Math 337, Summer 2010
Assignment 5

Dr. T Hillen, University of Alberta

Exercise 0.1. ✗
Consider Laplace's equation

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}=0$$

in a semi-circular disk of radius a centered at the origin with boundary conditions

$$\begin{aligned} &u(r,0)=0, \quad 0<r\leqslant a,\\ &u(r,\pi)=0, \quad 0<r\leqslant a,\\ &u(a,\theta)=\sin\theta, \quad 0\leqslant\theta\leqslant\pi,\\ &|u(r,\theta)|<\infty \quad \text{as} \quad r\to 0^+. \end{aligned}$$

Solve this problem using separation of variables.

Solution to Exercise 0.1: Assuming a solution of the form $u(r,\theta)=R(r)\cdot\varphi(\theta)$, and separating variables, we have the following problems for R and φ :

$$\begin{aligned} r(rR')'+\lambda R=0, &\quad 0<r\leqslant a, \\ |R(0)|<\infty, & \end{aligned} \qquad\qquad \begin{aligned} \varphi''+\lambda\varphi=0, &\quad 0\leqslant\theta\leqslant\pi,\\ \varphi(0)=0, &\\ \varphi(\pi)=0. & \end{aligned}$$

We solve the θ-problem first. The eigenvalues and corresponding eigenfunctions are

$$\lambda_n=n^2 \qquad \text{and} \qquad \varphi_n(x)=\sin n\theta$$

for $n\geqslant 1$.
The corresponding r-equation is the Cauchy-Euler equation

$$r^2R''+rR'-n^2R=0,$$

with general solution

$$R(r)=A_nr^n+B_nr^{-n}$$

for $n \geqslant 1$. The boundedness condition $|R(0)| < \infty$ requires that $B_n = 0$, so that

$$R_n(r) = A_n r^n$$

for $n \geqslant 1$.
Using the superposition principle, we write

$$u(r, \theta) = \sum_{n=1}^{\infty} A_n r^n \sin n\theta$$

for $0 < r \leqslant a,\ 0 \leqslant \theta \leqslant \pi$, and from the boundary condition we want

$$\sin\theta = u(a, \theta) = \sum_{n=1}^{\infty} A_n a^n \sin n\theta.$$

From the orthogonality of the eigenfunctions on the interval $[0, \pi]$, we have

$$aA_1 = 1 \qquad \text{and} \qquad A_n = 0,\ n \geqslant 2,$$

so the solution is

$$u(r, \theta) = \frac{r}{a} \sin\theta$$

for $0 \leqslant r \leqslant a,\ 0 \leqslant \theta \leqslant \pi$.

Exercise 0.2.
Assume that $f(x)$ is absolutely integrable and a is a given real constant. Show that

$$\mathcal{F}\left(e^{iax} f(x)\right)(\omega) = \widehat{f}(\omega - a).$$

Solution to Exercise 0.2: Since $|e^{iax}| = 1$ and f is absolutely integrable on $(-\infty, \infty)$, then $e^{iax} f(x)$ is also absolutely integrable on $(-\infty, \infty)$ and we have

$$\begin{aligned}
\mathcal{F}\left(e^{iax} f(x)\right)(\omega) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{iax} f(x) e^{-i\omega x}\, dx \\
&= \frac{1}{2\pi} \int_{-\infty}^{\infty} f(x) e^{-i(\omega - a)x}\, dx \\
&= \widehat{f}(\omega - a)
\end{aligned}$$

for all $\omega \in \mathbb{R}$.

