KRIVINE'S FUNCTION CALCULUS AND BOCHNER INTEGRATION

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ABSTRACT. We prove that Krivine's Function Calculus is compatible with integration. Let (Ω, Σ, μ) be a finite measure space, X a Banach lattice, $\boldsymbol{x} \in X^n$, and $f\colon \mathbb{R}^n \times \Omega \to \mathbb{R}$ a function such that $f(\cdot, \omega)$ is continuous and positively homogeneous for every $\omega \in \Omega$, and $f(\boldsymbol{s}, \cdot)$ is integrable for every $\boldsymbol{s} \in \mathbb{R}^n$. Put $F(\boldsymbol{s}) = \int f(\boldsymbol{s}, \omega) d\mu(\omega)$ and define $F(\boldsymbol{x})$ and $f(\boldsymbol{x}, \omega)$ via Krivine's Function Calculus. We prove that under certain natural assumptions $F(\boldsymbol{x}) = \int f(\boldsymbol{x}, \omega) d\mu(\omega)$, where the right hand side is a Bochner integral.

1. Motivation

In [Kal12], the author defines a real-valued function of two real or complex variable via $F(s,t) = \int_0^{2\pi} \lvert s + e^{i\theta} t \rvert d\theta$. This is a positively homogeneous continuous function. Therefore, given two vectors u and v in a Banach lattice X, one may apply Krivine's Function Calculus to F and consider $F(u,v)$ as an element of X. The author then claims that

$$F(u,v) = \int_0^{2\pi} \lvert u + e^{i\theta} v \rvert d\theta, \tag{1}$$

where the right hand side here is understood as a Bochner integral; this is used later in [Kal12] to conclude that $\lVert F(u,v) \rVert \leqslant \int_0^{2\pi} \lVert u + e^{i\theta} v \rVert d\theta$ because Bochner integrals have this property: $\lVert \int f \rVert \leqslant \int \lVert f \rVert$. A similar exposition is also found in [DGTJ84, p. 146]. Unfortunately, neither [Kal12] nor [DGTJ84] includes a proof of (1). In this note, we prove a general theorem which implies (1) as a special case.

2. Preliminaries

We start by reviewing the construction of Krivine's Function Calculus on Banach lattices; see [LT79, Theorem 1.d.1] for details. For Banach lattice terminology, we refer the reader to [AA02, AB06].

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Fix $n \in \mathbb{N}$. A function $F\colon \mathbb{R}^n \to \mathbb{R}$ is said to be ***positively homogeneous*** if

$$F(\lambda t_1, \dots, \lambda t_n) = \lambda F(t_1, \dots, t_n) \text{ for all } t_1, \dots, t_n \in \mathbb{R} \text{ and } \lambda \geqslant 0.$$

Let H_n be the set of all continuous positively homogeneous functions from $\mathbb{R}^n$ to $\mathbb{R}$. Let S_∞^n be the unit sphere of ℓ_∞^n, that is,

$$S_\infty^n = \bigl\{(t_1, \dots, t_n) \in \mathbb{R}^n \;:\; \max_{i=1,\dots,n} |t_i| = 1\bigr\}.$$

It can be easily verified that the restriction map $F \mapsto F_{|S_\infty^n}$ is a lattice isomorphism from H_n onto $C(S_\infty^n)$. Hence, we can identify H_n with $C(S_\infty^n)$. For each $i = 1, \dots, n$, the i-th coordinate projection $\pi_i\colon \mathbb{R}^n \to \mathbb{R}$ clearly belongs to H_n.

