WHEN ARE THE NONSTANDARD HULLS OF NORMED LATTICES DISCRETE OR CONTINUOUS?

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ABSTRACT. This note is a nonstandard analysis version of the paper "When are ultrapowers of normed lattices discrete or continuous?" by W. Wnuk and B. Wiatrowski.

In Functional Analysis, the *ultrapower* and the *nonstandard analysis* approaches are equivalent: results obtained by one of these two methods can usually be translated into the other. In this short note, we present nonstandard analysis versions of the main results of [WW06], where they were originally presented in the ultrapower language. We believe that in this new form the ideas of the proofs are more transparent.

Suppose that E is a Archimedean vector lattice. Recall that an element $0 < e \in E$ is said to be ***discrete*** if $0 \leqslant x \leqslant e$ implies that x is a scalar multiple of e or, equivalently, the interval $[0, e]$ doesn't contain two non-zero disjoint vectors (see [LZ71, Theorem 26.4]). We say that E is ***continuous*** if it contains no discrete elements and ***discrete*** if every non-zero positive vector dominates a discrete element or, equivalently, E has a complete disjoint system consisting of discrete elements (see [AB03, p. 40]).

If E is a normed space. We will write ${}^*E$ for the nonstandard extension of E and $\widehat{E}$ for the nonstandard hull of E. We refer the reader to [Em00, Wol97] for terminology and details on nonstandard hulls of normed spaces and normed lattices. We will use the following standard fact (see, e.g., [Tr04, Remark 4]).

Lemma 1. *Suppose that E is a normed lattice and $a, x, b \in {}^*E$ such that $a \leqslant b$ and $\hat{a} \leqslant \hat{x} \leqslant \hat{b}$. Then there exists $y \in {}^*E$ such that $y \approx x$ and $a \leqslant y \leqslant b$.*

The following is a variant of Theorem 2.2 of [WW06]:

Theorem 2. *Let E be a normed lattice. Then the following are equivalent.*

(i) *$\widehat{E}$ is continuous;*

(ii) $\exists \varepsilon > 0\ \forall x \in E_+\ \exists a, b \in [0, x] \quad a \perp b$ *and* $\|a\| \wedge \|b\| \geqslant \varepsilon \|x\|$.

Proof. (i)⇒(ii) Suppose that E fails (ii). Let ε be a positive infinitesimal. Then there exists a vector $x \in {}^*E_+$ such that for all $a, b \in {}^*[0, x]$ with $a \perp b$ we have $\|a\| \wedge \|b\| < \varepsilon \|x\|$.

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Without loss of generality, $\|x\| = 1$. Let $\hat{a}, \hat{b} \in [0, \hat{x}]$ and $\hat{a} \perp \hat{b}$. By Lemma 1, we may assume that $a, b \in {}^*[0, x]$. Furthermore, $\hat{a} \perp \hat{b}$ implies that $a \wedge b \approx 0$. Let $u = a - a \wedge b$ and $v = b - a \wedge b$, then $u, v \in {}^*[0, x]$ and $u \perp v$, so that $\|u\| \wedge \|v\| < \varepsilon$. It follows that either $\|u\|$ or $\|v\|$ is infinitesimal. Say, $u \approx 0$. Then $a = u + a \wedge b$ is infinitesimal as well, so that $\hat{a} = 0$. Thus, $\hat{x}$ is discrete in $\widehat{E}$.

(ii)⇒(i) Suppose that (ii) holds for some (standard) $\varepsilon > 0$. Let $\hat{x} \in \widehat{E}_+$, show that $\hat{x}$ is not discrete. Without loss of generality, $x \in {}^*E_+$ and $\|x\| = 1$. By (ii), we can find $a, b \in {}^*[0, x]$ such that $a \perp b$ and $\|a\| \wedge \|b\| \geqslant \varepsilon$. It follows that neither a nor b is infinitesimal, so that $\hat{a}, \hat{b}$ are two non-zero disjoint elements of $[0, \hat{x}]$. Hence, $\hat{x}$ is not discrete. □

Recall that a normed lattice satisfies the ***Fatou property*** if $0 \leqslant x_\alpha \uparrow x$ implies $\|x_\alpha\| \to \|x\|$, and the σ-Fatou property if $0 \leqslant x_n \uparrow x$ implies $\|x_n\| \to \|x\|$, see, e.g., [AB03]. We will use the following simple lemma.

Lemma 3. *Suppose that E is a normed lattice with the Fatou property and $S \subseteq E_+$ such that $x = \sup S$ exists. Then for every $\varepsilon > 0$ there is a finite subset γ of S such that $\|\sup \gamma\| \geqslant (1 - \varepsilon)\|x\|$. The same is true for countable families if E satisfies the σ-Fatou property.*

Proof. Let Λ be the collection of all finite subsets of S, ordered by inclusion. Clearly, $\sup_{\alpha \in \Lambda} \sup \alpha = x$. Let $x_\alpha = \sup \alpha$, then $(x_\alpha)_{\alpha \in \Lambda}$ is an increasing net and $0 \leqslant x_\alpha \uparrow x$. It follows from the Fatou property that $\|x_\alpha\| \to \|x\|$, so that there exists $\gamma \in \Lambda$ with $\|x_\gamma\| \geqslant (1 - \varepsilon)\|x\|$.

