ALMOST INVARIANT HALF-SPACES OF OPERATORS ON BANACH SPACES

GEORGE ANDROULAKIS, ALEXEY I. POPOV, ADI TCACIUC,
AND VLADIMIR G. TROITSKY

Abstract. We introduce and study the following modified version of the Invariant Subspace Problem: whether every operator T on an infinite-dimensional Banach space has an almost invariant half-space, that is, a subspace Y of infinite dimension and infinite codimension such that Y is of finite codimension in $T(Y)$. We solve this problem in the affirmative for a large class of operators which includes quasinilpotent weighted shift operators on ℓ_p $(1 \leqslant p < \infty)$ or c_0.

1. Introduction

Throughout the paper, X is a Banach space and by $\mathcal{L}(X)$ we denote the set of all (bounded linear) operators on X. By a "subspace" of a Banach space we always mean a "closed subspace". Given a sequence (x_n) in X, we write $[x_n]$ for the closed linear span of (x_n).

Definition 1.1. A subspace Y of a Banach space X is called a ***half-space*** if it is both of infinite dimension and of infinite codimension in X.

Definition 1.2. If $T \in \mathcal{L}(X)$ and Y is a subspace of X, then Y is called ***almost invariant*** under T, or T***-almost invariant***, if there exists a finite dimensional subspace F of X such that $T(Y) \subseteq Y + F$.

In this work, the following question will be referred to as the ***almost invariant half-space problem***: *Does every operator on an infinite-dimensional Banach space have an almost invariant half-space?* Observe that every subspace of X that is not a half-space is clearly almost invariant under any operator. Also, note that the almost invariant half-space problem is not weaker than the well known invariant subspace problem, because in the latter the invariant subspaces are not required to be half-spaces.

Date: May 19, 2009.
The third and the fourth authors were supported by NSERC.

The natural question whether the usual unilateral right shift operator acting on a Hilbert space has almost invariant half-spaces has an affirmative answer. Moreover, it is known that this operator has even invariant half-spaces. Indeed, by [6, Corollary 3.15], this operator has an invariant subspace with infinite-dimensional orthogonal complement (thus the invariant subspace is of infinite codimension). It is not hard to see that the space exhibited in the proof of this statement is in fact infinite dimensional.

It is natural to consider Donoghue operators as candidates for counterexamples to the almost invariant half-space problem, as their invariant subspaces are few and well understood. Recall that a Donoghue operator $D \in \mathcal{L}(\ell_2)$ is an operator defined by

$$De_0 = 0, \quad De_i = w_i e_{i-1}, \quad i \in \mathbb{N},$$

where (w_i) is a sequence of non-zero complex numbers such that $\big(|w_i|\big)$ is monotone decreasing and in ℓ_2. It is known that if D is a Donoghue operator then D has only invariant subspaces of finite dimension and D^* has only invariant subspaces of finite codimension (see [6, Theorem 4.12]). Hence neither D nor D^* have invariant half-spaces. In Section 3 we will employ the tools of Section 2 to show that, nevertheless, every Donoghue operator has almost invariant half-spaces. We do not know whether the operators constructed by Enflo [3] and Read [7] have almost invariant half-spaces.

The following result explains how almost invariant half-spaces of operators are related to invariant subspaces of perturbed operators.

Proposition 1.3. *Let $T \in \mathcal{L}(X)$ and $H \subseteq X$ be a half-space. Then H is almost invariant under T if and only if H is invariant under $T + K$ for some finite rank operator K.*

Proof. Suppose that T has an almost invariant half-space H. Let F be a subspace of the smallest dimension satisfying the condition in Definition 1.2. Then we have $H \cap F = \{0\}$. Define $P\colon H + F \to F$ by $P(h + f) = f$. Since P is a finite rank operator, we can extend it to a finite rank operator on X using Hahn-Banach theorem. That is, there exists $\widetilde{P}\colon X \to F$ such that $\widetilde{P}|_{H+F} = P$. Define $K\colon X \to X$ by $K := -\widetilde{P}T$. Clearly K has finite rank and for any $h \in H$ we have $Th = h' + f$ for some $h' \in H$ and $f \in F$, so that

$$(T + K)(h) = Th - \widetilde{P}Th = h' + f - \widetilde{P}(h' + f) = h' + f - f = h'$$

Therefore, $(T + K)H \subseteq H$, which shows that $T + K$ has an invariant half-space.

Conversely, from $(T+K)(H) \subseteq H$ it follows immediately that $T(H) \subseteq H+K(H)$,so that H is an almost invariant half space for T. $\square$

Finally we would like to point out that if an operator has almost invariant half-spaces, then so does its adjoint. For that we will need two simple lemmas. The proof of the first lemma is elementary.