Let X be a (real) Banach lattice and $\boldsymbol{x} = (x_1, \dots, x_n) \in X^n$. Let $e \in X_+$ be such that $x_1, \dots, x_n$ belong to I_e, the principal order ideal of e. For example, one could take $e = |x_1| \vee \dots \vee |x_n|$. By Kakutani's representation theorem, the ideal I_e equipped with the norm

$$\|x\|_e = \inf\bigl\{\lambda > 0 \;:\; |x| \leqslant \lambda e\bigr\}$$

is lattice isometric to $C(K)$ for some compact Hausdorff K. Let $F \in H_n$. Interpreting $x_1, \dots, x_n$ as elements of $C(K)$, we can define $F(x_1, \dots, x_n)$ in $C(K)$ as a composition. We may view it as an element of I_e and, therefore, of X; we also denote it by $\widetilde{F}$ or $\Phi(F)$. It may be shown that, as an element of X, it does not depend on the particular choice of e. This results in a (unique) lattice homomorphism $\Phi\colon H_n \to X$ such that $\Phi(\pi_i) = x_i$. The map Φ will be referred to as ***Krivine's function calculus***. This construction allows one to define expressions like $\left(\sum_{i=1}^n |x_i|^p\right)^{\frac{1}{p}}$ for $0 < p < \infty$ in every Banach lattice X; this expression is understood as $\Phi(F)$ where $F(t_1, \dots, t_n) = \left(\sum_{i=1}^n |t_i|^p\right)^{\frac{1}{p}}$. Furthermore,

$$\bigl\|F(\boldsymbol{x})\bigr\| \leqslant \|F\|_{C(S_\infty^n)} \cdot \Bigl\|\bigvee_{i=1}^n |x_i|\Bigr\|. \tag{2}$$

Let L_n be the sublattice of H_n or, equivalently, of $C(S_\infty^n)$, generated by the coordinate projections π_i as $i = 1, \dots, n$. It follows from the Stone-Weierstrass Theorem that L_n is dense in $C(S_\infty^n)$. It follows from $\Phi(\pi_i) = x_i$ that $\Phi(L_n)$ is the sublattice generated by $x_1, \dots, x_n$ in X, hence $\operatorname{Range}\Phi$ is contained in the closed sublattice of X generated by $x_1, \dots, x_n$. It follows from, e.g., Exercise 8 on [AB06, p.204] that this sublattice is separable.

Let (Ω, Σ, μ) be a finite measure space and X a Banach space. A function $f\colon \Omega \to X$ is *measurable* if there is a sequence (f_n) of simple functions from Ω to X such that $\lim_n \|f_n(\omega) - f(\omega)\| = 0$ almost everywhere. If, in addition, $\int \|f_n(\omega) - f(\omega)\| d\mu(\omega) \to 0$

then f is ***Bochner integrable*** with $\int_A f\,d\mu = \lim_n \int_A f_n\,d\mu$ for every measurable set A. In the following theorem, we collect a few standard facts about Bochner integral for future reference; we refer the reader to [DU77, Chapter II] for proofs and further details.

Theorem 2.1. *Let $f\colon \Omega \to X$.*

(i) *If f is the almost everywhere limit of a sequence of measurable functions then f is measurable.*
(ii) *If f is separable-valued and there is a norming set $\Gamma \subseteq X^*$ such that $x^* f$ is measurable for every $x^* \in \Gamma$ then f is measurable.*
(iii) *A measurable function f is Bochner integrable iff $\|f\|$ is integrable.*
(iv) *If $f(\omega) = u(\omega)x$ for some fixed $x \in X$ and $u \in L_1(\mu)$ and for all ω then f is measurable and Bochner integrable.*
(v) *If f is Bochner integrable and $T\colon X \to Y$ is a bounded operator from X to a Banach space Y then $T\big(\int f\,d\mu\big) = \int Tf\,d\mu$.*

3. Main theorem

Throughout the rest of the paper, we assume that (Ω, Σ, μ) is a finite measure space, $n \in \mathbb{N}$, and $f\colon \mathbb{R}^n \times \Omega \to \mathbb{R}$ is such that $f(\cdot, \omega)$ is in H_n for every $\omega \in \Omega$ and $f(\boldsymbol{s}, \cdot)$ is integrable for every $\boldsymbol{s} \in \mathbb{R}^n$. For every $\boldsymbol{s} \in \mathbb{R}^n$, put $F(\boldsymbol{s}) = \int f(\boldsymbol{s}, \omega) d\mu(\omega)$. It is clear that F is positively homogeneous.

