DISJOINTLY HOMOGENEOUS BANACH LATTICES: DUALITY AND COMPLEMENTATION

J. FLORES, F. L. HERNÁNDEZ, E. SPINU, P. TRADACETE, AND V. G. TROITSKY

Abstract. We study several properties of disjointly homogeneous Banach lattices with a special focus on two questions: the self-duality of this class and the existence of disjoint sequences spanning complemented subspaces. Various results around these problems are given. In particular, we provide examples of reflexive disjointly homogeneous spaces whose dual spaces are not disjontly homogeneous.

1. Introduction

This paper is devoted to the study of several properties of disjointly homogeneous Banach lattices. Recall that a Banach lattice E is *disjointly homogeneous* if two arbitrary sequences of normalized pairwise disjoint elements in E always have equivalent subsequences.

This class of spaces was first introduced in [10]. More recently, in [9], the general problem of obtaining compactness of the iterations of a strictly singular operator T on a Banach lattice E was considered. The motivation behind the study of this problem was a classical result by V. D. Milman ([21]) which states that strictly singular operators in $L_p(\mu)$, $1 \leq p \leq \infty$, have compact square. The key property needed for the results in [9] is that the space E is disjointly homogeneus (see [9, Thm. 2.9]).

Remarkably, $L_p(\mu)$ spaces exhibit a particularly strong version of disjoint homogeneity as every normalized disjoint sequence is equivalent to the unit vector basis of ℓ_p. Actually, this property characterizes L_p spaces (see also Proposition 2.2 and the explanations before it). Examples of disjointly homogeneous Banach lattices also include Lorentz spaces of the form $\Lambda(W,p)[0,1]$, for which every normalized disjoint sequence has a subsequence equivalent to the unit vector basis of ℓ_p (see [8]). Among other examples of disjointly homogeneus Banach lattices we can find Tsirelson and related spaces (see [10] and the

Date: December 15, 2013.
2010 *Mathematics Subject Classification.* Primary: 46B42. Secondary: 46E30 .
Key words and phrases. Banach lattice, disjoint sequence, Orlicz space, complementation.

last section of this paper). Moreover, a characterization of disjointly homogeneous Orlicz spaces $L_\varphi[0,1]$ was given in [9]: there exists some $1 \leqslant p < \infty$ such that every function in the set

$$E_\varphi^\infty = \bigcap_{s>0} \overline{\left\{ \tfrac{\varphi(rt)}{\varphi(r)} \ : \ r \geqslant s \right\}} \subset C(0,1)$$

is equivalent to the function t^p. In particular, Orlicz functions of the form $\varphi(t) = t^p \log^\alpha(1+t)$ yield more examples of disjointly homogeneous Orlicz spaces $L_\varphi[0,1]$.

The paper addresses two particular aspects concerning disjointly homogeneous spaces. Firstly, we consider the question of the self-duality of this property. In other words, is the dual E^* disjointly homogeneous if and only if E is? Secondly, we are interested in the existence of complemented subspaces generated by disjoint sequences.

Regarding the first aspect, some partial results in this direction were given in [10] : If E is ∞-disjointly homogeneous (i.e., every normalized pairwise disjoint sequence has a subsequence equivalent to the unit vector basis of c_0), then E^* is 1-disjointly homogenoeus (i.e., every normalized pairwise disjoint sequence has a subsequence equivalent to the unit vector basis of ℓ_1). However, the converse was shown untrue: the Lorentz space $L_{p,1}(0,1)$ is 1-disjointly homogeneous, while its dual, $L_{p',\infty}$, is not disjointly homogeneous.

Therefore, in the pursuit of positive results concerning the self duality of disjointly homogeneous Banach lattices, we will restrict to the reflexive case. The main obstacle we recurrently encounter is the bad behaviour of the Hahn-Banach extensions of the biorthogonal functionals of a given disjoint basic sequence in the space. Actually, we will present examples of reflexive disjointly homogeneous Orlicz spaces on $(0,\infty)$ whose dual spaces are not disjointly homogeneous, answering the above question in the negative. To do this, we first need to develop a criterion for an Orlicz space on $(0,\infty)$ to be disjointly homogeneous, in the spirit of [9, Theorem 4.1] for probability spaces (see Theorem 5.1).

Nevertheless, in an attempt to obtain partial positive results we isolate a property $\mathfrak{P}$ on the Banach lattice E (see Definition 3.1), which ensures that if E^* is disjointly homogeneous then E must also be. It turns out that property $\mathfrak{P}$ is very much connected with the second aspect considered in our study: the existence of disjoint sequences spanning complemented subspaces. In fact, among disjointly homogenous spaces, property $\mathfrak{P}$ characterizes those spaces which have a positive disjoint sequence spanning a complemented subspace.

Apparently, it is unkown whether an arbitrary Banach lattice always contains a positive disjoint sequence whose span is complemented. For a wide class of Banach lattices, such as rearrangement invariant spaces or those with an infinite atomic part, such a sequence always exists. But, to our knowledge, there is no criterion to decide what lattices have this property, if not all of them. However, as noticed above, in the class of disjontly homogenous Banach lattices a criterion can be given using property $\mathfrak{P}$ (see Theorem 4.4). In addition to this characterization we show that every disjoint sequence of a disjointly homogenous non-reflexive Banach lattice E has a subsequence whose span is complemented in E. Similarly, for $1 < p < \infty$, if a Banach lattice E is p-disjointly homogeneous (that is, every normalized disjoint sequence in E has a subsequence equivalent to the unit basis of ℓ_p), then every such subsequence is complemented under the additional assumption that E is p-convex.

Thus, a natural question arises: when does every disjoint sequence in E have a subsequence whose span is complemented? We introduce the class of *disjointly complemented* Banach lattices as those which meet that requirement. This class includes, beyond L_p-spaces, Lorentz spaces $\Lambda(W,p)[0,1]$ and some Orlicz spaces. We explore some connections between disjointly homogeneous and disjointly complemented spaces and obtain some partial results.

The paper is organized as follows. After some preliminaries and discussion on the basic notions of the paper, in Section 3 we present positive results concerning the stability under duality for the class of disjointly homogeneous Banach lattices (see Proposition 3.4 and Theorem 3.5.) Section 4 is devoted to the study of disjointly complemented Banach lattices and results on complemented disjoint sequences. In Section 5 we will present examples of reflexive p-disjointly homogeneous Banach lattices whose dual spaces are not disjointly homogeneous. Finally, the last sectionis devoted to other definitions close to that of disjointly homogeneous, and the connections between them. In particular, when the p-sum of p-disjointly homogeneous spaces is considered, it becomes clear that the constant in the definition of a disjointly homogeneous space cannot be overlooked, and this motivates the notion of *uniformly disjointly homogeneous* Banach lattice. Some connections are also established between disjointly homogeneous Banach lattices and the positive Schur property as well as with spaces having a Rosenthal basis.

Acknowledgments: The research of J. Flores, F.L. Hernández and P. Tradacete has been supported by Spanish grants MTM2008-02652, MTM2012-31286 and Grupo UCM 910346. P. Tradacete has also been partially supported by MTM2010-14946. V.G. Troitsky was supported by NSERC.

2. Basic definitions and preliminaries

We follow standard terminology concerning Banach spaces and Banach lattices as in the monographs [17, 18, 20].

For a sequence (x_n) in a Banach space, we write $[x_n]$ for the closed linear span of the sequence. Given basic sequences (x_n), (y_n), and $C > 0$, the notation $(x_n) \overset{C}{\sim} (y_n)$ means that for every scalars $(a_n)_{n=1}^{\infty}$

$$C^{-1}\Big\|\sum_{n=1}^{\infty} a_n y_n\Big\| \leqslant \Big\|\sum_{n=1}^{\infty} a_n x_n\Big\| \leqslant C\Big\|\sum_{n=1}^{\infty} a_n y_n\Big\|.$$

The notion of disjointly homogeneous Banach lattice has been previously considered in [9] and [10]. Let us recall its definition.

Definition 2.1. A Banach lattice E is ***disjointly homogeneous (DH)*** if for every pair (x_n), (y_n) of normalized disjoint sequences in E, there exist $C > 0$ and a common subsequence (n_k) such that $(x_{n_k}) \overset{C}{\sim} (y_{n_k})$.

Notice that in this definition it is equivalent to consider only positive disjoint normalized (or even semi-normalized) sequences.

Our interest here will mainly focus on Banach lattices for which there is $1 \leq p < \infty$ such that every normalized disjoint sequence has a subsequence equivalent to the unit vector basis of ℓ_p (respectively c_0). These form an important class of DH spaces, which will be denoted p***-disjointly homogeneous***, in short p-DH (resp. ∞-disjointly homogeneous, ∞-DH).

This class of spaces includes the spaces $L_p(\mu)$, Lorentz spaces of the form $L_{p,q}(\mu)$ or $\Lambda(W,p)(\mu)$, and some Orlicz spaces $L_\varphi(\mu)$ (see [9] for details).

Observe that in the definition of DH Banach lattice it is important to allow the possibility of passing to subsequences in order to get the required equivalence. Otherwise, the class of spaces reduces to the spaces $L_p(\mu)$ or $c_0(\Gamma)$, as the following shows.

Proposition 2.2. *Let E be an order continuous Banach lattice for which every pair of normalized disjoint sequences are equivalent. Then E is order isomorphic to $L_p(\mu)$ (for some $1 \leqslant p < \infty$) or $c_0(\Gamma)$.*

Proof. By [18, Lemma 1.b.13], it is enough to show that every normalized disjoint sequence in E is equivalent to the unit vector basis of ℓ_p or c_0. Indeed, let (x_n) be a normalized disjoint sequence in E. If (y_n) is a normalized block sequence of (x_n), then by hypothesis it must be equivalent to (x_n). Hence, (x_n) is equivalent to all its normalized block sequences, so by Zippin's theorem [17, Theorem 2.a.9], it follows that (x_n) is equivalent to the unit basis of ℓ_p or c_0 as desired. □

Recall that every reflexive Banach lattice is order continuous. We will routinely use the following standard representation technique.

Remark 2.3. Recall that an order continuous Banach lattice E with a weak unit can be represented as a dense order ideal in $L_1(\mu)$ for some probability measure μ [18, Theorem 1.b.14], so we can consider E as a Köthe function space. Suppose that (x_n) is a sequence in an order continuous Banach lattice E. Let $u = \sum_{n=1}^{\infty} \frac{x_n}{2^n \|x_n\|}$, then the closed ideal B_u generated by u is a projection band in E, $(x_n) \subset B_u$, and u is a weak unit in B_u; hence B_u can be represented as in [18, Theorem 1.b.14]. Thus, every sequence in E is contained in some Köthe function space. Furthermore, B_u^* is a projection band in E^*, and if E (or E^*) is disjointly homogeneous then so is B_u (resp., B_u^*).

We will also use the following standard fact.