Lemma 1.4. *Let X be a Banach space and Y be a subspace of X. Then Y is infinite codimensional if and only if $Y^\perp$ is of infinite dimension. Thus Y is a half-space if and only if both Y and $Y^\perp$ are of infinite dimension.*

Lemma 1.5. *A subspace Y of X is a half-space if and only if $Y^\perp$ is a half-space in X^*.*

Proof. Suppose Y is a half-space. By Lemma 1.4, $Y^\perp$ must be infinite-dimensional. Also $(Y^\perp)^\perp \supseteq j(Y)$ where $j\colon X \to X^{**}$ denotes the natural embedding. Thus $(Y^\perp)^\perp$ is infinite dimensional. Now Lemma 1.4 yields that $Y^\perp$ is a half-space.

Let's assume that $Y^\perp$ is a half-space. Since $Y^\perp$ is infinite codimensional Y must be infinite-dimensional (see, e.g. [1, Theorem 5.110]). On the other hand, since $Y^\perp$ is infinite dimensional, by Lemma 1.4 we obtain that Y is of infinite codimension, thus a half-space. □

Remark 1.6. The statement dual to that of Lemma 1.5 is not true in general. That is, if Z is a half-space in X^* then $Z_\perp$ need not be a half-space. For example, c_0 is a half-space in ℓ_∞ while $(c_0)_\perp = \{0\} \subseteq \ell_1$ is not.

Proposition 1.7. *Let T be an operator on a Banach space X. If T has an almost invariant half-space then so does its adjoint T^*.*

Proof. Let Y be a half-space in X such that Y is almost invariant under T, and F be a finite-dimensional subspace of X of smallest dimension such that $TY \subseteq Y + F$. Then $Y \cap F = \{0\}$. Thus there exists a subspace W of X such that $W + F = X$, $W \cap F = \{0\}$, and $Y \subseteq W$. In particular, $W^\perp$ is finite dimensional. Denote $Z = (Y + F)^\perp$. By Lemma 1.5, Z is a half-space in X^*. For every $z \in Z$ and $y \in Y$ we have $\langle y, T^*z\rangle = \langle Ty, z\rangle = 0$ since $Ty \in Y + F$. Therefore $T^*Z \subseteq Y^\perp$. To finish the proof, it suffices to show that $Y^\perp = Z + W^\perp$.

Indeed, by the definition of Z we have that $Z \subseteq Y^\perp$. Also since $Y \subseteq W$ we have $W^\perp \subseteq Y^\perp$. Thus $Z + W^\perp \subseteq Y^\perp$. On the other hand, since F is finite dimensional and $F \cap W = \{0\}$, we may choose a basis (f_i) of F with biorthogonal functionals (f_i^*) such that $f_i^* \in W^\perp$. Since $Y \subseteq W$ we have that $f_i^* \in Y^\perp$. Thus, if x^* is an arbitrary element of $Y^\perp$ then $x^* - \sum_i x^*(f_i) f_i^* \in (Y + F)^\perp = Z$, and therefore $x^* \in Z + W^\perp$. □

2. Basic tools

All Banach spaces in Sections 2, 3 and 4 are assumed to be complex. For a subset A of $\mathbb{C}$, we will write $A^{-1} = \bigl\{\frac{1}{\lambda} \colon \lambda \in A, \lambda \neq 0\bigr\}$. For a Banach space X and $T \in \mathcal{L}(X)$, we will use symbols $\sigma(T)$ for the spectrum of T, $r(T)$ for the spectral radius of T, and $\rho(T)$ for the resolvent set of T. For a nonzero vector $e \in X$ and $\lambda \in \rho(T)^{-1}$, define a vector $h(\lambda, e)$ in X by

$$h(\lambda, e) := \bigl(\lambda^{-1}I - T\bigr)^{-1}(e).$$

Note that if $|\lambda| < \frac{1}{r(T)}$ then[1] Neumann's formula yields

$$h(\lambda, e) = \lambda \sum_{n=0}^{\infty} \lambda^n T^n e. \tag{1}$$

Also, observe that $\bigl(\lambda^{-1}I - T\bigr)h(\lambda, e) = e$ for every $\lambda \in \rho(T)^{-1}$, so that

$$Th(\lambda, e) = \lambda^{-1}h(\lambda, e) - e. \tag{2}$$

The last identity immediately yields the following result.

Lemma 2.1. *Let X be a Banach space, $T \in \mathcal{L}(X)$, $0 \neq e \in X$, and $A \subseteq \rho(T)^{-1}$. Put*

$$Y = \overline{\operatorname{span}}\bigl\{h(\lambda, e) \colon \lambda \in A\bigr\}.$$

Then Y is a T-almost invariant subspace (which is not necessarily a half-space), with $TY \subseteq Y + \operatorname{span}\{e\}$.

Remark 2.2. *The Replacement procedure.* For any nonzero vector e in a Banach space X, we have

$$h(\lambda, e) - h(\mu, e) = (\mu^{-1} - \lambda^{-1})h\bigl(\lambda, h(\mu, e)\bigr)$$

whenever $\lambda, \mu \in \rho(T)^{-1}$.

