University of Alberta
Mathematical and Statistical Sciences
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Math 667, Topics in Differential Equations
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Assignment 5, due April 06, 2005, 9 AM

Exercise 16: (4)

For $\mu \in [0,1)$ we study the following dynamical system

$$\begin{aligned} \dot{y} &= -y \\ \dot{x} &= \begin{cases} \mu(1-x)^2, & 0 \le x < 1 \\ -(1-x)^2, & 1 \le x \end{cases} \end{aligned}$$

Find the global attractor for each $\mu \in [0,1)$. Show that the attractors are upper semicontinuous in 0 but not lower semicontinuous.

Exercise 17: (4)

Let $\Lambda(B) := \bigcup_{x\in B} \omega(x)$. Give an example for a dynamical system that satisfies

$$\Lambda(B) \neq \Lambda(\Lambda(B))$$

(Not the exmaple from Robinson, Exercise 10.3, p. 281!!)

Exercise 18: (2)

Show that the solution semigroup of $\dot{u} = u^{2/3}$ is not injective.

Exercise 19: (10)

The nonlinear Cattaneo system in one space dimension reads

$$\begin{aligned} u_t &= -\gamma v_x + f(u) \\ v_t &= -\gamma u_x - 2\mu v \end{aligned}$$

It is a model for correlated random walk on an interval $[0,l]$ of particles moving with speed γ and turning rate μ. The functions u and v are particle density and particle flux, respectively. We consider homogeneous Neumann boundary conditions, which have the form $v(0) = 0$, and $v(l) = 0$. For the nonlinearity f we assume

$$f \in C^2, \quad \|f'\|_\infty < 2\mu, \quad F(u) = \int_0^u f(s)ds, \quad \lim_{|u|\to\infty} F(u) = -\infty.$$

The solutions form a semigroup in $X = H^1([0,l]) \times H_0^1([0,l])$. We define

$$P(u,v) = \int_0^l F(u) + \mu v^2 + \gamma u_x v \, dx, \qquad Q(u,v) = \int_0^l u_t^2 + v_t^2 \, dx.$$

1. Show that there exists a $\lambda < 0$ such that $L = \lambda P + Q$ is a strong Lyapunov function for the Cattaneo system.
2. With λ chosen as in part 1. show that

   $$\lim_{(u,v)\to\infty \text{ in } X} L(u,v) = +\infty.$$

   You have to ensure that $L \to \infty$ for all of the following four limits: $u \to \infty$ in L^2, $u_x \to \infty$ in L^2, $v \to \infty$ in L^2, and $v_x \to \infty$ in L^2.
3. Show that the Cattaneo system has an attractor in L^2. If we assume that the set of all steady states $\mathcal{E}$ is finite, then show that

   $$\mathcal{A} = \bigcup_{z\in\mathcal{E}} \overline{W^u(z)}.$$