Exercise 0.3.
Assume that $f''(t)$ is absolutely integrable and

$$\lim_{t\to\infty} f(t) = 0 \qquad \text{and} \qquad \lim_{t\to\infty} f'(t) = 0.$$

Show that

$$\mathcal{F}_s\left(f''\right)(\omega) = -\omega^2 \mathcal{F}_s\left(f\right)(\omega) + \frac{2}{\pi}\,\omega f(0).$$

Solution of 0.3: Assuming that $\lim_{t\to\infty} f(t) = 0$, and $\lim_{t\to\infty} f'(t) = 0$, and integrating by parts we have

$$\begin{aligned}
\mathcal{F}_s\left(f''\right)(\omega) &= \frac{2}{\pi}\int_0^\infty f''(t)\sin\omega t\,dt \\
&= \frac{2}{\pi}\,f'(t)\sin\omega t\,\Big|_0^\infty - \frac{2}{\pi}\,\omega\int_0^\infty f'(t)\cos\omega t\,dt \\
&= -\frac{2}{\pi}\,\omega\int_0^\infty f'(t)\cos\omega t\,dt \\
&= -\frac{2}{\pi}\,\omega\, f(t)\cos\omega t\,\Big|_0^\infty + \frac{2}{\pi}\,\omega^2\int_0^\infty f(t)\sin\omega t\,dt \\
&= \frac{2}{\pi}\,\omega\, f(0) - \omega^2\mathcal{F}_s\left(f\right)(\omega).
\end{aligned}$$

We used the fact that

$$|f'(t)\sin\omega t| \leqslant |f'(t)| \longrightarrow 0 \quad \text{as} \quad t\to\infty,$$

and

$$|f(t)\cos\omega t| \leqslant |f(t)| \longrightarrow 0 \quad \text{as} \quad t\to\infty.$$

Exercise 0.4. ✗✗

Let

$$f(x) = \begin{cases} \cos x & |x| < \pi, \\ 0 & |x| > \pi. \end{cases}$$

(a) Find the Fourier integral of f.

(b) For which values of x does the integral converge to $f(x)$?

(c) Evaluate the integral

$$\int_0^\infty \frac{\lambda \sin \lambda\pi \cos \lambda x}{1 - \lambda^2}\, d\lambda$$

for $-\infty < x < \infty$.

Solution to 0.4:

(a) The Fourier integral representation of f is given by

$$f(x) \sim \int_0^\infty [A(\lambda) \cos \lambda x + B(\lambda) \sin \lambda x]\, d\lambda, \quad -\infty < x < \infty$$

where

$$A(\lambda) = \frac{1}{\pi} \int_{-\infty}^\infty f(t) \cos \lambda t\, dt \qquad \text{and} \qquad B(\lambda) = \frac{1}{\pi} \int_{-\infty}^\infty f(t) \sin \lambda t\, dt$$

for $\lambda \geqslant 0$.

Since $f(t)$ is an even function on the interval $-\infty < t < \infty$, then $B(\lambda) = 0$ for all $\lambda \geqslant 0$, and

$$A(\lambda) = \frac{1}{\pi} \int_{-\infty}^\infty f(t) \cos \lambda t\, dt = \frac{2}{\pi} \int_0^\infty f(t) \cos \lambda t\, dt$$

for all $\lambda \geqslant 0$.

Now, for $\lambda \neq 1$, we have

$$\begin{aligned} A(\lambda) &= \frac{2}{\pi} \int_0^{\pi} \cos t \cos \lambda t \, dt \\ &= \frac{2}{\pi} \int_0^{\pi} \frac{1}{2} \left[\cos(1+\lambda)t + \cos(1-\lambda)t\right] \, dt \\ &= \frac{\sin(1+\lambda)t}{\pi(1+\lambda)} \bigg|_0^{\pi} + \frac{\sin(1-\lambda)t}{\pi(1-\lambda)} \bigg|_0^{\pi} \\ &= \frac{\sin(1+\lambda)\pi}{\pi(1+\lambda)} + \frac{\sin(1-\lambda)\pi}{\pi(1-\lambda)} \\ &= -\frac{\sin \lambda\pi}{\pi(1+\lambda)} + \frac{\sin \lambda\pi}{\pi(1-\lambda)} \\ &= \frac{2\lambda \sin \lambda\pi}{\pi(1-\lambda^2)}, \end{aligned}$$

that is,

$$A(\lambda) = \frac{2\lambda \sin \lambda\pi}{\pi(1-\lambda^2)}$$

for $\lambda \geqslant 0, \ \lambda \neq 1$.