Indeed,

$$\begin{aligned} h(\lambda, e) - h(\mu, e) &= \bigl[(\lambda^{-1}I - T)^{-1} - (\mu^{-1}I - T)^{-1}\bigr](e) \\ &= (\mu^{-1} - \lambda^{-1})(\lambda^{-1}I - T)^{-1}(\mu^{-1}I - T)^{-1}(e) \\ &= (\mu^{-1} - \lambda^{-1})(\lambda^{-1}I - T)^{-1}h(\mu, e) \\ &= (\mu^{-1} - \lambda^{-1})h\bigl(\lambda, h(\mu, e)\bigr) \end{aligned}$$

Lemma 2.3. *Suppose that $T \in \mathcal{L}(X)$ has no eigenvectors. Then, for any nonzero vector $e \in X$ the set $\bigl\{h(\lambda, e) \,:\, \lambda \in \rho(T)^{-1}\bigr\}$ is linearly independent.*

---

[1] In case $r(T) = 0$ we take $\frac{1}{r(T)} = +\infty$.

Proof. We are going to use induction on n to show that for any nonzero vector $e \in X$ and any distinct $\lambda_1, \lambda_2, \dots, \lambda_n \in \rho(T)^{-1}$ the set

$$\big\{h(\lambda_1, e), h(\lambda_2, e), \dots, h(\lambda_n, e)\big\}$$

is linearly independent. The statement is clearly true for $n = 1$; we assume it is true for $n - 1$ and will prove it for n.

Fix $e \in X$ and distinct $\lambda_1, \lambda_2, \dots, \lambda_n \in \rho(T)^{-1}$. Let $a_1, a_2, \dots, a_n$ be scalars such that $\sum_{k=1}^n a_k h(\lambda_k, e) = 0$. It follows from (2) that

$$0 = T\Bigg(\sum_{k=1}^n a_k h(\lambda_k, e)\Bigg) = \sum_{k=1}^n a_k \lambda_k^{-1} h(\lambda_k, e) - \sum_{k=1}^n a_k e.$$

If $\sum_{k=1}^n a_k \neq 0$ then $e \in \operatorname{span}\big\{h(\lambda_k, e)\big\}_{k=1}^n$, so that $\operatorname{span}\big\{h(\lambda_k, e)\big\}_{k=1}^n$ is T-invariant by (2). This subspace is finite-dimensional, so that T has an eigenvalue, which is a contradiction. Therefore $\sum_{k=1}^n a_k = 0$, so that $a_1 = -\sum_{k=2}^n a_k$.

Using the Replacement Procedure we obtain

$$\begin{aligned} 0 = \sum_{k=1}^n a_k h(\lambda_k, e) &= \Big(-\sum_{k=2}^n a_k\Big) h(\lambda_1, e) + \sum_{k=2}^n a_k h(\lambda_k, e) \\ &= \sum_{k=2}^n a_k \big(h(\lambda_k, e) - h(\lambda_1, e)\big) \\ &= \sum_{k=2}^n a_k (\lambda_1^{-1} - \lambda_k^{-1}) h\big(\lambda_k, h(\lambda_1, e)\big). \end{aligned}$$

By the induction hypothesis, the set $\Big\{h\big(\lambda_k, h(\lambda_1, e)\big)\Big\}_{k=2}^n$ is linearly independent, hence $a_k(\lambda_1^{-1} - \lambda_k^{-1}) = 0$ for any $2 \leqslant k \leqslant n$. It follows immediately that $a_k = 0$ for any $1 \leqslant k \leqslant n$, and this concludes the proof. □

This gives us a natural way to try to construct almost invariant half-spaces. Indeed, suppose that T has no eigenvectors. Let $e \in X$ such that $e \neq 0$, and let (λ_n) be a sequence of distinct elements of $\rho(T)^{-1}$. Put $Y = \big[h(\lambda_n, e)\big]_{n=1}^\infty$. Then Y is almost invariant by Lemma 2.1 and infinite-dimensional by Lemma 2.3. However, the difficult part is to show that Y is infinite codimensional. Even passing to subsequences might not help, as there are sequences whose every subsequence spans a dense subspace (see, e.g., [8, page 58] and also [2]).

3. Weighted shift operators

In this section we give a sufficient condition for a quasinilpotent operator to have almost invariant half-spaces (Theorem 3.2). As an application, we show in Corollary 3.4

that quasinilpotent weighted shifts on ℓ_p or c_0 have invariant half-spaces. In particular, every Donoghue operator has an almost invariant half-space.

Recall that a sequence (x_i) in a Banach space is called ***minimal*** if $x_k \notin [x_i]_{i\neq k}$ for every k, (see also [4, section 1.f]). It is easy to see that this is equivalent to saying that for every k, the biorthogonal functional x_k^* defined on $\operatorname{span}\{x_i\}$ by $x_k^*\bigl(\sum_{i=0}^n \alpha_i x_i\bigr) = \alpha_k$ is bounded.