Lemma 2.4. *Suppose that X is a reflexive Banach space, (x_n) and (x_n^*) are two basic sequences in X and X^*, respectively, and $T \in L(X)$ is such that $Tx = \sum_{n=1}^{\infty} x_n^*(x)x_n$ for every $x \in X$. Then*

(i) *$T^*x^* = \sum_{n=1}^{\infty} x^*(x_n)x_n^*$ for each $x^* \in X^*$ (in particular, the series converges in X^*);*

(ii) *If (y_n) and (y_n^*) are two basic sequences in X and X^*, respectively, such that $(x_n) \overset{C_1}{\sim} (y_n)$ and $(x_n^*) \overset{C_2}{\sim} (y_n^*)$ with constants $C_1, C_2 > 0$, then $Sx := \sum_{n=1}^{\infty} y_n^*(x)y_n$ converges for every $x \in X$ and $\|S\| \leqslant C_1 C_2 \|T\|$.*

Proof. (i) Fix $x^* \in X^*$. For every $m \in \mathbb{N}$ and $x \in X$, we have

$$\Bigl|\bigl\langle\sum_{n=1}^m x^*(x_n)x_n^*, x\bigr\rangle\Bigr| = \Bigl|\sum_{n=1}^m x^*(x_n)x_n^*(x)\Bigr| = \Bigl|\bigl\langle x^*, \sum_{n=1}^m x_n^*(x)x_n\bigr\rangle\Bigr|$$
$$\leqslant \|x^*\|\Bigl\|\sum_{n=1}^m x_n^*(x)x_n\Bigr\| \leqslant K\|x^*\|\bigl\|Tx\bigr\| \leqslant K\|x^*\|\|T\|\|x\|,$$

where K is the basis constant of (x_n). It follows that $\Bigl\|\sum_{n=1}^m x^*(x_n)x_n^*\Bigr\| \leqslant K\|x^*\|\|T\|$ for each m. Since X is reflexive, (x_n^*) is boundedly complete, hence $\sum_{n=1}^\infty x^*(x_n)x_n^*$ converges; denote it Rx^*. For every $x \in X$, we have

$$\langle Rx^*, x\rangle = \sum_{n=1}^\infty x^*(x_n)x_n^*(x) = \langle x^*, Tx\rangle,$$

so that $R = T^*$.

(ii) It follows from the assumptions and from (i) that the following operators are bounded on E and E^*, respectively

$$x \mapsto \sum_{n=1}^\infty x_n^*(x)y_n, \quad x^* \mapsto \sum_{n=1}^\infty x^*(y_n)x_n^*, \quad x^* \mapsto \sum_{n=1}^\infty x^*(y_n)y_n^*, \text{ and } x \mapsto \sum_{n=1}^\infty y_n^*(x)y_n.$$

The norm estimate follows from the same chain. □

3. Stability under duality: positive results

In this section we present positive results concerning the stability under duality of reflexive DH Banach lattices.

We have mentioned in the introduction that $L_p(\mu)$ spaces are the simplest example of DH Banach lattices since every normalized disjoint sequence is equivalent to the unit basis of ℓ_p. It is clear that for these spaces, the dual is also DH. A similar fact holds for the Lorentz spaces $\Lambda(W,q)$ and $L_{p,q}$ (see [9]), since in these spaces every normalized disjoint sequence has a subsequence equivalent to the unit basis of ℓ_q.

It was proved in [9] that an Orlicz space $L_\varphi[0,1]$ is disjointly homogeneous if and only if every function in the associated set

$$E_\varphi^\infty = \bigcap_{s>0} \overline{\bigl\{\tfrac{\varphi(rt)}{\varphi(r)} \;:\; r \geqslant s\bigr\}} \subset C(0,1)$$

is equivalent to the function t^p for some fixed $1 \leqslant p < \infty$. In this case, it is immediate that for the conjugate Orlicz function φ', every function in $E^\infty_{\varphi'}$ is equivalent to the function $t^{p'}$, which is tantamount to the space $L_\varphi[0,1]^* = L_{\varphi'}[0,1]$ being disjointly homogeneous.

Therefore, for all the examples given above, we have stability under duality for the DH property. Let us give now some motivation for the general case. Suppose that E is a reflexive Banach lattice such that E^* is DH. Given two disjoint normalized sequences (x_n) and (y_n) in E, we would like to find subsequences that are equivalent. One can find two disjoint normalized sequences (x_n^*) and (y_n^*) in E^* such that $x_n^*(x_m) = y_n^*(y_m) = \delta_{nm}$ for each $n, m \in \mathbb{N}$. Since E^* is DH, after passing to subsequences we may assume that (x_n^*) and (y_n^*) are equivalent in E^*. On the other hand, for each m, we can consider x_m^* as a functional on $[x_n]$ (formally speaking, we are taking the restriction of x_m^* to $[x_n]$); moreover, since E is reflexive, (x_m^*) is a basis of $[x_n]^*$. Then for any coefficients $\alpha_1, \ldots, \alpha_m$ we have

$$\begin{aligned}\Big\|\sum_{i=1}^m \alpha_i x_i\Big\| &= \sup\Big\{\Big|\Big\langle\sum_{i=1}^m \alpha_i x_i, \sum_{i=1}^m \beta_i x_i^*\Big\rangle\Big| \,:\, \Big\|\sum_{i=1}^m \beta_i x_i^*\Big\|_{[x_n]^*} \leqslant 1\Big\}\\ &= \sup\Big\{\Big|\sum_{i=1}^m \alpha_i\beta_i\Big| \,:\, \Big\|\sum_{i=1}^m \beta_i x_i^*\Big\|_{[x_n]^*} \leqslant 1\Big\}.\end{aligned}$$

In general, clearly $\big\|\sum_{i=1}^m \beta_i x_i^*\big\|_{[x_n]^*} \leqslant \big\|\sum_{i=1}^m \beta_i x_i^*\big\|_{E^*}$. However, if we could somehow control the converse estimate, we could continue, using the equivalence of (x_n^*) and (y_n^*) in E^*:

$$\begin{aligned}\Big\|\sum_{i=1}^m \alpha_i x_i\Big\| &\sim \sup\Big\{\Big|\sum_{i=1}^m \alpha_i\beta_i\Big| \,:\, \Big\|\sum_{i=1}^m \beta_i x_i^*\Big\|_{E^*} \leqslant 1\Big\}\\ &\sim \sup\Big\{\Big|\sum_{i=1}^m \alpha_i\beta_i\Big| \,:\, \Big\|\sum_{i=1}^m \beta_i y_i^*\Big\|_{E^*} \leqslant 1\Big\} \sim \Big\|\sum_{i=1}^m \alpha_i y_i\Big\|,\end{aligned}$$

which would imply that (x_n) and (y_n) are equivalent. In particular, such an argument would work if we could find a bounded operator $S\colon [x_n]^* \to E^*$ such that $Sx_m^* = x_m^*$ for each m and a similar operator for (y_n). This observation motivates the following definition.

Definition 3.1. A Banach lattice E has ***property*** $\mathfrak{P}$ if for every disjoint positive normalized sequence $(f_n) \subset E$ there exists an operator $T : E \to [f_n]$, such that some subsequence $(T^*f^*_{n_k})$ is equivalent to a seminormalized disjoint sequence in E^* (here (f^*_n) denote the corresponding biorthogonal functionals in $[f_n]^*$).

Remark 3.2. Given a disjoint sequence (f_n) as in the above definition, we can consider $Px = \sum_{k=1}^\infty f^*_{n_k}(x)f_{n_k}$, the canonical projection from $[f_n]$ onto $[f_{n_k}]$ (which has $\|P\| = 1$ because (f_n) is 1-unconditional). If E has property $\mathfrak{P}$, then we can now view

$$PTx = \sum_{k=1}^\infty f^*_{n_k}(Tx)f_{n_k} = \sum_{k=1}^\infty \big(T^*f^*_{n_k}\big)(x)f_{n_k}$$

as a bounded operator on E.

Next, we provide several equivalent characterizations of property $\mathfrak{P}$.

Proposition 3.3. *Let E be a reflexive Banach lattice. The following are equivalent:*

(i) *For every disjoint positive normalized sequence $(f_n) \subset E$ there exists a positive operator $T : E \to [f_n]$, with $\liminf_n \operatorname{dist}\big(f_n, T(B_E)\big) < 1$.*

(ii) *For every disjoint positive normalized sequence $(f_n) \subset E$ there exists a positive operator $T : E \to [f_n]$, such that $\|T^*f^*_n\| \nrightarrow 0$.*

(iii) *E has property $\mathfrak{P}$.*

Proof. (i)$\Rightarrow$(ii): Indeed, let (f_n) be a positive normalized disjoint sequence in E and suppose (i) holds for some positive operator T. Hence, we can consider a sequence (x_k) in B_E, and a subsequence (f_{n_k}) such that $\|f_{n_k} - Tx_k\| \leqslant \alpha$ for some $\alpha < 1$. Now, for every $k \in \mathbb{N}$ it follows that

$$\begin{aligned}\|T^*f^*_{n_k}\| &= \sup\big\{\langle T^*f^*_{n_k}, x\rangle \ : \ x \in B_E\big\} \geqslant \langle f^*_{n_k}, Tx_k\rangle \\ &= \langle f^*_{n_k}, f_{n_k}\rangle - \langle f^*_{n_k}, f_{n_k} - Tx_k\rangle \geqslant 1 - \|f^*_{n_k}\|\|f_{n_k} - Tx_k\| \geqslant 1 - \alpha > 0.\end{aligned}$$

Thus, it follows that (ii) holds.

(ii)$\Rightarrow$(iii): Suppose that (f_n) is a positive normalized disjoint sequence in E. As in Remark 2.3 we may assume that E is a Köthe function space. Consider the biorthogonal functionals (f^*_n) which are a basis of the dual $[f_n]^*$. Let T be as in (ii). Since (f^*_n) is weakly null and $T^* : [f_n]^* \to E^*$ is positive, it follows that the sequence $(T^*f^*_n)$ is weakly null and positive. In particular, we have that $\|T^*f^*_n\|_{L_1} \to 0$. By hypothesis, $\|T^*f^*_n\| \nrightarrow 0$, it follows, by Kadec-Pełczyński dichotomy [18, Prop. 1.c.8], that some subsequence of $(T^* f_n^*)$ is equivalent to a seminormalized disjoint subsequence. Hence E has property $\mathfrak{P}$.

(iii)$\Rightarrow$(i): Given a sequence of normalized positive disjoint elements (f_n) in E, let $Y = [f_n]$. By hypothesis there is an operator $T : E \to Y$ such that $(T^* f_{n_k}^*)$ is equivalent to some disjoint seminormalized sequence (h_k^*) in E^*. Replacing h_k^* with $|h_k^*|$ if necessary, we may assume $h_k^* \geqslant 0$. Passing to a subsequence and scaling, we may assume, in addition, that $\|h_k^*\| = 1$.

Combining Remark 3.2 with Lemma 2.4(ii), we conclude that $Rx = \sum_{k=1}^\infty h_k^*(x) f_{n_k}$ defines a bounded operator on E. For each $k \in \mathbb{N}$, pick $h_k \in E_+$ with $\|h_k\| \leqslant 1$, $h_k^*(h_k) \geqslant \|h_k^*\|/2$ and $h_k^*(h_j) = 0$ for $j \neq k$. Hence, we have that

$$\liminf_n \operatorname{dist}(f_n, R(B_E)) \leqslant \lim_k \|f_{n_k} - Rh_k\| \leqslant \lim_k \big(1 - \|h_k^*\|/2\big) < 1.$$

Thus, R satisfies the required conditions for (i). $\square$

Let us see now a partial positive result for the stability under duality of DH property.