Now suppose that E satisfies σ-Fatou property and $x = \bigvee_{i=1}^\infty x_i$. Let $z_k = \bigvee_{i=1}^k x_i$, then $x_k \leqslant z_k \leqslant x$, so that $x = \bigvee_{k=1}^\infty z_k$. Now σ-Fatou property guarantees that $\|z_k\| \to \|x\|$, so that $(1 - \varepsilon)\|x\| \leqslant \|z_m\| = \|x_1 \vee \cdots \vee x_m\|$ for some m. □

The following is a variant of Theorem 3.2 of [WW06].

Theorem 4. *Let E be a discrete normed lattice, and $\mathcal{D}$ the set of all discrete elements of norm one in E. If E satisfies the Fatou property (or the σ-Fatou property if $\mathcal{D}$ is countable) then the discrete elements of $\widehat{E}$ are exactly the positive scalar multiples of the elements of $\{\hat{e} : e \in {}^*\mathcal{D}\}$.*

Proof. It suffices to show that given $x \in {}^*E$ with $\|x\| = 1$, then $\hat{x}$ is discrete in $\widehat{E}$ if and only if $\hat{x} = \hat{e}$ for some $e \in {}^*\mathcal{D}$. Suppose that $\hat{x} = \hat{e}$ for some $e \in {}^*\mathcal{D}$. Take any $a \in {}^*E$ such that $0 \leqslant \hat{a} \leqslant \hat{x}$. By Lemma 1, we may assume that $0 \leqslant a \leqslant e$. It follows that a is a scalar multiple of e, hence $\hat{a}$ is a scalar multiple of $\hat{x}$.

Conversely, suppose that $\hat{x}$ is discrete in $\widehat{E}$. Note that the set D is a complete disjoint system in E. By [AB03, Theorem 1.75], we have $x = \sup\{P_e x : e \in {}^*\mathcal{D}\}$. For every $e \in {}^*\mathcal{D}$, the vector $P_e x$ is a scalar multiple of e, and $0 \leqslant P_e x \leqslant x$, hence $0 \leqslant \widehat{P_e x} \leqslant \hat{x}$. Therefore, if $P_e x$ is not infinitesimal for some $e \in {}^*\mathcal{D}$ then $\hat{x}$ is a scalar multiple of $\widehat{P_e x}$, hence of $\hat{e}$.

Suppose now that $P_e x$ is infinitesimal for every $e \in {}^*\mathcal{D}$. It follows from $x = \sup\{P_e x : e \in {}^*\mathcal{D}\}$ and Lemma 3 that there exist $n \in {}^*\mathbb{N}$ and $e_1, \dots, e_n \in {}^*\mathcal{D}$ such that $\|z\| \geqslant \frac{3}{4}$, where $z = \big\|P_{e_1} x \vee \dots \vee P_{e_n} x\big\|$. Choose $k \leqslant n$ in ${}^*\mathbb{N}$ so that $\big\|P_{e_1} x \vee \dots \vee P_{e_{k-1}} x\big\| < \frac{1}{4}$, while $\big\|P_{e_1} x \vee \dots \vee P_{e_k} x\big\| \geqslant \frac{1}{4}$. Put $u = P_{e_1} x \vee \dots \vee P_{e_k} x = P_{e_1} x + \dots + P_{e_k} x$. Then

$$\tfrac{1}{4} \leqslant \|u\| \leqslant \big\|P_{e_1} x \vee \dots \vee P_{e_{k-1}} x\big\| + \|P_{e_k} x\| \lesssim \tfrac{1}{4},$$

hence $\|u\| \approx \frac{1}{4}$. Put $v = z - u$, then $u \perp v$, $0 \leqslant u, v \leqslant z$, and $\|u\|, \|v\| \geqslant \frac{1}{4}$. Therefore, $\hat{u}$ and $\hat{v}$ are non-zero and disjoint elements of $[0, \hat{x}]$; a contradiction. □

Corollary 5. *Suppose that E is an AM-space with a strong unit, and H is a discrete regular sublattice of E. Then $\widehat{H}$ is discrete.*

Proof. Let $\mathcal{D}$ be a complete disjoint system of discrete elements of norm one in H. Suppose that $\hat{x} \in \widehat{H}_+$. We will show that $\hat{x}$ majorizes a discrete vector. Without loss of generality, $x \in {}^*H_+$ with $\|x\| = 1$. Then $x = \sup\{P_e x : e \in {}^*\mathcal{D}\}$ by [AB03, Theorem 1.75]. Since E is an AM-space, we can apply Lemma 3 with $\varepsilon \approx 0$ and find $n \in {}^*\mathbb{N}$ and $e_1, \dots, e_n \in {}^*\mathcal{D}$ such that $\big\|P_{e_1} x \vee \dots \vee P_{e_n} x\big\| \geqslant (1 - \varepsilon)\|x\| \approx 1$. Again, since E is an AM-space, we have $\big\|P_{e_1} x \vee \dots \vee P_{e_n} x\big\| = \|P_{e_1} x\| \vee \dots \vee \|P_{e_n} x\|$, so that $\|P_{e_k} x\| \approx 1$ for some $k \leqslant n$. Then $\widehat{P_{e_k} x}$ is non-zero. It is discrete by Theorem 4 because $P_{e_k} x$ is a multiple of e_k. Finally, notice that $\widehat{P_{e_k} x} \leqslant \hat{x}$. □

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