We will use the following numerical lemma.

Lemma 3.1. *Given a sequence (r_i) of positive reals, there exists a sequence (c_i) of positive reals such that the series $\sum_{i=0}^\infty c_i r_{i+k}$ converges for every k.*

Proof. For every i take $c_i = \frac{1}{2^i}\min\{\frac{1}{r_1},\dots,\frac{1}{r_{2i}}\}$. For every $i \geqslant k$ we have $k+i \leqslant 2i$, so that $c_i r_{i+k} \leqslant \frac{1}{2^i}$. It follows that

$$\sum_{i=0}^{\infty} c_i r_{i+k} \leqslant \sum_{i=0}^{k-1} c_i r_{i+k} + \sum_{i=k}^{\infty} \tfrac{1}{2^i} < +\infty.$$

□

Theorem 3.2. *Let X be a Banach space and $T \in \mathcal{L}(X)$ satisfying the following conditions:*

(i) *T has no eigenvalues.*
(ii) *The unbounded component of $\rho(T)$ contains $\{z \in \mathbb{C} : 0 < |z| < \varepsilon\}$ for some $\varepsilon > 0$.*
(iii) *There is a vector whose orbit is a minimal sequence.*

Then T has an almost invariant half-space.

Proof. Let $e \in X$ be such that $(T^i e)_{i=0}^\infty$ is minimal. For each i put $x_i = T^i e$. Then for each k, the biorthogonal functional x_k^* defined on $\operatorname{span} x_i$ by $x_k^*\bigl(\sum_{i=0}^n \alpha_i x_i\bigr) = \alpha_k$ is bounded. Let $r_k = \|x_k^*\|$. Let (c_i) be a sequence of positive real numbers as in Lemma 3.1, so that $\beta_k := \sum_{i=0}^\infty c_i r_{i+k} < +\infty$ for every k. By making c_i's even smaller, if necessary, we may assume that $\sqrt[i]{c_i} \to 0$.

Consider a function $F : \mathbb{C} \to \mathbb{C}$ defined by $F(z) = \sum_{i=0}^\infty c_i z^i$. Evidently, F is entire. Observe that we may assume that the set $\bigl\{z \in \mathbb{C} : F(z) = 0\bigr\}$ is infinite. Indeed, by the Picard Theorem there exists a negative real number d such that the set $\bigl\{z \in \mathbb{C} : F(z) = d\bigr\}$ is infinite. Now replace c_0 with $c_0 - d$. This doesn't affect our other assumptions on the sequence (c_i).

Fix a sequence of distinct complex numbers (λ_n) such that $F(\lambda_n) = 0$ for every n. Since F is non-constant, the sequence (λ_n) has no accumulation points. Hence, $|\lambda_n| \to +\infty$.

Note that (ii) can be restated as follows: $\rho(T)^{-1}$ has a connected component $\mathcal{C}$ such that $0 \in \overline{\mathcal{C}}$ and $\mathcal{C}$ contains a neighbourhood of ∞. Thus by passing to a subsequence of λ_n's and relabeling, if necessary, we can assume that $\lambda_n \in \mathcal{C}$ for all n.

Observe that the condition $\lambda_n \in \rho(T)^{-1}$ for every n implies that $h(\lambda_n, e)$ is defined for each n. Put $Y = [h(\lambda_n, e)]_{n=1}^{\infty}$. Then Y is almost invariant under T by Lemma 2.1 and $\dim Y = \infty$ by Lemma 2.3. We will prove that Y is actually a half-space by constructing a sequence of linearly independent functionals (f_n) such that every f_n annihilates Y.

For every $k = 0, 1, \dots,$ put $F_k(z) = z^k F(z)$. Let's write $F_k(z)$ in the form of Taylor series, $F_k(z) = \sum_{i=0}^{\infty} c_i^{(k)} z^i$. Then

$$c_i^{(k)} = \begin{cases} 0 & \text{if } i < k, \text{ and} \\ c_{i-k} & \text{if } i \geqslant k. \end{cases}$$

Define a functional f_k on $\operatorname{span}\{T^i e\}_{i=0}^{\infty}$ via $f_k(T^i e) = c_i^{(k)}$. Since T has no eigenvalues, the orbit of T is linearly independent thus f_k is well-defined. We will show now that f_k is bounded. Let $x \in \operatorname{span}\{T^i e\}_{i=0}^{\infty}$, then $x = \sum_{i=0}^{n} x_i^*(x) T^i e$ for some n, so that

$$\begin{aligned} |f_k(x)| = \Big| f_k\Big(\sum_{i=0}^{n} x_i^*(x) T^i e \Big) \Big| &\leqslant \Big(\sum_{i=0}^{n} \|x_i^*\| c_i^{(k)} \Big) \|x\| \\ &= \Big(\sum_{i=k}^{n} r_i c_{i-k} \Big) \|x\| \leqslant \Big(\sum_{i=k}^{\infty} r_i c_{i-k} \Big) \|x\| = \beta_k \|x\|, \end{aligned}$$

so that $\|f_k\| \leqslant \beta_k$. Hence, f_k can be extended by continuity to a bounded functional on $[T^i e]_{i=1}^{\infty}$, and then by the Hahn-Banach Theorem to a bounded functional on all of X.