Suppose, in addition, that F is continuous. Let X be a Banach lattice, $\boldsymbol{x} \in X^n$, and $\Phi\colon H_n \to X$ the corresponding function calculus. Since $F \in H_n$, $\widetilde{F} = F(\boldsymbol{x}) = \Phi(F)$ is defined as an element of X. On the other hand, for every ω, the function $\boldsymbol{s} \in \mathbb{R}^n \mapsto f(\boldsymbol{s}, \omega)$ is in H_n, hence we may apply Φ to it. We denote the resulting vector by $\tilde{f}(\omega)$ or $f(\boldsymbol{x}, \omega)$. This produces a function $\omega \in \Omega \mapsto f(\boldsymbol{x}, \omega) \in X$.

Theorem 3.1. *Suppose that F is continuous and the function $M(\omega) := \big\|f(\cdot, \omega)\big\|_{C(S_\infty^n)}$ is integrable. Then $f(\boldsymbol{x}, \omega)$ is Bochner integrable as a function of ω and $F(\boldsymbol{x}) = \int f(\boldsymbol{x}, \omega) d\mu(\omega)$, where the right hand side is a Bochner integral.*

Proof. Special case: $X = C(K)$ for some compact Hausdorff K. By uniqueness of function calculus, Krivine's function calculus Φ agrees with "point-wise" function calculus. In particular,

$$\widetilde{F}(k) = F\big(x_1(k), \dots, x_n(k)\big) \text{ and } \big(\tilde{f}(\omega)\big)(k) = f\big(x_1(k), \dots, x_n(k), \omega\big)$$

for all $k \in K$ and $\omega \in \Omega$. We view $\tilde{f}$ as a function from Ω to $C(K)$.

We are going to show that $\tilde{f}$ is Bochner integrable. It follows from $\tilde{f}(\omega) \in \operatorname{Range}\Phi$ that $\tilde{f}$ a separable-valued function. For every $k \in K$, consider the point-evaluation functional $\varphi_k \in C(K)^*$ given by $\varphi_k(x) = x(k)$. Then

$$\varphi_k\big(\tilde{f}(\omega)\big) = \big(\tilde{f}(\omega)\big)(k) = f\big(x_1(k), \dots, x_n(k), \omega\big).$$

for every $k \in K$. By assumptions, this function is integrable; in particular, it is measurable. Since the set $\big\{\varphi_k : k \in K\big\}$ is norming in $C(K)^*$, Theorem 2.1(ii) yields that $\tilde{f}$ is measurable.

Clearly, $\big|\big(\tilde{f}(\omega)\big)(k)\big| \leqslant M(\omega)$ for every $k \in K$ and $\omega \in \Omega$, so that $\|\tilde{f}(\omega)\|_{C(K)} \leqslant M(\omega)$ for every ω. It follows that $\int \|\tilde{f}(\omega)\|_{C(K)}\, d\mu(\omega)$ exists and, therefore, $\tilde{f}$ is Bochner integrable by Theorem 2.1(iii).

Put $h := \int \tilde{f}(\omega)\, d\mu(\omega)$, where the right hand side is a Bochner integral. Applying Theorem 2.1(v), we get

$$\begin{aligned} h(k) = \varphi_k(h) = \int \varphi_k\big(\tilde{f}(\omega)\big)\, d\mu(\omega) &= \int f\big(x_1(k), \dots, x_n(k), \omega\big)\, d\mu(\omega) \\ &= F\big(x_1(k), \dots, x_n(k)\big) = \widetilde{F}(k). \end{aligned}$$

for every $k \in K$. It follows that $\int \tilde{f}(\omega)\, d\omega = \widetilde{F}$.