Proposition 3.4. *Let E be a reflexive Banach lattice satisfying an upper p-estimate. If E^* is q-DH (with $\frac{1}{p} + \frac{1}{q} = 1$), then E is p-DH.*

Proof. Let (x_n) be a disjoint normalized sequence in E and let us see that it has a subsequence equivalent to the unit vector basis of ℓ_p. To this end, let us choose a disjoint normalized sequence (x_n^*) in E^* such that $x_n^*(x_m) = \delta_{nm}$ for all $n, m \in \mathbb{N}$. Since E^* is q-DH, we may assume, after passing to a subsequence, that the sequence (x_n^*) is C-equivalent to the unit vector basis of ℓ_q. Given any coefficients $\alpha_1, \dots, \alpha_m$, since E has an upper p-estimate, we have $\big\|\sum_{n=1}^m \alpha_n x_n\big\| \leqslant M\big(\sum_{n=1}^m |\alpha_n|^p\big)^{1/p}$. On the other hand,

$$\begin{aligned}
\Big\|\sum_{n=1}^m \alpha_n x_n\Big\| &= \sup\Big\{\big\langle \sum_{n=1}^m \alpha_n x_n, x^*\big\rangle \;:\; x^* \in B_{E^*}\Big\} \\
&\geqslant \sup\Big\{\big\langle \sum_{n=1}^m \alpha_n x_n, \sum_{n=1}^m \beta_n x_n^*\big\rangle \;:\; \big\|\sum_{n=1}^m \beta_n x_n^*\big\| \leqslant 1\Big\} \\
&\geqslant C \sup\Big\{\sum_{n=1}^m \alpha_n \beta_n \;:\; \Big(\sum_{n=1}^m |\beta_n|^q\Big)^{1/q} \leqslant 1\Big\} \\
&= C\Big(\sum_{n=1}^m |\alpha_n|^p\Big)^{1/p}.
\end{aligned}$$

$\square$

In particular, if a reflexive Banach lattice E is p-DH and satisfies a lower p-estimate, for some $1 < p < \infty$, then E^* is q-DH (with $\frac{1}{p} + \frac{1}{q} = 1$).

We provide next a duality result for Banach lattices with property $\mathfrak{P}$ which generalizes in a certain sense the previous result. In particular, this can be applied to Banach lattices in which every disjoint positive sequence has a subsequence whose span is complemented by a positive projection (see Section 4). Examples of these include L_p spaces, Lorentz function spaces $\Lambda(W, p)$ for $p < \infty$, Tsirelson's space, etc.

Theorem 3.5. *Let E be a reflexive Banach lattice with property $\mathfrak{P}$. If E^* is DH, then E is DH. Moreover, in the particular case when E^* is p-DH, for some $1 < p < \infty$, then E is q-DH with $\frac{1}{p} + \frac{1}{q} = 1$.*

Proof. Let (x_n) and (y_n) be two disjoint normalized sequences in E. As in Remark 2.3, we may assume that E is a Köthe function space. Let (x_n^*) be the biorthogonal functionals to (x_n) in $[x_n]^*$, that is, $x_n^*(x_m) = \delta_{mn}$. By hypothesis, there exists an operator $T : E \to [x_n]$ such that some subsequence $(T^* x_{n_k}^*)$ is equivalent to a seminormalized disjoint subsequence (g_k^*).

Using the Köthe function space representation of E, we can find a normalized disjoint sequence (h_k^*) in E^* which is biorthogonal to (y_{n_k}), that is, $h_m^*(y_{n_k}) = \delta_{mk}$. Since E^* is DH, there is a subsequence (k_j) such that the sequences $(g_{k_j}^*)$ and $(h_{k_j}^*)$ are equivalent. It follows that $(T^* x_{n_{k_j}}^*) \overset{C}{\sim} (h_{k_j}^*)$ for some constant $C > 0$.

Given scalars $\alpha_1,\dots,\alpha_m$, following the argument at the beginning of the section we get

$$\begin{aligned}
\Big\|\sum_{j=1}^m \alpha_j x_{n_{k_j}}\Big\| &= \sup\Big\{\Big|\langle\sum_{j=1}^m \alpha_j x_{n_{k_j}},\sum_{j=1}^m \beta_j x^*_{n_{k_j}}\rangle\Big| \,:\, \|\sum_{j=1}^m \beta_j x^*_{n_{k_j}}\|_{[x_n]^*}\leqslant 1\Big\}\\
&= \sup\Big\{\Big|\sum_{j=1}^m \alpha_j\beta_j\Big| \,:\, \|\sum_{j=1}^m \beta_j x^*_{n_{k_j}}\|_{[x_n]^*}\leqslant 1\Big\}\\
&\leqslant \sup\Big\{\Big|\sum_{j=1}^m \alpha_j\beta_j\Big| \,:\, \|\sum_{j=1}^m \beta_j T^* x^*_{n_{k_j}}\|_{E^*}\leqslant \|T^*\|\Big\}\\
&\leqslant C\|T^*\|\sup\Big\{\Big|\sum_{j=1}^m \alpha_j\beta_j\Big| \,:\, \|\sum_{j=1}^m \beta_j h^*_{k_j}\|\leqslant 1\Big\}\\
&= C\|T^*\|\sup\Big\{\Big|\langle\sum_{j=1}^m \alpha_j y_{n_{k_j}},\sum_{j=1}^m \beta_j h^*_{k_j}\rangle\Big| \,:\, \|\sum_{j=1}^m \beta_j h^*_{k_j}\|\leqslant 1\Big\}\\
&\leqslant C\|T^*\|\Big\|\sum_{j=1}^m \alpha_j y_{n_{k_j}}\Big\|.
\end{aligned}$$

An analogous argument starting now with $(y_{n_{k_j}})$ and $(x_{n_{k_j}})$ yields a further subsequence $(n_{k_{j_i}})$ such that $\|\sum_{i=1}^m \alpha_i y_{n_{k_{j_i}}}\|\leqslant C'\|\sum_{i=1}^m \alpha_i x_{n_{k_{j_i}}}\|$ for certain further subsequences and some constant $C'>0$ independent on the scalars $\alpha_1,\dots,\alpha_m$. Therefore, $(x_{n_{k_{j_i}}})$ and $(y_{n_{k_{j_i}}})$ are equivalent, which proves that E is DH.

Notice that in the particular case when E^* is p-DH, then in the above chain of inequalities, the disjoint sequence $(h^*_{k_j})$ can be taken to be equivalent to the unit vector basis of ℓ_p, thus we have:

$$\sup\Big\{\Big|\sum_{j=1}^m \alpha_j\beta_j\Big| \,:\, \Big\|\sum_{j=1}^m \beta_j h^*_{k_j}\Big\|\leqslant 1\Big\}\approx\Big(\sum_{j=1}^m|\alpha_j|^q\Big)^{1/q}.$$

In this case, the argument above shows that E is q-DH as claimed. □

4. Complemented disjoint sequences

Recall that a sequence (x_n) is said to be complemented in E if there is a projection P on E with $\operatorname{Range} P=[x_n]$.

Notice that if P is a positive projection onto the span of a disjoint sequence $(x_n)\subset E$, and (x_n^*) denote the biorthogonal functionals, then, in general, the sequence $(P^*x_n^*)$ need not be disjoint in E^*:

Example 4.1. Take $E = \mathbb{R}^3$ and let

$$x_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad x_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \quad \text{and} \quad P = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}.$$

Note that $Pe_1 = x_1$, $Pe_2 = x_2$, and $Pe_3 = x_1 + x_2$. It follows from $(P^*x_n^*)_i = \langle P^*x_n^*, e_i \rangle = \langle x_n^*, Pe_i \rangle$ that

$$P^*x_1^* = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \quad \text{and} \quad P^*x_2^* = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix},$$

so that $P^*x_1^*$ and $P^*x_2^*$ are not disjoint.

Nevertheless, if a disjoint positive sequence spans a complemented subspace, then we will see that one can always find a positive projection whose adjoint sends the biorthogonal functionals to a disjoint sequence.

Proposition 4.2. *Let E be a reflexive Banach lattice, (f_n) a positive disjoint sequence, and $R \in \mathcal{L}(E)$ a projection onto $[f_n]$. Then there exists a positive disjoint sequence (g_n^*) in E^* with $\langle g_n^*, f_m \rangle = \delta_{n,m}$ such that the operator $Px = \sum_{n=1}^\infty g_n^*(x) f_n$ defines a positive projection onto $[f_n]$ with $\|P\| \leqslant \|R\|$.*

Proof. For $x \in E$, we can write $Rx = \sum_{n=1}^\infty h_n^*(x) f_n$, where $h_n^* \in E^*$ satisfy $\langle h_n^*, f_m \rangle = \delta_{n,m}$. Let P_n denote the band projection onto the band generated by f_n. We claim that $Qx = \sum_{n=1}^\infty P_n^* h_n^*(x) f_n$ defines a bounded operator $Q \in \mathcal{L}(E)$ with $\|Q\| \leqslant \|R\|$.

Indeed, let P_n^d denote the projection onto the orthogonal band to f_n. Consider the operators $R_1 = (P_1 - P_1^d) R (P_1 - P_1^d)$ and $Q_1 = (R + R_1)/2$. Since $P_1 - P_1^d$ is an isometry, we have $\|Q_1\| \leqslant \|R\|$. It follows from $Q_1 = P_1 R P_1 + P_1^d R P_1^d$ that

$$Q_1 x = P_1^* h_1^*(x) f_1 + \sum_{n=2}^\infty h_n^*(P_1^d x) f_n$$

for every $x \in E$. Proceeding inductively, for $n > 1$, we consider $R_n = (P_n - P_n^d) Q_{n-1} (P_n - P_n^d)$, and $Q_n = (Q_{n-1} + R_n)/2$ which is given by

$$Q_n x = \sum_{k=1}^n P_k^* h_k^*(x) f_k + \sum_{k>n} h_k^*(P_n^d \dots P_1^d x) f_k$$

and $\|Q_n\| \leqslant \|R\|$. In particular,

$$\Big\| \sum_{k=1}^n P_k^* h_k^*(x) f_k \Big\| \leqslant \|Q_n x\| \leqslant \|R\|$$

for every n and every $x \in B_E$. Since (f_n) is a disjoint sequence and E is reflexive, it follows that the series $\sum_{k=1}^{\infty} P_k^* h_k^*(x) f_k$ converges and $\left\| \sum_{k=1}^{\infty} P_k^* h_k^*(x) f_k \right\| \leqslant \|R\|$. This proves the claim.