Now we show that each f_k annihilates Y. Fix k. Recall that for each $\lambda \in \rho(T)^{-1}$ such that $|\lambda| < \frac{1}{r(T)}$ we have $h(\lambda, e) = \lambda \sum\limits_{i=0}^{\infty} \lambda^i T^i e$. Therefore

$$f_k\big(h(\lambda, e)\big) = f_k\Big(\lambda \sum_{i=0}^{\infty} \lambda^i T^i e \Big) = \lambda \sum_{i=0}^{\infty} \lambda^i c_i^{(k)} = \lambda F_k(\lambda) = \lambda^{k+1} F(\lambda).$$

for every $\lambda \in \mathcal{C}$ such that $|\lambda| < \frac{1}{r(T)}$ (recall $0 \in \overline{\mathcal{C}}$). The map $\lambda \mapsto h(\lambda, e)$ and, therefore, the map $\lambda \mapsto f_k\big(h(\lambda, e)\big)$, is analytic on the set $\rho(T)^{-1}$. Therefore, by the principle of uniqueness of analytic function, the functions $f_k\big(h(\lambda, e)\big)$ and $\lambda^{k+1} F(\lambda)$ must agree on $\mathcal{C}$. Since $\lambda_n \in \mathcal{C}$ for all n, we have $f_k\big(h(\lambda_n, e)\big) = \lambda_n^{k+1} F(\lambda_n) = 0$ for all n. Thus, Y is annihilated by every f_k.

It is left to prove the linear independence of $\{f_k\}_{k=1}^{\infty}$. Observe that $f_k \neq 0$ for all k since $f_k(T^i e) \neq 0$ for $i \geqslant k$. Suppose that $f_N = \sum_{k=M}^{N-1} a_k f_k$ with $a_M \neq 0$. However $f_N(T^M e) = 0$ by definition of f_N while $\sum_{k=M}^{N-1} a_k f_k(T^M e) = a_M c_0 \neq 0$, contradiction. □

Remark 3.3. Note that condition (ii) of Theorem 3.2 is satisfied by many important classes of operators. For example, it is satisfied if $\sigma(T)$ is finite (in particular, if T is quasinilpotent) or if 0 belongs in the unbounded component of $\rho(T)$.

Corollary 3.4. *Suppose that $X = \ell_p$ ($1 \leqslant p < \infty$) or c_0 and $T \in \mathcal{L}(X)$ is a weighted right shift operator with weights converging to zero but not equal to zero. Then both T and T^* have almost invariant half-spaces.*

Proof. It can be easily verified that T is quasinilpotent. Clearly, T has no eigenvalues, and the orbit of e_1 is evidently a minimal sequence. By Theorem 3.2 and Remark 3.3, T has almost invariant half-spaces. Finally, Proposition 1.7 yields almost invariant half-spaces for T^*. □

The following statement is a special case of Corollary 3.4.

Corollary 3.5. *If D is a Donoghue operator then both D and D^* have almost invariant half-spaces.*

Recall that a subset $\mathcal{D}$ of $\mathbb{C}$ is called a cone if $\mathcal{D}$ is closed under addition and multiplication by positive scalars.

Remark 3.6. Condition (ii) in Theorem 3.2 can be weakened as follows: instead of requiring that $\rho(T)$ contains a punctured disk centered at zero, we may only require that it contains a non-trivial sector of this disk, i.e., the intersection of the punctured disk with a non-empty open cone. Equivalently, $\rho(T)^{-1}$ has a connected component $\mathcal{C}$ such that $0 \in \overline{\mathcal{C}}$, and there exists an open cone $\mathcal{D}$ in $\mathbb{C}$ and $M > 0$ such that $\big\{z \in \mathcal{D} \ : \ |z| \geqslant M\big\} \subseteq \mathcal{C}$. Indeed, suppose that such $\mathcal{D}$ and M exist. Choose ν in this cone with $|\nu| = M$. Also, suppose that, as in the proof of Theorem 3.2, we have already found a sequence (λ_m) of zeros of F. The set $\{M\frac{\lambda_m}{|\lambda_m|}\}_{m=0}^{\infty}$ has an accumulation point, say μ. By passing to a subsequence, we may assume that $M\frac{\lambda_m}{|\lambda_m|} \to \mu$. Note that the spectrum of $\frac{\mu}{\nu}T$ is obtained by rotating the spectrum of T. Thus $\rho\left(\frac{\mu}{\nu}T\right)^{-1}$ has a connected component $\mathcal{C}'$ such that $0 \in \overline{\mathcal{C}'}$ and there exists an open cone $\mathcal{D}'$ in $\mathbb{C}$

which contains a neighborhood of μ and $\{z \in \mathcal{D}' : |z| \geqslant M\} \subseteq \mathcal{C}'$. Thus by passing to a subsequence of (λ_m) we can assume that $\lambda_m \in \mathcal{C}'$ for all m. Replace in the proof T with $\frac{\mu}{\nu}T$. Note that this doesn't affect the assumptions on the operator and the definitions of c_i's, F, and λ_m's. Finally, multiplying an operator by a non-zero number does not affect its almost invariant half-spaces.