General case. Let $e = |x_1| \vee \dots |x_n|$. Then $\big(I_e, \|\cdot\|_e\big)$ is lattice isometric to $C(K)$ for some compact Hausdorff K. Note also that $|x| \leqslant \|x\|_e e$ for every $x \in I_e$; this yields $\|x\| \leqslant \|x\|_e \|e\|$, hence the inclusion map $T\colon \big(I_e, \|\cdot\|_e\big) \to X$ is bounded. Identifying I_e with $C(K)$, we may view T as a bounded lattice embedding from $C(K)$ into X.

By the construction on Krivine's Function Calculus, Φ actually acts into I_e, i.e., $\Phi = T\Phi_0$, where Φ_0 is the $C(K)$-valued function calculus. By the special case, we know that $\int \tilde{f}(\omega)\, d\mu(\omega) = \widetilde{F}$ in $C(K)$. Applying T, we obtain the same identity in X by Theorem 2.1(v). ☐

Finally, we analyze whether any of the assumptions may be removed. Clearly, one cannot remove the assumption that F is continuous; otherwise, $\widetilde{F}$ would make no sense. The following example shows that, in general, F need not be continuous.

Example 3.2. Let $n = 2$, let μ be a measure on $\mathbb{N}$ given by $\mu\big(\{k\}\big) = 2^{-k}$. For each k, we define $f_k = f(\cdot, k)$ as follows. Note that it suffices to define f_k on S^2_∞. Let I_k be the straight line segment connecting $(1, 0)$ and $(1, 2^{-k+1})$. Define f_k so that it vanishes on $S^2_\infty \setminus I_k$, $f_k(1, 0) = f_k(1, 2^{-k+1}) = 0$, $f_k(1, 2^{-k}) = 2^k$, and is linear on each half of I_k. Then $f_k \in H_2$ and $F(\boldsymbol{s})$ is defined for every $\boldsymbol{s} \in \mathbb{R}^2$. It follows from $F(\boldsymbol{s}) = \sum_{k=1}^\infty 2^{-k} f_k(\boldsymbol{s})$ that $F(1, 0) = 0$ and $F(1, 2^{-k}) \geqslant 2^{-k} f_k(1, 2^{-k}) = 1$, hence F is discontinuous at $(1, 0)$.

The assumption that M is integrable cannot be removed as well. Indeed, consider the special case when $X = C(S_\infty^n)$ and $x_i = \pi_i$ as $i = 1, \dots, n$. In this case, Φ is the identity map and $\tilde{f}(\omega) = f(\cdot, \omega)$. It follows from Theorem 2.1(iii) that $\tilde{f}$ is Bochner integrable iff $\|\tilde{f}\|$ is integrable iff M is integrable.

Finally, the assumption that $f(\cdot, \omega)$ is in H_n for every ω may clearly be relaxed to "for almost every ω".

4. Direct proof

In the previous section, we presented a proof of Theorem 3.1 using representation theory. In this section, we present a direct proof. However, we impose an additional assumption: we assume that $f(\cdot, \omega)$ is continuous on S_∞^n uniformly on ω, that is,

$$\text{for every } \varepsilon > 0 \text{ there exists } \delta > 0 \text{ such that } \bigl|f(\boldsymbol{s}, \omega) - f(\boldsymbol{t}, \omega)\bigr| < \varepsilon \\ \text{for all } \boldsymbol{s}, \boldsymbol{t} \in S_\infty^n \text{ and all } \omega \in \Omega \text{ provided that } \|\boldsymbol{s} - \boldsymbol{t}\|_\infty < \delta. \tag{3}$$

In Theorem 3.1, we assumed that F was continuous and M was integrable. Now these two conditions are satisfied automatically. In order to see that F to is continuous, fix $\varepsilon > 0$; let δ be as in (3), then

$$\bigl|F(\boldsymbol{s}) - F(\boldsymbol{t})\bigr| \leqslant \int \bigl|f(\boldsymbol{s}, \omega) - f(\boldsymbol{t}, \omega)\bigr| d\mu(\omega) < \varepsilon \mu(\Omega) \tag{4}$$

whenever $\boldsymbol{s}, \boldsymbol{t} \in S_\infty^n$ with $\|\boldsymbol{s} - \boldsymbol{t}\|_\infty < \delta$. The proof of integrability of M will be included in the proof of the theorem.