Since the sequence $(P_n^* h_n^*)$ is disjoint, it is 1-equivalent to $\left(|P_n^* h_n^*|\right)$ in E^*. It follows from Lemma 2.4(ii) that $Sx = \sum_{n=1}^{\infty} |P_n^* h_n^*|(x) f_n$ defines a bounded operator on E with $\|S\| \leqslant \|R\|$. Note that for each n we have $\left\langle |P_n^* h_n^*|, f_n \right\rangle \geqslant \left\langle P_n^* h_n^*, f_n \right\rangle = 1$. Put

$$g_n^* = \frac{1}{\left\langle |P_n^* h_n^*|, f_n \right\rangle} |P_n^* h_n^*|.$$

Then (g_n^*) is a positive disjoint sequence in E^* with $\langle g_n^*, f_m \rangle = \delta_{mn}$ and $0 \leqslant g_n^* \leqslant |P_n^* h_n^*|$. It follows that $Px = \sum_{n=1}^{\infty} g_n^*(x) f_n$ defines a positive operator with $P \leqslant S$; hence $\|P\| \leqslant \|R\|$. It is easy to see that P is a projection onto $[f_n]$. $\square$

Corollary 4.3. *Given a positive disjoint sequence (e_n) in a reflexive Banach lattice E, the following are equivalent:*

(i) *The subspace $[e_n]$ is complemented in E.*

(ii) *There exists a disjoint positive sequence (e_n^*) in E^* with $\langle e_n^*, e_m \rangle = \delta_{mn}$ such that $\sum_{n=1}^{\infty} e_n^*(x) e_n$ converges for each $x \in E$. In this case, the map $P\colon x \mapsto \sum_{n=1}^{\infty} e_n^*(x) e_n$ defines a positive projection from E onto $[e_n]$.*

Note also that if the sequence (e_n) is normalized then for each n we have $\|e_n^*\| \geqslant |e_n^*(e_n)| = 1$ and $|e_n^*(x)| = \left\| e_n^*(x) e_n \right\| \leqslant \|Px\| \leqslant \|P\| \|x\|$, so that the sequence (e_n^*) is seminormalized.

Question. Does every reflexive Banach lattice contain a complemented positive disjoint sequence?

This question has a positive answer for most examples of Banach lattices considered in the literature. For instance, let E be a Banach lattice with a Schauder basis such that the order is compatible with the basis. It is easy to see that such a basis is 1-unconditional, and it is clear in this case that E has a positive disjoint complemented sequence.

Another family of spaces satisfying the same property is that of rearrangement invariant spaces. Using the averaging projection, it is well known that any sequence of normalized positive characteristic functions over a family of disjoint sets is complemented in any rearrangement invariant space (cf. [18, Theorem 2.a.4]).

On the other hand, it is well known that in a non-atomic order continuous Banach lattice E, every unconditional basic sequence (u_n) spanning a complemented subspace is equivalent to a disjoint sequence (f_n) spanning also a complemented subspace provided, that $[u_n]$ is lattice anti-euclidean (that is, $[u_n]$ does not contain uniformly complemented lattice copies of ℓ_2^n for every n, see [3, Theorem 3.4]).

Notice also that according to Corollary 4.3 and Lemma 2.4(i), it follows that a reflexive Banach lattice E contains a complemented positive disjoint sequence if and only if E^* does.

Theorem 4.4. *Let E be a DH Banach lattice. E has property $\mathfrak{P}$ if and only if E contains a complemented positive disjoint sequence.*

Proof. Suppose first that E contains a complemented positive disjoint sequence (e_n). Now, let (e_n^*), and P be as in Corollary 4.3, and let (f_n) be a normalized disjoint sequence in E. Passing to subsequences, we may assume that (f_n) is equivalent to (e_n), so that $Re_n = f_n$ defines an isomorphism operator from $[e_n]$ onto $[f_n]$. Put $T = RP$. Then $Tx = \sum_{n=1}^{\infty} e_n^*(x) f_n$ for every $x \in E$. It follows that $T^*x^* = \sum_{n=1}^{\infty} x^*(f_n) e_n^*$ for each $x^* \in [f_n]^*$. In particular, $T^* f_n^* = e_n^*$. Therefore, E satisfies property $\mathfrak{P}$.

Conversely, suppose E has property $\mathfrak{P}$ and pick a normalized disjoint sequence (f_n) in E_+. By hypothesis, there is an operator $T : E \to [f_n]$ such that $(T^* f_{n_k}^*)$ is equivalent to some disjoint normalized sequence (g_k^*). Let (g_k) be a disjoint normalized sequence in E such that $g_k^*(g_m) = \delta_{km}$. Passing to a further subsequence, if necessary, we may assume that (g_k) is equivalent to (f_{n_k}). Using Lemma 2.4(ii), we conclude that the map $x \mapsto \sum_{k=1}^{\infty} g_k^*(x) g_k$ is in $\mathcal{L}(E)$. It is easy to see that this is a projection onto $[g_k]$. $\square$

In most instances of DH spaces, like the L_p, Lorentz spaces and some Orlicz spaces it holds that every disjoint sequence has a complemented subsequence. This motivates the study of the following class.

Definition 4.5. A Banach lattice E is called ***disjointly complemented*** (DC) if every disjoint sequence (x_n) has a subsequence whose span is complemented in E.

We will study next the relation between DC and DH Banach lattices. More precisely, we are interested in studying whether DH Banach lattices are also DC.

4.1. **Non-reflexive case.** Let us consider first the case of non-reflexive Banach lattices. Recall that if E is non-reflexive, then E either contains a lattice copy of c_0 or of ℓ_1 (cf. [20, Theorem 2.4.15]). Therefore, if E is DH and non-reflexive, it follows that it is either 1-DH or ∞-DH.

Proposition 4.6. *If E is a separable Banach lattice and ∞-DH then it is DC.*

Proof. Let (x_n) be a disjoint normalized sequence in E. Since E is ∞-DH, we can find a subsequence (x_{n_k}) which is equivalent to c_0. Then $[x_{n_k}]$ is complemented in E by Sobczyk's theorem [1, 2.5.9] that in a separable Banach space every subspace isomorphic to c_0 is complemented. □

For the 1-DH case, we will use the following fact [20, Lemma 2.3.11]: *If a positive disjoint sequence in a Banach lattice is equivalent to the unit vector basis of ℓ_1 then its closed span is complemented.* It follows easily that if E is 1-DH, then every positive disjoint sequence has a complemented subsequence. The next proposition shows that this remains true even for non-positive disjoint sequences.

Proposition 4.7. *If E is a 1-DH Banach lattice, then E is DC.*

Proof. Suppose that (x_n) is a normalized disjoint sequence in E. Passing to a subsequence, we may assume that (x_n) is equivalent to the unit vector basis (e_n) of ℓ_1.

Suppose that $x_{n_k}^- \to 0$ for some subsequence (n_k). Passing to a subsequence and using the Principle of Small Perturbations (cf. [1, Theorem 1.3.9]), we may assume that (x_n) is equivalent to (x_n^+) and (x_n) is complemented whenever (x_n^+) is complemented. However, (x_n^+) is complemented by [20, Lemma 2.3.11]. The argument for the case when $x_{n_k}^+ \to 0$ for some subsequence (n_k) is similar.

Suppose now that neither (x_n^+) nor (x_n^-) has a null subsequence. In particular, there exists $C > 0$ such that $C^{-1} \leqslant \|x_n^\pm\| \leqslant C$ for all n. Passing to subsequences of (x_n) twice, we may assume that both (x_n^+) and (x_n^-) are c-equivalent to the unit vector basis (e_n) of ℓ_1 for some $c > 0$.

Claim: the sequence (z_n) given by

$$z_n = \begin{cases} x_k^+ & \text{if } n = 2k-1 \\ \\ x_k^- & \text{if } n = 2k \end{cases}$$

is still equivalent to the unit vector basis of ℓ_1. Indeed, consider a linear combination $z := \sum_{n=1}^{2m} \lambda_n z_n$. Then $\|z\| \leqslant C \sum_{n=1}^{2m} |\lambda_n|$. On the other hand, since (z_n) is disjoint, we have $\big\|\sum_{n=1}^{m} \lambda_{2n-1} x_n^+\big\| \leqslant \|z\|$ and $\big\|\sum_{n=1}^{m} \lambda_{2n} x_n^-\big\| \leqslant \|z\|$, so that

$$\sum_{n=1}^{2m}|\lambda_n| = \sum_{n=1}^{m}|\lambda_{2n-1}| + \sum_{n=1}^{m}|\lambda_{2n}| \leqslant c\Big(\Big\|\sum_{n=1}^{m} \lambda_{2n-1} x_n^+\Big\| + \Big\|\sum_{n=1}^{m} \lambda_{2n} x_n^-\Big\|\Big) \leqslant 2c\|z\|.$$

This proves the claim.

As mentioned above, by [20, Lemma 2.3.11], there exists a projection P on E with $\operatorname{Range} P = [z_n]$. Since (z_n) and (x_n) are both equivalent to (e_n), the map

$$R\Big(\sum_{n=1}^{\infty} a_n z_n\Big) = \sum_{n=1}^{\infty} \frac{a_{2n-1} - a_{2n}}{2} x_n$$

defines a bounded operator $R : [z_n] \to [x_n]$. It is easy to check that RP is a projection of E onto $[x_n]$. $\square$

Combining the preceding propositions, we get the following.

Corollary 4.8. *If E is a separable non-reflexive Banach lattice which is DH, then E is DC.*

Clearly, the separability of E is essential here: ℓ_∞ is non-reflexive and DH, however it is not DC. In fact, every normalized disjoint sequence is equivalent to the unit vector basis of c_0 and by Phillips-Sobczyk's theorem (cf. [1, Theorem 2.5.5], [17, Theorem 2.a.7]), the space ℓ_∞ does not contain any complemented subspace isomorphic to c_0.

It turns out that 1-DH Banach lattices have been considered previously under a different approach. Recall that a Banach lattice E has the *positive Schur property* if every weakly null sequence (x_n) of positive vectors is norm convergent, see [13, 23, 24, 25]. It follows from, e.g., [20, Corollary 2.3.5] that it suffices to verify this condition for disjoint sequences.

Proposition 4.9. *A Banach lattice E is 1-DH if and only if E has the positive Schur property .*

Proof. Assume that E fails the positive Schur property. Then there exist a positive disjoint normalized weakly null sequence. Clearly, it has no subsequences equivalent to the unit vector basis of ℓ_1. Thus E is not 1-DH.

Conversely, assume that E has the positive Schur property. Take a normalized disjoint positive sequence (x_n) in E and apply Rosenthal's ℓ_1 Theorem. Suppose that (x_{n_k}) is a weakly Cauchy subsequence of (x_n). Since E has the positive Schur property, it does not contain c_0, hence it is weakly sequentially complete. Therefore, (x_{n_k}) converges weakly. Then (x_{n_k}) is weakly null by, e.g., [1, Lemma 1.6.1]. It follows that (x_{n_k}) is norm null, which is a contradiction. This shows that (x_n) has no weakly Cauchy subsequences and, therefore, by Rosenthal's ℓ_1 Theorem, (x_n) has a subsequence equivalent to the unit vector basis of ℓ_1. Hence E is 1-DH. $\square$

4.2. **Reflexive case.** Recall that in Theorem 3.5 it was proved that a reflexive Banach lattice E with property $\mathfrak{P}$ is disjointly homogeneous provided so is E^*.