Note that every operator T with $\sigma(T) \subseteq \mathbb{R}$ satisfies this weaker version of condition (ii). In particular, it is satisfied by self-adjoint operators on Hilbert spaces.

Corollary 3.7. *Suppose that $T \in \mathcal{L}(X)$ such that T has no eigenvectors, $\sigma(T) \subseteq \mathbb{R}$, and there is a vector whose orbit is a minimal sequence. Then T has an almost invariant half-space.*

4. Non-quasinilpotent operators

In this section we will modify the argument of Theorem 3.2 to extend its statement to another class of operators having non-zero spectral radius. It is a standard fact that if (x_i) is a minimal sequence then $\frac{1}{\|x_n^*\|} = \operatorname{dist}\big(x_n, [x_i]_{i\neq n}\big)$ for every n.

Theorem 4.1. *Let X be a Banach space and $T \in \mathcal{L}(X)$ be an operator with $r(T) \leqslant 1$ having no eigenvectors. Let $e \in X$; put $x_n = T^n e$ for $n \in \mathbb{N}$. If (x_n) is a minimal sequence and $\sum_{n=1}^{\infty} \frac{\|x_n^*\|}{n} < \infty$ then T has an almost invariant half-space.*

Proof. Let D stand for the unit disk in $\mathbb{C}$. For a sequence $(\lambda_n) \subset D$ such that

$$\sum_{n=1}^{\infty}\big(1 - |\lambda_n|\big) < \infty. \tag{3}$$

the corresponding Blaschke product is defined by

$$B(z) = \prod_{n=1}^{\infty} \frac{|\lambda_n|}{\lambda_n} \frac{\lambda_n - z}{1 - \overline{\lambda_n} z}. \tag{4}$$

It is well known that B is a bounded analytic function on D with zeros exactly at (λ_n). According to [5, Theorem 2] we can choose a sequence $(\lambda_n) \subset D$ satisfying (3) such that $\frac{B^{(n)}(0)}{n!} = \mathrm{O}\big(\frac{1}{n+1}\big)$. Thus $B^{(n)}(0) = \mathrm{O}\big(\frac{n!}{n+1}\big)$. For $m \in \mathbb{N}$ set $F_m(z) = z^m B(z)$. Obviously the functions (F_m) are linearly independent. It follows from

$$F_m(z) = z^m B(z) = \sum_{n=0}^{\infty} \frac{B^{(n)}(0)}{n!} z^{n+m} = \sum_{n=m}^{\infty} \frac{B^{(n-m)}(0)}{(n-m)!} z^n$$

that

$$F_m^{(n)}(0) = \begin{cases} 0 & \text{for } n < m, \text{ and} \\ \frac{n!}{(n-m)!} B^{(n-m)}(0) \leqslant C \frac{n!}{n-m+1} & \text{for } n \geqslant m. \end{cases} \tag{5}$$

Put $Y = [h(\lambda_n, e)]_{n=1}^{\infty}$. By Lemma 2.1, Y is almost invariant under T and $\dim Y = \infty$ by Lemma 2.3. As in the proof of Theorem 3.2, we will show that under the conditions of the Theorem 4.1 there is a sequence of linearly independent functionals annihilating Y.

Define a linear functional f_m on $\operatorname{span}\{x_n\}$ by $f_m(x_n) = \frac{F_m^{(n)}(0)}{n!}$. Since T has no eigenvectors, the orbit of T is linearly independent, so f_m is well defined.

Let's prove that f_m is bounded for every $m \in \mathbb{N}$. Take any $x := \sum \alpha_n x_n \in \operatorname{span}\{x_n\}$. Using (5), we obtain

$$\bigl|f_m(x)\bigr| = \Bigl|\sum \alpha_n \frac{F_m^{(n)}(0)}{n!}\Bigr| \leqslant C \sum_{n \geqslant m} \frac{|\alpha_n|}{n-m+1}$$
$$= C \sum_{n \geqslant m} \frac{|x_n^*(x)|}{n-m+1} \leqslant C\|x\| \sum_{n \geqslant m} \frac{\|x_n^*\|}{n-m+1}.$$