Theorem 4.1. *Suppose that $f(\cdot, \omega)$ is continuous on S_∞^n uniformly on ω. Then $f(\boldsymbol{x}, \omega)$ is Bochner integrable as a function of ω and $F(\boldsymbol{x}) = \int f(\boldsymbol{x}, \omega) d\mu(\omega)$.*

Proof. Without loss of generality, by scaling μ and $\boldsymbol{x}$, we may assume that μ is a probability measure and $\left\|\bigvee_{i=1}^n |x_i|\right\| = 1$; this will simplify computations. In particular, (2) becomes $\|H(\boldsymbol{x})\| \leqslant \|H\|_{C(S_\infty^n)}$ for every $H \in C(S_\infty^n)$. Note also that $\boldsymbol{x}$ in the theorem is a "fake" variable as $\boldsymbol{x}$ is fixed. It may be more accurate to write $\widetilde{F}$ and $\tilde{f}(\omega)$ instead of $F(\boldsymbol{x})$ and $f(\boldsymbol{x}, \omega)$, respectively. Hence, we need to prove that $\tilde{f}$ as a function from Ω to X is Bochner integrable and its Bochner integral is $\widetilde{F}$.

Fix $\varepsilon > 0$. Let δ be as in (3). It follows from (4) that

$$\bigl|F(\boldsymbol{s}) - F(\boldsymbol{t})\bigr| < \varepsilon \text{ whenever } \boldsymbol{s}, \boldsymbol{t} \in S_\infty^n \text{ with } \|\boldsymbol{s} - \boldsymbol{t}\|_\infty < \delta. \tag{5}$$

Each of the $2n$ faces of S_∞^n is a translate of the $(n-1)$-dimensional unit cube B_∞^{n-1}. Partition each of these faces into $(n-1)$-dimensional cubes of diameter less than δ, where the diameter is computed with respect to the $\|\cdot\|_\infty$-metric. Partition each of these cubes into simplices. Therefore, there exists a partition of the entire S^n_∞ into finitely many simplices of diameter less than δ. Denote the vertices of these simplices by $\boldsymbol{s}_1,\dots,\boldsymbol{s_m}$. Thus, we have produced a triangularization of S^n_∞ with nodes $\boldsymbol{s}_1,\dots,\boldsymbol{s_m}$.

Let $\boldsymbol{a}\in\mathbb{R}^m$. Define a function $L\colon S^n_\infty\to\mathbb{R}$ by setting $L(\boldsymbol{s_j})=a_j$ as $j=1,\dots,m$ and then extending it to each of the simplices linearly; this can be done because every point in a simplex can be written in a unique way as a convex combination of the vertices of the simplex. We write $L=T\boldsymbol{a}$. This gives rise to a linear operator $T\colon\mathbb{R}^m\to C(S^n_\infty)$. For each $j=1,\dots,m$, let e_j be the j-th unit vector in $\mathbb{R}^m$; put $d_j=Te_j$. Clearly,

$$T\boldsymbol{a}=\sum_{j=1}^m a_jd_j \text{ for every } \boldsymbol{a}\in\mathbb{R}^m. \tag{6}$$

Let $H\in C(S^n_\infty)$. Let $L=T\boldsymbol{a}$ where $a_j=H(\boldsymbol{s_j})$. Then L agrees with H at $\boldsymbol{s}_1,\dots,\boldsymbol{s}_m$. We write $L=SH$; this defines a linear operator $S\colon C(S^n_\infty)\to C(S^n_\infty)$. Clearly, this is a linear contraction.