Question. Is every reflexive DH Banach lattice DC?

We have the following partial result in this direction.

Proposition 4.10. *Let E be a reflexive Banach lattice containing a complemented positive disjoint sequence. If E and E^* are DH, then E is DC.*

Proof. Let (x_n) be a disjoint normalized sequence in E, and let (x_n^*) be a sequence of biorthogonal functionals to (x_n) in E^*, which without loss of generality can be taken as a disjoint normalized sequence. Consider now (e_n) and (e_n^*) as in Corollary 4.3. Since E is DH, passing to subsequences we have that $(e_{n_j}) \sim (x_{n_j})$.

Now, since E^* is DH, it follows that for some further subsequence $(e^*_{n_{j_k}}) \sim (x^*_{n_{j_k}})$. Note that, the map $x \mapsto \sum_{k=1}^{\infty} e^*_{n_{j_k}}(x) e_{n_{j_k}}$ is still a bounded projection in $\mathcal{L}(E)$. By Lemma 2.4(ii), the map $x \mapsto \sum_{k=1}^{\infty} x^*_{n_{j_k}}(x) x_{n_{j_k}}$ is a bounded operator on E. Clearly, this is a projection onto $[x_{n_{j_k}}]$. $\square$

Let us recall the p-convexification $E^{(p)}$ and p-concavification $E_{(p)}$ of a Banach lattice E (see [18, pp. 53,54]). Recall that, given a Banach lattice E and $1 \leqslant p < \infty$, we may define new vector space operations on E via $x \oplus y = \big(x^{1/p} + y^{1/p}\big)^p$ and $\alpha \odot x = \alpha^p x$ and the norm $\|x\|_{E^{(p)}} = \|x\|^{1/p}$; this results in a Banach lattice denoted $E^{(p)}$ and called the p-convexification of E (the lattice operations are the same as in E). Conversely, the p-concavification $E_{(p)}$ of E is defined as E with new vector space operations $x \oplus y = \big(x^p + y^p\big)^{1/p}$ and $\alpha \odot x = \alpha^{1/p} x$ and the same lattice operations; $E_{(p)}$ is a vector lattice.

If E is p-convex with constant M then $E_{(p)}$ becomes a Banach lattice under a certain norm, which will be denoted by $\|\cdot\|_{E_{(p)}}$, which satisfies $\frac{1}{M^p}\|x\|^p \leqslant \|x\|_{E_{(p)}} \leqslant \|x\|^p$ for every $x \in E$.

Recall that if $x \perp y$ then $x \oplus y = x + y$ in both $E^{(p)}$ and $E_{(p)}$. It follows that if (x_n) is a disjoint sequence in E then for linear combinations of x_n's in $E^{(p)}$ we have

$$\Big\|\bigoplus_{k=1}^{n} \alpha_k \odot x_k\Big\|_{E^{(p)}} = \Big\|\sum_{k=1}^{n} \alpha_k^p x_k\Big\|^{1/p}.$$

This immediately yields the following proposition.

Proposition 4.11. *Let E be a Banach lattice. It holds that:*

(i) *E is DH if and only if $E^{(p)}$ is DH. Moreover, E is q-DH if and only if $E^{(p)}$ is pq-DH.*
(ii) *Suppose that E is p-convex, then E is DH if and only if $E_{(p)}$ is DH. Similarly, in this case E is q-DH if and only if $E_{(p)}$ is q/p-DH.*

The following lemma is based on [5, Lemma 2.6] (see also the paragraph in [5] preceding Lemma 2.5).

Lemma 4.12. *Suppose that $1 < p < \infty$ and E is a p-convex order continuous Banach lattice with a weak unit and (x_n) is a disjoint sequence in E equivalent to the unit vector basis of ℓ_p. Then there is a one-to-one lattice homomorphism $J\colon E \to L_p(\mu)$ for some measure μ such that the restriction of J to $[x_n]$ is an isomorphism.*

Proof. Let $E_{(p)}$ be the p-concavification of E; we write $\oplus$ and $\odot$ for the vector operations in $E_{(p)}$. We claim that in $E_{(p)}$, the sequence (x_n) is equivalent to the unit vector basis of ℓ_1. Indeed, suppose that M is the p-convexity constant of E, and (x_n) is C-equivalent to the unit vector basis of ℓ_p. For any scalars $\alpha_1, \dots, \alpha_n$ we have

$$\Big\|\oplus\text{-}\sum_{k=1}^{n} \alpha_k \odot x_k\Big\|_{E_{(p)}} \geqslant \frac{1}{M^p}\Big\|\oplus\text{-}\sum_{k=1}^{n} \alpha_k \odot x_k\Big\|^p = \frac{1}{M^p}\Big\|\sum_{k=1}^{n} \alpha_k^{1/p} x_k\Big\|^p \geqslant \frac{C}{M^p}\sum_{k=1}^{n}|\alpha_k|.$$

The reverse estimate follows from the triangle inequality, since $E_{(p)}$ is a Banach lattice.

Clearly, the sequence $\big(|x_n|\big)$, viewed as a sequence in $E_{(p)}$ is still equivalent to the unit vector basis of ℓ_1. Define a functional f on the linear span of $\big(|x_n|\big)$ in $E_{(p)}$ via $f\big(\oplus\text{-}\sum_{k=1}^n \alpha_k \odot |x_k|\big) = \sum_{k=1}^n \alpha_k$. Clearly, f is continuous, hence it extends to a functional in $E^*_{(p)}$. Being an order continuous Banach lattice with a weak unit, $E_{(p)}$ admits a strictly positive functional $g \in E^*_{(p)}$. Let $h = |f| \vee g$; then $h \in E^*_{(p)}$ is strictly positive and $h\big(|x_n|\big) \geqslant f\big(|x_n|\big) = 1$ for every n.

It is easy to see that the map $x \mapsto h\big(|x|\big)$ defines a lattice norm on $E_{(p)}$. It follows, as in [18, p. 53], that the map $|||x||| := \big[h\big(|x|\big)\big]^{1/p}$ defines a lattice norm on the p-convexification of $E_{(p)}$, which, as a vector lattice, coincides with E, so we view $|||\cdot|||$ as a norm on E. If $x \perp y$ in E then

$$|||x+y|||^p = h\big(|x+y|\big) = h\big(|x| \oplus |y|\big) = h\big(|x|\big) + h\big(|y|\big) = |||x|||^p + |||y|||^p .$$

It follows that the completion of $\big(E, |||\cdot|||\big)$ is an AL_p-space; hence is lattice isometric to $L_p(\mu)$ for some measure μ. Let $J\colon E \to L_p(\mu)$ be the natural embedding. Clearly, J is a lattice homomorphism. In particular, (Jx_n) is a disjoint sequence in $L_p(\mu)$. Furthermore, it follows from $\|Jx_n\| = |||x_n||| \geqslant 1$, that (Jx_n) is semi-normalized; hence it is equivalent to the unit vector basis of ℓ_p. Since both (x_n) and (Jx_n) are equivalent to the unit vector basis of ℓ_p, it follows that J is an isomorphism on $[x_n]$. □

Corollary 4.13. *If E is a p-convex order continuous Banach lattice with $1 < p < \infty$ and (x_n) is a disjoint sequence in E equivalent to the unit vector basis of ℓ_p then $[x_n]$ is complemented.*

Proof. Without loss of generality, E has a weak unit; otherwise replace E with the (projection) band generated by (x_n). Let $J\colon E \to L_p(\mu)$ be as in Lemma 4.12. Then (Jx_n) is a disjoint seminormalized sequence in $L_p(\mu)$, hence there is a projection $P\colon L_p(\mu) \to [Jx_n]$. Now the operator $J^{-1}PJ$ is a projection from E to $[x_n]$. □

Corollary 4.14. *Let E be a p-DH Banach lattice which is p-convex with $1 < p < \infty$. Then E is DC.*

Proof. Since E is p-DH, it is reflexive, hence order continuous. Let (x_n) be a disjoint normalized sequence in E. Passing to a subsequence, we may assume that (x_n) is equivalent to the unit vector basis of ℓ_p. Now apply Corollary 4.13. □

5. Reflexive DH spaces with non-DH dual

In this section, we present examples of reflexive DH Banach lattices whose duals are not DH. The examples are given in the class of Orlicz function spaces on the interval $(0,\infty)$ and the class of weighted Orlicz sequence spaces.

Recall that given an Orlicz function φ on $[0,\infty)$ (i.e. a continuous convex increasing function with $\varphi(0)=0$ and $\varphi(t)\to\infty$ as $t\to\infty$), the Orlicz space $L^\varphi(0,\infty)$ is the space of measurable functions $f:[0,\infty)\to\mathbb{R}$ such that

$$\int_0^\infty \varphi\big(|f(t)|/s\big)\,dt<\infty,$$

for some $s>0$. This space becomes a Banach lattice with the usual operations and the norm

$$\|f\|=\inf\Big\{s>0:\int_0^\infty \varphi\big(|f(t)|/s\big)\,dt\leqslant 1\Big\}.$$

Recall also that an Orlicz function φ is said to satisfy the Δ_2-condition if there is $C>0$ such that $\varphi(2t)\leqslant C\varphi(t)$ for every $t\geqslant 0$. Note that an Orlicz space $L^\varphi(0,\infty)$ is separable if and only if φ satisfies the Δ_2-condition (cf. [18]).

Now, let us give a useful characterization of p-DH Orlicz function spaces $L^\varphi(0,\infty)$ (a similar result in the case of Orlicz function spaces on the $[0,1]$ interval was given in [9]). Recall that $C_\varphi(0,\infty)=\overline{\operatorname{conv}}\,E_\varphi(0,\infty)$ in the space $C(0,1)$, where $E_\varphi(0,\infty)$ is the closure of the set

$$\Big\{F\in C(0,1)\ :\ F(\cdot)=\frac{\varphi(s\cdot)}{\varphi(s)}\,,\ \text{for some}\ s\in(0,\infty)\Big\}.$$

Theorem 1.1 in [19] asserts that if an Orlicz function F is equivalent to a function in $C_\varphi(0,\infty)$ then $L^\varphi(0,\infty)$ contains a lattice copy of ℓ^F and, conversely, every normalized disjoint sequence in $L^\varphi(0,\infty)$ contains a subsequence equivalent to the unit vector basis of ℓ^F for some $F\in C_\varphi(0,\infty)$. See [19] for more details.

Theorem 5.1. *Let $L_\varphi(0,\infty)$ be a separable Orlicz space. Given $1\leqslant p<\infty$, the space $L^\varphi(0,\infty)$ is p-DH if and only if $C_\varphi(0,\infty)\cong\{t^p\}$.*

Proof. Let us show first the sufficient part. Let (f_n) be a sequence of disjoint normalized functions in $L^\varphi(0,\infty)$. Then by Theorem 1.1(i) in [19] there exists a subsequence (f_{n_k}) such that (f_{n_k}) is equivalent to the unit vector basis (e_k) of an Orlicz sequence space ℓ^F

for some function $F \in C_\varphi(0,\infty)$. Hence, as $C_\varphi(0,\infty) \simeq \{t^p\}$ we conclude that $L^\varphi(0,\infty)$ is p-DH.

