Assume that (x_k) is weakly Cauchy. Then (Bx_k) is weakly Cauchy. Since (Bx_k) is a sequence in ℓ_1, it must converge to some $z \in \ell_1$ by the Schur property. Then

$\|A'_{N_k}Bx_k - A'_{N_k}z\| \leqslant \|A'_{N_k}\| \cdot \|Bx_k - z\| \leqslant \|A\| \cdot \|Bx_k - z\| \to 0$. Since $A'_{N_k} \to 0$ in SOT, it follows that $A'_{N_k}Bx_k \to 0$, contrary to the assumption. Therefore (x_k) must be equivalent to (f_n).

Since the entries of the matrix of A'_N are all less than $\frac{\|A\|}{s_N}$, the coordinates of the vector $A'_{N_k}Bx_k$ are all less than $\frac{\|A\|}{s_{N_k}}\|B\| \to 0$. Hence, passing to a subsequence, we may assume that $(A'_{N_k}Bx_k)$ is equivalent to a block sequence (u_k) of (e_n) such that each u_k is a restriction of $A'_{N_k}Bx_k$. In particular, the coordinates of (u_k) converge to zero. Passing to a further subsequence, we may assume by Remark 1.4 that $(A'_{N_k}Bx_k) \sim (f_n)$.

The sequence (Tx_k) cannot have subsequences equivalent to (f_n) since T is strictly singular. Therefore, by Rosenthal's ℓ_1-theorem, we may assume that (Tx_k) is weakly Cauchy. Since $d_{w,1}$ is weakly sequentially complete (Remark 1.5), the sequence (Tx_k) weakly converges to a vector $y \in d_{w,1}$. Since the positive cone in a Banach lattice is weakly closed, $y \geqslant 0$.

Note that $Tx_k \geqslant A'_{N_k}Bx_k \geqslant u_k \geqslant 0$ for every k. Since (u_k) is a seminormalized block sequence of (e_n), it follows that (Tx_k) is not norm convergent. Write $Tx_k = y + h_k$; then (h_k) converges to zero weakly but not in norm. Therefore, passing to a subsequence, we may assume that (h_k) is seminormalized and basic (but not, necessarily, positive).

Let $r_k = A'_{N_k}Bx_k - (A'_{N_k}Bx_k \wedge y) \geqslant 0$, $k \in \mathbb{N}$. Observe that $A'_{N_k}Bx_k \wedge y \in [0, y]$ for all k. Since $d_{w,1}$ has order continuous norm and the order in $d_{w,1}$ is defined by a 1-unconditional basis, order intervals in $d_{w,1}$ are compact (see, e.g., [24, Theorem 6.1]). Therefore, passing to a subsequence of (x_{n_k}), we may assume that $(A'_{N_k}Bx_k \wedge y)$ is convergent, hence, passing to a further subsequence, (r_k) is equivalent to (f_n) by Lemma 6.11 and Theorem 1.2.

It follows from $y + h_k \geqslant A'_{N_k}Bx_k \geqslant 0$ that $|h_k| \geqslant r_k$ for all k. Passing to a subsequence, we may assume by Lemma 6.2 that $(h_k) \succeq (r_k) \sim (f_n)$. By Lemma 6.10, in fact $(h_k) \sim (f_n)$, and, hence, by Lemma 6.11, $(ABx_k) \sim (f_n)$. Since also $(x_k) \sim (f_n)$, this contradicts to $T = AB \in \mathcal{SS}(d_{w,1})$. □

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(A. Kaminska) Department of Mathematical Sciences The University of Memphis, Memphis, TN 38152-3240. USA

E-mail address: `kaminska@memphis.edu`

(A. I. Popov, E. Spinu, and V. G. Troitsky) Department of Mathematical and Statistical Sciences, University of Alberta, Edmonton, AB, T6G 2G1. Canada

E-mail address: `apopov@math.ualberta.ca, spinu@ualberta.ca`

E-mail address: `troitsky@ualberta.ca`

(A. Tcaciuc) Mathematics and Statistics Department, Grant MacEwan University, Edmonton, AB, T5J P2P, Canada

E-mail address: `atcaciuc@ualberta.ca`