Now, for $\lambda = 1$, we have

$$A(1) = \frac{2}{\pi} \int_0^{\pi} \cos^2 t \, dt = \frac{2}{\pi} \int_0^{\pi} \frac{1}{2} \left[1 + \cos 2t\right] \, dt = \frac{1}{\pi} \left[t + \frac{1}{2} \sin 2t\right] \bigg|_0^{\pi} = \frac{1}{\pi} \cdot \pi = 1.$$

Therefore

$$A(\lambda) = \begin{cases} \dfrac{2\lambda \sin \lambda\pi}{\pi(1-\lambda^2)} & \text{for} \quad \lambda \geqslant 0, \ \lambda \neq 1 \\ 1 & \text{for} \quad \lambda = 1. \end{cases}$$

(b) Since $f(x)$ is continuous for all $x \neq \pm\pi$, then from Dirichlet's theorem, the Fourier integral representation converges to $f(x)$ for all such x, that is,

$$f(x) = \int_0^{\infty} A(\lambda) \cos \lambda x \, d\lambda = \int_0^{\infty} \frac{2\lambda \sin \lambda\pi}{\pi(1-\lambda^2)} \cos \lambda x \, d\lambda = \begin{cases} \cos x & \text{for} \quad |x| < \pi \\ 0 & \text{for} \quad |x| > \pi. \end{cases}$$

for all $x \neq \pm\pi$.

When $x = \pm\pi$, from Dirichlet's theorem the Fourier integral representation converges to

$$\frac{f(\pi^+) + f(\pi^-)}{2} = \frac{0-1}{2} = -\frac{1}{2} \qquad \text{and} \qquad \frac{f(-\pi^+) + f(-\pi^-)}{2} = \frac{-1+0}{2} = -\frac{1}{2}.$$

(c) From part (b) above, we have

$$\int_0^\infty \frac{\lambda \sin \lambda\pi \cos \lambda x}{1-\lambda^2}\, d\lambda = \begin{cases} \dfrac{\pi}{2}\cos x & \text{for} \quad |x| < \pi \\ \quad 0 & \text{for} \quad |x| > \pi \\ -\dfrac{\pi}{4} & \text{for} \quad |x| = \pi. \end{cases}$$

Exercise 0.5. ✗
Besides linear equations, some nonlinear equations can also result in *traveling wave solutions* of the form

$$u(x,t) = \phi(x-ct).$$

Fisher's equation, which models the spread of an advantageous gene in a population, where $u(x,t)$ is the density of the gene in the population at time t and location x, is given by

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + u(1-u).$$

Show that Fisher's equation has a solution of this form if ϕ satisfies the nonlinear ordinary differential equation

$$\phi'' + c\phi' + \phi(1-\phi) = 0.$$

Solution to Exercise 0.5: If $u(x,t) = \phi(x-ct)$, then

$$\begin{aligned} \frac{\partial u}{\partial x} &= \phi'(x-ct) \\ \frac{\partial^2 u}{\partial x^2} &= \phi''(x-ct) \\ \frac{\partial u}{\partial t} &= -c\phi'(x-ct), \end{aligned}$$

and Fisher's equation becomes

$$-c\phi'(x-ct) = \phi''(x-ct) + \phi(x-ct)\left(1-\phi(x-ct)\right),$$

for all x and t, so that if ϕ satisfies the nonlinear ordinary differential equation

$$\phi''(s) + c\phi'(s) + \phi(s)(1-\phi(s)) = 0, \quad -\infty < s < \infty,$$

then $u(x,t) = \phi(x-ct)$ is a traveling wave solution to Fisher's equation.