Let us now show now the necessity part. Suppose that there exists a function $F \in C_\varphi(0,\infty)$ which is not equivalent to the function t^p on $[0,1]$. Then, applying Theorem 1.1.(ii) in [19], we get that there exists a disjoint normalized sequence (f_n) in $L^\varphi(0,\infty)$ such that (f_n) is equivalent to the unit vector basis (e_n) in ℓ^F. If $L^\varphi(0,\infty)$ is p-DH then there exists a subsequence (f_{n_k}) which is equivalent to the canonical basis of ℓ_p. Hence by the symmetry of the basis of ℓ^F, we conclude that $\ell^F = \ell^p$, so the function F is equivalent to t^p, which is a contradiction. $\square$

Theorem 5.2. *Let $1 < p < \infty$ and an Orlicz function $\varphi(t)$ agrees with t^p on $[0,1]$ and $\varphi(t) \simeq t^p \log(1+t)$ on $[1,\infty)$. Then the Orlicz space $L^\varphi(0,\infty)$ is a reflexive p-DH Banach lattice whose dual is not DH.*

Proof. First, let us prove that $L^\varphi(0,\infty)$ is p-DH by showing that $C_\varphi(0,\infty) \simeq \{t^p\}$.

Indeed, by [19, p. 242], every function $F \in C_\varphi(0,\infty)$ can be expressed as a convex combination of functions $F = aF_1 + bF_2 + cF_3$, where $F_1 \in C_{\varphi,1}$, $F_2 \in C_\varphi^\infty$ and

$$F_3(t) = \int_1^\infty \frac{\varphi(st)}{\varphi(s)} d\mu(s)$$

for $t \in [0,1]$, where μ is a probability measure in $[1,\infty)$ with $\mu(\{1\}) = 0$.

Now, from the hypothesis we clearly have $F_1 \sim t^p$ at 0, and also $F_2 \sim t^p$ at 0 since $\lim_{s\to\infty} \frac{s\varphi'(s)}{\varphi(s)} = p$.

Hence it remains only to show that $F_3 \sim t^p$ at 0. Since φ is equivalent to a p-convex function, there exists a constant $M > 0$ such that

$$\sup_{s>0} \frac{\varphi(st)}{\varphi(s)} \leqslant Mt^p$$

for t near 0 (cf. [22]). Thus, $F_3(t) \leqslant Mt^p$ near 0. We also have $D\varphi(t) \leqslant F_3(t)$ at 0, where $D = \int_1^\infty \frac{d\mu(s)}{\varphi(s)} < \infty$. Therefore the function F_3 is equivalent to t^p at 0. Thus, using Theorem 5.1, we conclude that $L^\varphi(0,\infty)$ is p-DH.

Let us prove now that the dual space of $L^\varphi(0,\infty)$ is not q-DH. First notice that the dual space is the Orlicz space $L^\psi(0,\infty)$ where

$$\psi(t) = t^q \text{ for } t \in [0,1] \quad \text{and} \quad \psi(t) \simeq \frac{t^q}{\log^{q-1}(1+t)} \text{ for } t \in [1,\infty)$$

(see [15, p. 58]). We claim that the set $C_\psi(0,\infty)$ contains the family of Orlicz functions $G_\alpha(t) \sim t^q |\log t|^\alpha$ for every $0 < \alpha < q-1$.

Let us consider now the family of finite measures μ_α on $[1,\infty)$ defined by

$$\mu_\alpha([1,\infty)) = \int_1^\infty \frac{ds}{s \log^{q-\alpha}(1+s)}$$

for $0 < \alpha < q-1$. Then the functions

$$F_\alpha(t) = \int_1^\infty \frac{\psi(st)}{\psi(s)} d\mu_\alpha(s),$$

which belong to $C_\psi(0,\infty)$ as mentioned above, satisfy

$$\begin{aligned} F_\alpha(t) &\simeq \int_1^{1/t} t^q \log^{q-1}(1+s) d\mu_\alpha(s) + \int_{1/t}^\infty \frac{t^q \log^{q-1}(1+s)}{\log^{q-1}(1+ts)} d\mu_\alpha(s) \\ &= t^q \Bigg(\underbrace{\int_1^{1/t} \frac{ds}{s \log^{1-\alpha}(1+s)}}_{I_\alpha(t)} + \underbrace{\int_{1/t}^\infty \frac{ds}{s \log^{q-1}(1+ts) \log^{1-\alpha}(1+s)}}_{J_\alpha(t)} \Bigg). \end{aligned}$$

Now, for $0 < \alpha < \min\{1, q-1\}$, we have

$$I_\alpha(t) = \int_1^{1/t} \frac{ds}{s \log^{1-\alpha}(1+s)} \sim |\log t|^\alpha$$

for t near 0. Furthermore, as

$$J_\alpha(t) = \int_{1/t}^\infty \frac{ds}{s \log^{q-1}(1+ts) \log^{1-\alpha}(1+s)} = \int_1^\infty \frac{du}{u \log^{q-1}(1+u) \log^{1-\alpha}(1+\frac{u}{t})},$$

we get $\lim_{t\to 0} J_\alpha(t) = 0$.

Hence, the function F_α is equivalent to $t^q|\log t|^\alpha$ at 0. Thus, the function $t^q|\log t|^\alpha$ belongs (up to equivalence) to the set $C_\psi(0,\infty)$ for every $0 < \alpha < \min\{1, q-1\}$, and we conclude, using Theorem 5.1 that $L^\psi(0,\infty)$ is not DH. □

We pass now to give examples of *atomic* reflexive p-DH Banach lattices whose duals are not DH. These will be given within the class of weighted Orlicz sequence spaces (cf. [11]).

Recall that given a sequence of positive numbers $w = (w_n)$ and an Orlicz function φ, the *weighted Orlicz sequence space* $\ell^\varphi(w)$ is the space of all sequences (x_n) such that $\sum_{n=1}^\infty \varphi(\frac{|x_n|}{s}) w_n < \infty$ for some $s > 0$, endowed with the Luxemburg norm. Notice that the unit vectors form an unconditional basis of $\ell^\varphi(w)$ when φ satisfies the Δ_2-condition.

Theorem 5.3. *Let $w = (w_n)$ be a sequence of positive numbers such that there is a subsequence (w_{n_k}) with $w_{n_k} \to 0$ and $\sum_{k=1}^{\infty} w_{n_k} = \infty$. If φ is an Orlicz function as in the previous theorem then the weighted Orlicz sequence space $\ell^{\varphi}(w)$ is p-DH but its dual is not DH.*

Proof. It is clear that $\ell^{\varphi}(w)$ is p-DH Banach lattice since $\ell^{\varphi}(w)$ is lattice isomorphic to the sublattice $[\chi_{A_n}]$ of the space $L^{\varphi}(0,\infty)$ for any sequence of disjoint sets (A_n) with $\mu(A_n) = w_n$.

Let us show now that the dual of $\ell^{\varphi}(w)$, which is canonically identified with the weighted Orlicz sequence space $\ell^{\psi}(w)$ ([11, Prop. 5]), is not DH. For that we will show that for every function F_α in the set $C_{\psi}^{\infty}(0,\infty)$, the Orlicz sequence space ℓ^{F_α} is lattice isomorphic to a sublattice of $\ell^{\psi}(w)$ (in particular, for the functions $F_\alpha \sim t^q |\log t|^\alpha$ at 0).

Indeed, given a function $F \in C_{\psi}(0,\infty)$, there exist a disjoint normalized function sequence (f_n) in $L^{\psi}(0,\infty)$ such that (f_n) is equivalent to the canonical basis (e_k) of ℓ^{F_α}. Now there exist disjoint simple functions $h_n = \sum_{i=1}^{r_n} a_{i,n}\chi_{A_{i,n}}$ such that the sequence (h_n) is equivalent to (f_n).

Consider now the family of all characteristic disjoint functions $(\chi_{A_{i,n}})_{i=1,\dots,r_n}^{n=1,\dots,\infty}$. It is clear that the space generated by this family is lattice isomorphic to the weighted Orlicz sequence space $\ell^{\psi}(w^{\circ})$ where w° is the weight sequence defined by $w^{\circ} = (\mu(A_{i,n}))$. Now using the universal property of the sequence space $\ell^{\varphi}(w)$ (cf. [6, Prop. 3.1]) we conclude that $\ell^{\psi}(w)$ has a sublattice which is lattice isomorphic to $\ell^{\psi}(w^{\circ})$. Hence $\ell^{\psi}(w)$ also has a sublattice lattice isomorphic to the space ℓ^{F_α} and thus $\ell^{\psi}(w)$ is not DH. $\square$

Observe that this same kind of example cannot be constructed for Orlicz spaces over a probability space (see the begining of Section 3, and [9]). This motivates the following.

Question. Is there a reflexive p-DH rearrangement invariant function space on the interval $[0,1]$ whose dual is not DH?

A different kind of an example of a p-DH space whose dual is not DH will be presented in Subsection 6.2 .

6. Further remarks

6.1. **Quasi-DH spaces.** Notice that in [10] the definition of DH Banach lattices was slightly different. So, in order to avoid any confusion of terminology, let us introduce here the following.

Definition 6.1. A Banach lattice E is ***quasi disjointly homogeneous (quasi-DH)*** if for every pair (x_n), (y_n) of normalized disjoint sequences in E, there exist subsequences (x_{n_k}) and (y_{m_k}), and some constant $C > 0$ such that $(x_{n_k}) \overset{C}{\sim} (y_{m_k})$.

Notice that there is a subtle difference between this notion and that of DH. For quasi-DH Banach lattices, unlike for the DH case, the subsequences (n_k) and (m_k) need not be the same. In particular, every DH Banach lattice is quasi-DH. However, we do not know if these notions coincide in general.

Our terminology is consistent with that used in [9]. We remark that, however, Definition 6.1 was given as the definition of a DH space in [10], but the proofs of Theorems 3.2 and 3.6 in [10] only work for DH spaces under the terminology of the current paper.

Notice that for most classical Banach lattices both notions coincide. For instance, this is the case for stable Banach lattices (in the sense of Krivine-Maurey, [16], see also [2]).