Suppose that $H\in C(S^n_\infty)$ is such that $\bigl|H(\boldsymbol{s})-H(\boldsymbol{t})\bigr|<\varepsilon$ whenever $\|\boldsymbol{s}-\boldsymbol{t}\|_\infty<\delta$. Let $L=SH$. We claim that $\bigl\|L-H\bigr\|_{C(S^n_\infty)}<\varepsilon$. Indeed, fix $\boldsymbol{s}\in S^n_\infty$. Let $\boldsymbol{s}_{j_1},\dots,\boldsymbol{s}_{j_n}$ be the vertices of a simplex in the triangularization of S^n_∞ that contains $\boldsymbol{s}$. Then $\boldsymbol{s}$ can be written as a convex combination $\boldsymbol{s}=\sum_{k=1}^n\lambda_k\boldsymbol{s}_{j_k}$. Note that $\|\boldsymbol{s}-\boldsymbol{s}_{j_k}\|_\infty<\delta$ for all $j=1,\dots,n$. It follows that

$$\bigl|L(\boldsymbol{s})-H(\boldsymbol{s})\bigr|=\Bigl|\sum_{k=1}^n\lambda_kL(\boldsymbol{s}_{j_k})-\sum_{k=1}^n\lambda_kH(\boldsymbol{s})\Bigr|\leqslant\sum_{j=1}^n\lambda_k\bigl|H(\boldsymbol{s}_{j_k})-H(\boldsymbol{s})\bigr|<\varepsilon.$$

This proves the claim.

Let $G=SF$. It follows from (5) and the preceding observation $\|G-F\|_{C(S^n_\infty)}<\varepsilon$, so that

$$\bigl\|G(\boldsymbol{x})-F(\boldsymbol{x})\bigr\|<\varepsilon. \tag{7}$$

Similarly, for every $\omega\in\Omega$, apply S to $f(\cdot,\omega)$ and denote the resulting function $g(\cdot,\omega)$. In particular, $g(\boldsymbol{s}_j,\omega)=f(\boldsymbol{s}_j,\omega)$ for every $\omega\in\Omega$ and every $j=1,\dots,m$. It follows also that

$$\bigl\|f(\cdot,\omega)-g(\cdot,\omega)\bigr\|_{C(S^n_\infty)}<\varepsilon \tag{8}$$

for every ω, and, therefore

$$\bigl\|\tilde f(\omega)-\tilde g(\omega)\bigr\|=\bigl\|f(\boldsymbol{x},\omega)-g(\boldsymbol{x},\omega)\bigr\|<\varepsilon, \tag{9}$$

where $\tilde{g}(\omega) = g(\boldsymbol{x}, \omega)$ is the image under Φ of the function $\boldsymbol{s} \in S^n_\infty \mapsto g(\boldsymbol{s}, \omega)$. Note that

$$G(\boldsymbol{s}_j) = F(\boldsymbol{s}_j) = \int f(\boldsymbol{s}_j, \omega) d\mu(\omega) = \int g(\boldsymbol{s}_j, \omega) d\mu(\omega) \tag{10}$$

for every $j = 1, \dots, m$. Since $G = SF = T\boldsymbol{a}$ where $a_j = F(\boldsymbol{s}_j) = G(\boldsymbol{s}_j)$ as $j = 1, \dots, m$, it follows from (6) that

$$G = \sum_{j=1}^{m} G(\boldsymbol{s}_j) d_j. \tag{11}$$

Similarly, for every $\omega \in \Omega$, we have

$$g(\cdot, \omega) = \sum_{j=1}^{m} g(\boldsymbol{s}_j, \omega) d_j. \tag{12}$$

Applying Φ to (11) and (12), we obtain $\widetilde{G} = G(\boldsymbol{x}) = \sum_{j=1} G(\boldsymbol{s}_j) d_j(\boldsymbol{x})$ and

$$\tilde{g}(\omega) = g(\boldsymbol{x}, \omega) = \sum_{j=1}^{m} g(\boldsymbol{s}_j, \omega) d_j(\boldsymbol{x}) = \sum_{j=1}^{m} f(\boldsymbol{s}_j, \omega) d_j(\boldsymbol{x}).$$