It suffices to show that $\sum_{n \geqslant m} \frac{\|x_n^*\|}{n-m+1} < \infty$. Note that

$$m(n-m+1) = (m-1)(n-m) + n \geqslant n$$

whenever $n \geqslant m$, so that

$$\sum_{n=m}^{\infty} \frac{\|x_n^*\|}{n-m+1} = m \sum_{n=m}^{\infty} \frac{\|x_n^*\|}{m(n-m+1)} \leqslant m \sum_{n=m}^{\infty} \frac{\|x_n^*\|}{n} < \infty$$

by assumption. Hence, f_m is bounded, so that we can extend it to X. Observe that if $|\lambda| < 1$ and $m \in \mathbb{N}$ then (1) yields that

$$f_m\bigl(h(\lambda, e)\bigr) = f_m\Bigl(\lambda \sum_{n=0}^{\infty} \lambda^n T^n e\Bigr) = \lambda \sum_{n=0}^{\infty} \lambda^n \frac{F_m^{(n)}(0)}{n!} = \lambda F_m(\lambda),$$

Thus $f_m\bigl(h(\lambda_k, e)\bigr) = \lambda_k F_m(\lambda_k) = 0$ for all $m, k \in \mathbb{N}$, hence each f_m annihilates Y.

Finally, the set $\{f_m\}_{m=1}^{\infty}$ is linearly independent. Indeed if it was linearly dependent and a certain linear non-zero linear combination of them vanished, then by writing the Taylor expansion of each F_m on D we see that the same linear combination of F_m's would vanish. This is a contradiction, since the F_m's are linear independent. $\square$

5. Invariant subspaces of operators with many almost invariant half-spaces

Let X be a Banach space and $T\colon X \to X$ be a bounded operator. It is well known that if every subspace of X is invariant under T then T must be a multiple of the identity. In this section we will obtain a result of the same spirit for almost invariant half-spaces.

Proposition 5.1. *Let X be a Banach space and $T \in \mathcal{L}(X)$. Suppose that every half-space of X is almost invariant under T. Then T has a non-trivial invariant subspace of finite codimension. Iterating, one can get a chain of such subspaces.*

Proof. Let's assume that T has no non-trivial invariant subspaces of finite codimension. We will now construct by an inductive procedure a half-space that is not almost invariant under T.

Put $Y_0 = X$. Fix an arbitrary non-zero $z_1 \in X$. Choose $f_1 \in X^*$ such that $f_1(z_1) \neq 0$ and put $Y_1 = \ker f_1$.

Since Y_1 is not invariant under T, there exists $z_2 \in Y_1$ such that $f_1(Tz_2) \neq 0$. Define $g_2 \in Y_1^*$ by $g_2(y) = f_1(Ty)$. Let P_1 be a projection along $\operatorname{span}\{z_1\}$ onto Y_1. Define $f_2 = g_2 \circ P_1 \in X^*$. Now put $Y_2 = \ker g_2 = \ker f_2 \cap Y_1$. Then we have $Y_1 = Y_2 \oplus \operatorname{span}\{z_2\}$. Since $f_1(Ty) = g_2(y) = 0$ for all $y \in Y_2$, we have $TY_2 \subseteq Y_1$.

Continuing inductively with this procedure, we will build sequences (z_n) of vectors, (f_n) of functionals, and (Y_n) of subspaces such that

(i) $z_{n+1} \in Y_n$,

(ii) $Y_{n+1} = \ker f_{n+1} \cap Y_n = \bigcap_{k=1}^{n+1} \ker f_k$,

(iii) $f_{n+1}(y) = f_n(Ty)$ for all $y \in Y_n$,

(iv) $Y_n = Y_{n+1} \oplus \operatorname{span}\{z_{n+1}\}$,

(v) $TY_{n+1} \subseteq Y_n$, and

(vi) $f_n(z_i) = 0 \Leftrightarrow i \neq n$,

for all $n \in \mathbb{N}$. Indeed, suppose we have defined Y_i, z_i, and f_i, $1 \leqslant i \leqslant n$, satisfying (i)–(vi). Define $g_{n+1} \in Y_n^*$ by $g_{n+1}(y) = f_n(Ty)$ and put $f_{n+1} = g_{n+1} \circ P_n \in X^*$ where P_n is a projection along $[z_k]_{k=1}^n$ onto Y_n $\big($take $P_n(x) = x - \sum_{k=1}^n \frac{f_k(x)}{f_k(z_k)} z_k\big)$. Again, there is $z_{n+1} \in Y_n$ such that $f_{n+1}(z_{n+1}) \neq 0$. Put $Y_{n+1} = \ker f_{n+1} \cap Y_n$. Evidently, (i)–(vi) are then satisfied.

It is easily seen that the sequence (z_k) is linearly independent. Put $Z = [z_{2k}]_{k=1}^\infty$. Clearly $\dim Z = \infty$. It is also easy to see that $f_{2k-1}|_Z = 0$ for all $k \in \mathbb{N}$. Thus, Z is actually a half-space.