Proposition 6.2. *Let E be a stable Banach lattice. Then E is quasi-DH if and only if it is p-DH for some $1 \leq p \leq \infty$.*

Proof. Let (x_n) be a disjoint normalized sequence in E. Since E is stable, so is $[x_n]$. Therefore, by [16] there is some $1 \leq p \leq \infty$ such that ℓ_p is isomorphic to a subspace of $[x_n]$. In particular, there exists a block sequence (y_n) of (x_n), which is equivalent to the unit vector basis of ℓ_p. Since (x_n) is a disjoint sequence, so is (y_n), hence we have that E contains a disjoint normalized sequence equivalent to the unit vector basis of ℓ_p. Since E is quasi-DH, every disjoint normalized sequence in E has a subsequence equivalent to a subsequence of (y_n). Thus, E is p-DH. $\square$

Although we do not know if DH and quasi-DH Banach lattices form the same class, several of the results given in the previous sections can be translated to the framework of quasi-DH spaces. For instance, we can study the duality for quasi-DH Banach lattices, and in particular, one can wonder if a result like Theorem 3.5 holds for this class. It turns out that the proof of Theorem 3.5 is not entirely spoiled for quasi-DH spaces. If we start with two disjoint normalized sequences (x_n) and (y_n) in a Banach lattice E, whose dual is quasi-DH, then the problem arises when we pass to subsequences of the corresponding disjoint sequences (g_k^*) and (h_k^*) in E^*, since these may be different subsequences. Then the computations in the proof of Theorem 3.5 show that (x_{n_k}) is dominated (as a basic sequence) by *some* subsequence (y_{m_k}). Iterating this process, some further subsequence of (y_{m_k}) is also dominated by another subsequence of (x_{n_k}). However, this need imply that they are equivalent.

Notice also that the first part of the proof of Theorem 4.4 also works for quasi-DH Banach lattices, so in this case, property $\mathfrak{P}$ follows from the fact that the space contains a complemented positive disjoint sequence.

In [10], we showed that Tsirelson space provides an example of quasi-DH Banach lattice which is not p-DH for any $1 \leq p \leq \infty$. We will show now that the argument can be improved to show that actually, Tsirelson space is in fact DH.

Example 6.3. Tsirelson space T is a DH Banach lattice which is not p-DH, for any $1 \leq p \leq \infty$.

Proof. Let us denote (t_j) the unit basis of the space T (which is an unconditional basis for T). Every normalized disjoint sequence in T has a subsequence equivalent to a block basis, which, by [4, Prop. II.4], is equivalent to a certain subsequence of the unit basis. Therefore, it suffices to show that any two subsequences (t_{k_m}) and (t_{j_m}) have a common equivalent subsequence. To prove this we need the following Ramsey-type result:

Claim: Given two infinite subsequences (k_m) and (j_m) of $\mathbb{N}$, there is an infinite subsequence (m_n) such that either

(i) $k_{m_n} < j_{m_n}$ *for every n; or*
(ii) $k_{m_n} > j_{m_n}$ *for every n.*

To prove this fact, let us suppose we already have $m_1, \ldots, m_n$ such that $k_{m_i} < j_{m_i}$ for every $i = 1, \ldots, n$. Now, suppose there always exists $m > m_n$ such that $k_m < j_m$, then we let

$$m_{n+1} = \min\{m > m_n : k_m < j_m\},$$

and keep the induction going on, so we get condition (i). If on the contrary, at some step n, there is no such m, it follows that for every $m > m_n$ we have $k_m > j_m$, and this provides an infinite set satisfying (ii). Thus, the claim is proved.

For simplicity, let us assume that we have (m_n) satisfying (i), i.e. $k_{m_n} < j_{m_n}$ for every n. Now, let $n_1 = 1$, so $k_{m_{n_1}} < j_{m_{n_1}}$. Next, let n_2 be large enough so that $k_{m_{n_2}} > j_{m_{n_1}}$, and in general, for each i let n_i such that $k_{m_{n_i}} > j_{m_{n_{i-1}}}$. Thus we get a subsequence (m_{n_i}) satisfying

$$j_{m_{n_{i-1}}} < k_{m_{n_i}} < j_{m_{n_i}}.$$

Using [4, Prop. II.4] once more, we get that $(t_{j_{m_{n_i}}})$ and $(t_{k_{m_{n_i}}})$ are equivalent. Thus, T is DH.

Since T contains no subspace isomorphic to ℓ_p for $1 \leq p < \infty$, nor c_0, it follows clearly that T is not p-DH.

$\square$

Notice that a similar argument shows that T^* is also DH (see [4, p. 23]. In particular T is DC by Proposition 4.10 (this also follows from [4, Prop. II.6]).

6.2. **Uniformly DH spaces.** The constant appearing in the definition of a DH space (see Def. 2.1) plays a significant role when ℓ_p-sums of p-DH spaces are considered. This is clarified with the following example

Example 6.4. The ℓ_p-sum of p-DH spaces need not be p-DH.

Proof. Given $n \in \mathbb{N}$, let X_n denote the completion of the space of all eventually zero sequences c_{00} with respect to the norm

$$\big\|(a_k)\big\|_{X_n} = \sup\Big\{\sum_{i=1}^{n}|a_{k_i}| + \Big(\sum_{i>n}|a_{k_i}|^p\Big)^{\frac{1}{p}} \,:\, k_1 < k_2 < \ldots < k_i < \ldots\Big\}.$$

It is clear that $\|\cdot\|_{X_n}$ defines a lattice norm, which is actually equivalent to the ℓ_p norm. Indeed, we have

$$\big\|(a_k)\big\|_{X_n} \geqslant \sum_{k=1}^{n}|a_k| + \Big(\sum_{k>n}|a_k|^p\Big)^{\frac{1}{p}} \geqslant \big\|(a_k)\big\|_{\ell_p},$$

while for every increasing sequence (k_i) we have

$$\sum_{i=1}^{n}|a_{k_i}| + \Big(\sum_{i>n}|a_{k_i}|^p\Big)^{\frac{1}{p}} \leqslant (n^{\frac{1}{q}}+1)\Big(\sum_{i=1}^{\infty}|a_{k_i}|^p\Big)^{\frac{1}{p}} \leqslant (n^{\frac{1}{q}}+1)\big\|(a_k)\big\|_{\ell_p},$$

where $\frac{1}{p}+\frac{1}{q}=1$. Hence, taking suprema on the left hand side of the previous inequality we get $\big\|(a_k)\big\|_{X_n} \leqslant (n^{\frac{1}{q}}+1)\big\|(a_k)\big\|_{\ell_p}$. In particular, the space X_n is clearly p-DH.

Let us denote $Y := \Big(\bigoplus_{n=1}^{\infty} X_n\Big)_{\ell_p}$ endowed with the ℓ_p-sum of the corresponding norms $\|\cdot\|_{X_n}$. We claim that, Y is not p-DH. Indeed, let $(e_k^{(n)})$ denote the canonical basis of X_n. For a fixed $n \in \mathbb{N}$, the vectors $(e_k^{(n)})$ form a normalized disjoint sequence in Y which is equivalent to the unit vector basis of ℓ_p. Now, let

$$x_k = \sum_{n=3}^{\infty} \frac{1}{(n\log^2 n)^{1/p}} e_k^{(n)}.$$

It follows that (x_k) is a semi-normalized disjoint sequence in Y. However, given any subsequence (x_{k_i}) and $m \geq 1$ we have

$$\|\sum_{i=1}^{m} x_{k_i}\|_Y \geq m\Big(\sum_{n=m}^{\infty}\frac{1}{n\log^2 n}\Big)^{\frac{1}{p}} \geq \frac{m}{\log^{1/p}(m+1)},$$

where we used the fact that for $n \geq m$, $\|\sum_{i=1}^{m} e_{k_i}^{(n)}\|_{X_n} = m$. Thus, (x_{k_i}) is not equivalent to the unit vector basis of ℓ_p. Hence, Y is not p-DH.

□

After the previous example it seems only natural to introduce the following

Definition 6.5. A Banach lattice E is ***uniformly disjointly homogeneous*** if there is a constant $C>0$ such that every two disjoint normalized sequences (x_n) and (y_n) in E, have subsequences such that $(x_{n_k}) \overset{C}{\sim} (y_{n_k})$.

Clearly, every uniformly disjointly homogeneous Banach lattice E is DH. Next, we are going to investigate whether the converse of this is true. As usual, by a **Banach lattice ordered by basis** we mean a Banach lattice with a basis (x_n) such that $\sum_{n=1}^{\infty}\alpha_n x_n \geqslant 0$ iff $\alpha_i \geq 0$ for each n. Clearly, in this case, (x_n) is disjoint and, therefore, unconditional.

Proposition 6.6. *Suppose that E is a Banach lattice ordered by a basis, $1<p,q<\infty$ and $\frac{1}{p}+\frac{1}{q}=1$. If E is p-DH and E^* is q-DH then E is uniformly DH.*

Proof. The proposition follows easily from [14, Corollary 2(i)]. Observe that E satisfies the (S_p) condition of [14]. Indeed, suppose that (x_n) is a weakly null sequence in E. It is easy to see that if (x_n) is norm null then it trivially has a subsequence which satisfies an upper ℓ_p-estimate. On the other hand, if (x_n) is not norm null then by [17, Proposition 1.a.12] it has a subsequence equivalent to a disjoint sequence, hence, by assumption, equivalent to the unit vector basis of ℓ_p. It follows that E satisfies (S_p). Being p-DH for $p > 1$, E is reflexive, hence E^* is also a Banach lattice ordered by a basis. Applying a similar reasoning we conclude that E^* satisfies (S_q). Since E^* is q-DH and $q > 1$, E^* does not contain a (lattice) copy of ℓ_1, hence the conclusion now follows from [14, Corollary 2(i)]. □

Note that under the assumptions of Proposition 6.6, it also follows from [14, Corollary 2(i)] that E is (uniformly) DC.

Theorem 6.7. *For every $1 < p < \infty$, there exists a super-reflexive Banach lattice E_p ordered by a basis which is p-DH but not uniformly DH. Moreover, E_p^* is not DH.*

Proof. Let us take first $1 < p < 2$. By [12, Example 1], there exists a subspace Y of L_p such that every weakly null normalized sequence in Y has a subsequence equivalent to the unit vector basis of ℓ_p, but without a uniform bound on the constant of equivalence. Therefore, for each $n \in \mathbb{N}$ we can consider a normalized weakly null sequence $(x_k^n)_{k\in\mathbb{N}} \subset Y$ such that no subsequence is n-equivalent to the unit vector basis of ℓ_p. Notice that by passing to subsequences and using a standard peturbation argument, we can assume that for every $n, k \in \mathbb{N}$ there are integers $p_k^n < q_k^n < p_{k+1}^n$ and scalars $(a_j^{(n,k)})_{j\in[p_k^n,q_k^n]}$ such that $\|x_k^n - w_k^n\| \underset{k}{\to} 0$, where

$$w_k^n = \sum_{j=p_k^n}^{q_k^n} a_j^{(n,k)} h_j$$

and (h_j) denotes the Haar system (which is an unconditional basis in L_p.) Let now fix a surjective function $f : \mathbb{N} \to \mathbb{N}$ with the property that for every $n \in \mathbb{N}$ the set $f^{-1}(n)$ is infinite. We construct now a block basic sequence (e_m) of (h_j) inductively as follows: Let $k_0 = 0$ and

$$e_0 = w_0^{f(0)}.$$

For $m \geq 1$, let k_m be large enough so that $q_{k_{m-1}}^{f(m-1)} < p_{k_m}^{f(m)}$ and $\|w_{k_m}^{f(m)} - x_{k_m}^{f(m)}\| < 2^{-m}$, and define

$$e_m = w_{k_m}^{f(m)}.$$

By construction (e_m) is a weakly null unconditional basic sequence which is equivalent to some sequence in Y. Let E_p denote the span of (e_m) in L_p, which becomes a super-reflexive Banach lattice with the order induced by the basis.