Together with Theorem 2.1(iv), this yields that $\tilde{g}$ is measurable and Bochner integrable. It now follows from (10) and (11) that

$$\begin{aligned} G(\boldsymbol{x}) = \sum_{j=1} G(\boldsymbol{s}_j) d_j(\boldsymbol{x}) &= \sum_{j=1}^{m} \Big(\int g(\boldsymbol{s}_j, \omega) d\mu(\omega) \Big) d_j(\boldsymbol{x}) \\ &= \int \Big(\sum_{j=1}^{m} g(\boldsymbol{s}_j, \omega) d_j(\boldsymbol{x}) \Big) d\mu(\omega) = \int g(\boldsymbol{x}, \omega) d\mu(\omega). \end{aligned} \tag{13}$$

We will show next that $\tilde{f}$ is Bochner integrable. It follows from (9) and the fact that ε is arbitrary that $\tilde{f}$ can be approximated almost everywhere (actually, everywhere) by measurable functions; hence $\tilde{f}$ is measurable by Theorem 2.1(i). Next, we claim that there exists $\lambda \in \mathbb{R}_+$ such that $\big| f(\boldsymbol{s}, \omega) - f(\boldsymbol{1}, \omega) \big| \leqslant \lambda$ for all $\boldsymbol{s} \in S^n_\infty$ and all $\omega \in \Omega$. Here $\boldsymbol{1} = (1, \dots, 1)$. Indeed, let $\boldsymbol{s} \in S^n_\infty$ and $\omega \in \Omega$. Find $j_1, \dots, j_l$ such that $\boldsymbol{s}_{j_1} = \boldsymbol{1}$, $\boldsymbol{s}_{j_k}$ and $\boldsymbol{s}_{j_{k+1}}$ belong to the same simplex as $k = 1, \dots, l-1$, and $\boldsymbol{s}_{j_l}$ is a vertex of a simplex containing $\boldsymbol{s}$. It follows that

$$\big| f(\boldsymbol{s}, \omega) - f(\boldsymbol{1}, \omega) \big| \leqslant \big| f(\boldsymbol{s}, \omega) - f(\boldsymbol{s}_{j_l}, \omega) \big| + \sum_{k=1}^{l-1} \big| f(\boldsymbol{s}_{j_{k+1}}, \omega) - f(\boldsymbol{s}_{j_k}, \omega) \big| \leqslant l\varepsilon \leqslant m\varepsilon.$$

This proves the claim with $\lambda = m\varepsilon$. It follows that

$$\big\| \tilde{f}(\omega) \big\| \leqslant \big\| f(\cdot, \omega) \big\|_{C(S^n_\infty)} = \sup_{\boldsymbol{s} \in S^n_\infty} \big| f(\boldsymbol{s}, \omega) \big| \leqslant \big| f(\boldsymbol{1}, \omega) \big| + \lambda.$$

Since $\bigl|f(\mathbf{1},\omega)\bigr|+\lambda$ is an integrable function of ω, we conclude that $\|\tilde{f}\|$ is integrable, hence $\tilde{f}$ is Bochner integrable by Theorem 2.1(iii). It now follows from (9) that

$$\Bigl\|\int f(\boldsymbol{x},\omega)d\mu(\omega)-\int g(\boldsymbol{x},\omega)d\mu(\omega)\Bigr\|\leqslant\int\bigl\|f(\boldsymbol{x},\omega)-g(\boldsymbol{x},\omega)\bigr\|d\mu(\omega)<\varepsilon \tag{14}$$

Finally, combining (7), (13), and (14), we get

$$\Bigl\|F(\boldsymbol{x})-\int f(\boldsymbol{x},\omega)d\mu(\omega)\Bigr\|<2\varepsilon.$$

Since $\varepsilon>0$ is arbitrary, this proves the theorem. □

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