By assumption of the theorem, there exists F with $\dim F = m < \infty$ such that $TZ \subseteq Z + F$. For each $k \in \mathbb{N}$, pick $u_k \in Z$ and $v_k \in F$ such that $Tz_{2k} = u_k + v_k$. By (iv), we have $z_{2k} \in Y_{2k-1}$. Applying (iii) with $n = 2k-1$, we get $f_{2k}(z_{2k}) = f_{2k-1}(Tz_{2k})$. Now (vi) yields $f_{2k-1}(Tz_{2k}) \neq 0$.

On the other hand, if $1 \leqslant i < k$ then $z_{2k} \in Y_{2i-1}$, so that analogously $f_{2i}(z_{2k}) = f_{2i-1}(Tz_{2k})$. Therefore $f_{2i-1}(Tz_{2k}) = 0$.

Since $f_{2j-1}|_Z = 0$ for all $j \in \mathbb{N}$, we have $f_{2j-1}(u_k) = 0$ for all j and k. Therefore, for all $k \in \mathbb{N}$ and $1 \leqslant i < k$, we have $f_{2k-1}(v_k) \neq 0$ and $f_{2i-1}(v_k) = 0$. This implies, however, that F is infinite dimensional. □

Using a similar technique, we obtain the following result.

Proposition 5.2. *For all $T \in \mathcal{L}(X)$ and $n \in \mathbb{N}$ there exists a subspace Y of X with* $\operatorname{codim} Y = n$ *and a vector $e_Y \in X$ such that $TY \subseteq Y + \operatorname{span}\{e_Y\}$.*

Proof. Proof is by induction on n. For $n = 1$, any hyperplane satisfies the conclusion of the statement. Suppose that the statement is valid for all $k < n$.

Suppose that X contains a subspace Y of codimension $j \leqslant n$ that is invariant under T. If $j = n$ then we are done. If $j < n$ then by the induction assumption we can find $Z \subseteq Y$ such that Z has codimension $n - j$ in Y and $TZ \subseteq Z + [y]$ for some $y \in Y$. Indeed, consider the restriction T' of T to Y. Now we apply the induction assumption to T' and to $n - j$ and produce a subspace $Z \subseteq Y$ invariant under T of codimension j. But then Z has codimension n in X and still $TZ \subseteq Z + [y]$, so that Z satisfies the conclusion.

Therefore, we can assume that Z has no invariant subspaces of codimension $k \leqslant n$. Thus we can use the argument of Proposition 5.1 to show that there exist (finite) sequences of vectors $(z_k)_{k=1}^{n+1}$, functionals $(f_k)_{k=1}^{n}$, and subspaces $(Y_k)_{k=1}^{n}$ such that the conditions (i)–(vi) are satisfied. In particular, we get:

$$Y_n = \bigcap_{k=1}^{n} \ker f_k,$$

and (f_k) are linearly independent, so that $\operatorname{codim} Y_n = n$. Finally, by (vi) and (iv), we have $TY_n \subseteq Y_{n-1} = Y_n + [z_n]$. □

Acknowledgments. We would like to thank Heydar Radjavi for helpful discussions and suggestions. A part of the work on this paper was done while the first and the fourth authors were attending SUMIRFAS in the summer of 2008; we would like to express our thanks to its organizers.

References

[1] C.D. Aliprantis, K.C. Border, *Infinite dimensional analysis. A hitchhiker's guide. Third edition.* Springer, Berlin, 2006.

[2] I. Chalendar, E. Fricain, and J.R. Partington, Overcompleteness of sequences of reproducing kernels in model spaces. *Integral Equations Operator Theory*, 56 (2006), no. 1, 45–56.

[3] P. Enflo, On the invariant subspace problem for Banach spaces, *Acta Math.*, 158 (34) (1987) 213–313.

[4] J. Lindenstrauss, L. Tzafriri, *Classical Banach spaces. I. Sequence spaces.* Ergebnisse der Mathematik und ihrer Grenzgebiete, Vol. 92. Springer-Verlag, Berlin-New York, 1977.
[5] D.J. Newman, H.S. Shapiro, The Taylor coefficients of inner functions, *Michigan J. Math.* 9 (1962) 249–255.
[6] H. Radjavi, P. Rosenthal, *Invariant subspaces, Second edition.* Dover Publications, Inc., Mineola, NY, 2003.
[7] C.J. Read, A solution to the invariant subspace problem, *Bull. London Math. Soc.*, 16 (4) (1984) 337–401.
[8] I. Singer, *Bases in Banach spaces II*, Springer-Verlag, Berlin, 1981.

(G.Androulakis) Department of Mathematics, University of South Carolina, Columbia, SC 29208, USA
E-mail address: giorgis@math.sc.edu

(A. I. Popov, and V. G. Troitsky) Department of Mathematical and Statistical Sciences, University of Alberta, Edmonton, AB, T6G 2G1. Canada
E-mail address: apopov@math.ualberta.ca, vtroitsky@math.ualberta.ca

(A. Tcaciuc) Mathematics and Statistics Department, Grant MacEwan College, Edmonton, Alberta, Canada T5J P2P, Canada
E-mail address: tcaciuc@math.ualberta.ca