Since E_p is isomorphic to a subspace of Y, it follows that every weakly null, normalized sequence in E_p has a subsequence equivalent to the unit vector basis of ℓ_p. Hence, E_p is p-DH, but for every $n \in \mathbb{N}$ the sequence $(e_k)_{k \in f^{-1}(n)}$ has no subsequence n-equivalent to the unit vector basis of ℓ_p. Hence, E_p is not uniformly DH. Thus, we have shown that for each $1 < p < 2$, there exists a Banach lattice E_p ordered by a basis, which is p-DH but not uniformly DH.

Now for $2 \leqslant p < \infty$ take $E_{3/2}$ as above and call E_p the $(2p/3)$-convexification of $E_{3/2}$. By Proposition 4.11, we get that E_p is p-DH. As in Proposition 4.11, it is easy to see that a Banach lattice E is uniformly r-DH iff $E^{(p)}$ is uniformly rp-DH, so that E_p cannot be uniformly DH. It is also easy to see that the basis of $E_{3/2}$ becomes a (disjoint) basis of E_p.

We have now shown that for every $1 < p < \infty$, there exists a Banach lattice E_p ordered by an unconditional basis (e_n) which is p-DH but not uniformly DH. We claim that E_p^* is not DH. Suppose it is. Since E_p is reflexive, (e_n^*) is a basis of E_p^*. Since E_p is p-DH, some subsequence (e_{n_k}) is equivalent to the unit vector basis of ℓ_p. It follows that $(e_{n_k}^*)$ is equivalent to the unit vector basis of ℓ_q. Hence, E_p^* is q-DH. It now follows from Proposition 6.6 that E_p is uniformly DH, a contradiction. □

6.3. **DH property and Rosenthal bases.** In Theorem 5.3 we have constructed examples of reflexive atomic Banach lattices (with the order induced by a 1-unconditional basis), which are DH, but whose dual are not. In this section, we study the case of DH Banach lattices ordered by a subsymmetric basis. Recall that a basis (x_n) is called subsymmetric if it is unconditional and every subsequence (x_{n_i}) is equivalent to (x_n) (cf. [17, Chapter 3]).

Also, recall that a normalized basis (e_n) in a Banach space X is said to be a ***Rosenthal basis*** if every normalized block-sequence of (e_n) contains a subsequence equivalent to (e_n). It is an open question whether such a basis is necessarily equivalent to the unit basis of ℓ_p or c_0, see [7] for further details and partial results in this direction. In particular, it was observed in [7, p. 397] that a Rosenthal basis (x_n), always satisfies that every subsequence (x_{n_i}) is equivalent to (x_n).

Proposition 6.8. *Let E be a reflexive Banach lattice ordered by a subsymmetric basis (e_n). Then E is DH if and only if (e_n) is a Rosenthal basis.*

Proof. Suppose that E is DH, and let (x_k) be a block-sequence of (e_n). Then there is a subsequence of (x_k) which is equivalent to a subsequence of (e_n) and, therefore, to (e_n) itself because (e_n) is subsymmetric. Hence, (e_n) is a Rosenthal basis.

Conversely, suppose that (e_n) is a Rosenthal basis. Pick any two disjoint seminormalized sequences (x_k) and (y_k). By Bessaga-Pelzcynski's selection principle, we may assume that (x_k) and (y_k) are block-sequences of (e_n). By the assumption, there exists a subsequence (x_{k_i}) which is equivalent to (e_n). Furthermore, there is a subsequence $(y_{k_{i_j}})$ which is equivalent to (e_n). Since (e_n) is subsymmetric, $(x_{k_{i_j}}) \sim (e_n)$, so that $(x_{k_{i_j}}) \sim (y_{k_{i_j}})$. $\square$

Let X be a Banach space with a Rosenthal basis (e_n). It was proved in [7, Theorem 1, Proposition 7] that (e_n) is equivalent to the unit basis of ℓ_p or c_0 if (e_n) is "uniformly" Rosenthal or if E^* also has a Rosenthal basis. In view of Proposition 6.8, we can now restate these statements in terms of disjoint homogenuity as follows.

Proposition 6.9. *Let E be a reflexive Banach lattice ordered by a subsymmetric normalized basis (e_n). Then (e_n) is equivalent to the unit basis of ℓ_p for some $1 < p < \infty$ if any of the following conditions is satisfied:*

(i) *E is uniformly DH, or*
(ii) *E and E^* are both DH.*

If in particular, (e_n) is symmetric, then Proposition 6.9 (ii) also follows from [17, Theorem 3.a.10] due to Altshuler.

It is an open question ([7]) whether every reflexive Banach lattice with a subsymmetric basis which is DH must be isomorphic to ℓ_p for some $1 < p < \infty$. In this direction, if we consider the symmetric version of Tsirelson space (see [4, Chapter X, B]), which does not contain ℓ_p subspaces, then it is not hard to see that this space fails being DH. However, we can give the following result.

Corollary 6.10. *Let E be a reflexive Banach lattice with property $\mathfrak{P}$ such that E contains a disjoint subsymmetric sequence. If E^* is DH, then E is p-DH for some $1 < p < \infty$.*

Proof. By Theorem 3.5, E is DH. It follows from Theorem 4.4 that E contains a complemented disjoint sequence, say (e_i). By assumption, we may assume that (e_i) is subsymmetric. Let (e_i^*) be as in Corollary 4.3. It is easy to see that, being a sublattice of E^*, the space $[e_i^*]$ is DH. On the other hand, it is easy to see that $[e_i^*]$ can be identified with $[e_i]^*$. Applying Proposition 6.9(ii) to $[e_i]$ we conclude that (e_i) is equivalent to the unit vector basis of ℓ_p for some p. Therefore, E is p-DH. □

We would like to thank an anonymous referee for correcting a mistake in the initial version of Example 6.4 and for suggesting Proposition 6.6 and Theorem 6.7 to us.

References

[1] F. Albiac and N.J. Kalton, *Topics in Banach space theory.* Graduate Texts in Mathematics **233**, Springer, (2006).

[2] S. Argyros, S. Negrepontis and Th. Zachariades, Weakly stable Banach spaces. *Israel J. Math.* 57 (1987), 68–88.

[3] P.G. Casazza and N.J. Kalton, Uniqueness of unconditional bases in Banach spaces. *Israel J. Math.* 103 (1998), 141–175.

[4] P.G. Casazza and T.J. Shura, *Tsirelson's Space.* Lecture notes in Math. 1363, Springer-Verlag (1989).

[5] D. Chen, W.B. Johnson, and B. Zheng, Commutators on $(\sum \ell_q)_p$. *Studia Math.* 206 (2011), no. 2, 175–190.

[6] L. Drewnowski, F-spaces with a basis which is shrinking but not hyper-shrinking. *Studia Math.* 64 (1979), no. 2, 97–104.

[7] V. Ferenczi, A.M. Pelczar, and C. Rosendal, On a question of Haskell P. Rosenthal concerning a characterization of c_0 and ℓ_p. *Bull. London Math. Soc.* 36 (2004), no. 3, 396–406.

[8] T. Figiel, W.B. Johnson and L. Tzafriri, On Banach lattices and spaces having local unconditional structure with applications to Lorentz funtion spaces. *J. Approximation Theory* 13 (1975), 395–412.

[9] J. Flores, F.L. Hernández, E.M. Semenov, and P. Tradacete, Strictly singular and power-compact operators on Banach lattices. *Israel J. Math.* 188 (2012), 323–352.

[10] J. Flores, P. Tradacete, and V.G. Troitsky, Disjointly homogeneous Banach latices and compact product of operators. *J. Math. Anal. Appl.* 354 (2009), 657–663.

[11] F. Fuentes and F.L. Hernández, On weighted Orlicz sequence spaces and their subspaces. *Rocky Mountain J. Math.* 18 (1988), no. 3, 585–599.

[12] W. B. Johnson and E. Odell, Subspaces of L_p which embed into ℓ_p. *Compositio Math.* 28 (1974), 37–49.

[13] A. Kamińska and M. Mastylo, The Schur and (weak) Dunford-Pettis properties in Banach lattices. J. Aust. Math. Soc. **73** (2002), no. 2, 251–278.

[14] H. Knaust and E. Odell, Weakly null sequences with upper ℓ_p-estimates. *Functional analysis* (Austin, TX, 1987/1989), 85–107, Lecture Notes in Math., 1470, Springer, Berlin, 1991.

[15] M.A. Krasnoselskiĭ and Ja.N. Rutickiĭ, *Convex functions and Orlicz spaces.* Izdat. Fiz.-Mat. Lit., Moscow, 1958. English translation Noordhoff Ltd., Groningen 1961.

[16] J.L. Krivine and B. Maurey, Espaces de Banach stables. *Israel J. Math.* 39 (1981), no. 4, 273–295.
[17] J. Lindenstrauss and L. Tzafriri, *Classical Banach spaces. I.* Springer-Verlag, Berlin, 1977, Sequence spaces.
[18] J. Lindenstrauss and L. Tzafriri, *Classical Banach spaces. II.* Springer-Verlag, Berlin, 1979, Function spaces.
[19] N.J. Nielsen, On the Orlicz function spaces $L_M(0,\infty)$. *Israel J. Math.* 20 (1975), no. 3–4, 237–259.
[20] P. Meyer-Nieberg, *Banach lattices.* Springer-Verlag, Berlin, 1991.
[21] V.D. Milman, Operators of class C_0 and C_0^*. (in Russian) *Teor. Funkcii Funkcional. Anal. i Prilozen.* 10 (1970), 15–26.
[22] P. Turpin, Convexités dans les espaces vectoriels topologiques généraux. (in French) *Dissertationes Math.* 131 (1976).
[23] W. Wnuk, A note on the positive Schur property. *Glasgow Math. J.* 31 (1989) 169–172.
[24] W. Wnuk, Some characterizations of Banach lattices with the Schur property, *Rev. Mat. Univ. Complut. Madrid* 2 (1989), suppl., 217–224.
[25] W. Wnuk, Banach lattices with properties of the Schur type–a survey. *Confer. Sem. Mat. Univ. Bari* 249 (1993).

J. Flores, Department of Applied Mathematics, Escet, Universidad Rey Juan Carlos, 28933, Móstoles, Madrid, Spain.
E-mail address: `julio.flores@urjc.es`

F. L. Hernández, Departamento de Análisis Matemático, Universidad Complutense de Madrid, 28040, Madrid, Spain.
E-mail address: `pacoh@mat.ucm.es`

E. Spinu and V.G. Troitsky, Department of Mathematical and Statistical Sciences, University of Alberta, Edmonton, AB, T6G 2G1. Canada.
E-mail address: `espinu@math.ualberta.ca,troitsky@ualberta.ca`

P. Tradacete, Department of Mathematics, Universidad Carlos III de Madrid, 28911, Leganés, Madrid, Spain.
E-mail address: `ptradace@math.uc3m.es`