Math 337, Summer 2010
Assignment 2

Dr. T Hillen, University of Alberta

Exercise 0.1. ✗

Let $f(x) = \cos^2 x, \quad 0 \leqslant x \leqslant \pi$, and $f(x+2\pi) = f(x)$ otherwise.

(a) Find the Fourier sine series for f on the interval $[0, \pi]$.

Hint: For $n \geqslant 1$

$$\int \cos^2 x \sin nx \, dx = -\frac{1}{2n} \cos nx + \frac{1}{4} \int \left[\sin(n+2)x + \sin(n-2)x \right] \, dx.$$

(b) Find the Fourier cosine series for f on the interval $[0, \pi]$.

(c) For which values of x in $[0, \pi]$ do the series in (a) and (b) converge to $f(x)$?

Solution to Exercise 0.1:

(a) Writing $f(x) = \cos^2 x \sim \sum_{n=1}^{\infty} b_n \sin nx$, the coefficients b_n in the Fourier sine series are computed as follows:

$$\begin{aligned} b_n &= \frac{2}{\pi} \int_0^\pi \cos^2 x \sin nx \, dx = \frac{2}{\pi} \int_0^\pi \left(\frac{1}{2} + \frac{1}{2} \cos 2x \right) \sin nx \, dx \\ &= \frac{1}{\pi} \left(-\frac{1}{n} \cos nx \Big|_0^\pi \right) + \frac{1}{\pi} \int_0^\pi \cos 2x \sin nx \, dx \\ &= \frac{1}{n\pi} \left(1 - (-1)^n\right) + \frac{1}{2\pi} \int_0^\pi \left[\sin(n-2)x + \sin(n+2)x \right] \, dx \\ &= \frac{1-(-1)^n}{2\pi} \left(\frac{2}{n} + \frac{1}{n-2} + \frac{1}{n+2} \right) = 0 \end{aligned}$$

if $n \neq 2$ is even, while if $n = 2$, since $\sin 2x = 2 \sin x \cos x$, we have

$$\begin{aligned} b_2 &= \frac{2}{\pi} \int_0^\pi \cos^2 x \sin 2x \, dx = \frac{4}{\pi} \int_0^\pi \sin x \cos^3 x \, dx \\ &= -\frac{4}{\pi} \cos^4 x \Big|_0^\pi = 0. \end{aligned}$$

Therefore, $b_n = 0$ for all even $n \geqslant 2$.

If n is odd,

$$\begin{aligned} b_n &= \frac{2}{n\pi} + \frac{1}{\pi}\left[\frac{1}{n-2} + \frac{1}{n+2}\right] \\ &= \frac{2}{n\pi} + \frac{1}{\pi}\frac{2n}{n^2-4}. \end{aligned}$$

The Fourier sine series for f is therefore

$$\cos^2 x \sim \frac{2}{\pi}\sum_{k=1}^{\infty}\left\{\frac{1}{2k-1} + \frac{2k-1}{(2k-1)^2-4}\right\}\sin(2k-1)x$$

for $0 < x < \pi$.

(b) Using the double angle formula, we have

$$\cos^2 x = \frac{1}{2} + \frac{1}{2}\cos 2x,$$

which is the Fourier cosine series for f. If you integrate $\cos^2 x \cos nx$, you will find

$$a_0 = \frac{1}{2}, \quad a_2 = \frac{1}{2}, \quad \text{and} \quad a_k = 0 \quad \text{for} \quad k \neq 0,\ 2.$$

(c) From Dirichlet's theorem, the Fourier sine series in part (a) converges to $\cos^2 x$ for all $x \in (0, \pi)$ and converges to 0 for $x = 0$ and $x = \pi$. The Fourier cosine series in part (b) converges to $\cos^2 x$ for all $x \in [0, \pi]$ since the series is actually finite.

Exercise 0.2. ✗✗

Given the following initial boundary value problem for the heat equation on $[0,1]$.

$$\begin{aligned}\frac{\partial u}{\partial t} &= \frac{1}{9}\frac{\partial^2 u}{\partial x^2} - 2u\\ u(0,t) &= 0,\\ u(1,t) &= 0\\ u(x,0) &= 7\sin 3\pi x\end{aligned}$$

(a) If $u(x,t)$ is the solution to the problem above, find an initial boundary value problem satisfied by

$$w(x,t) = e^{2t}u(x,t).$$

(b) Solve the problem found in part (a) for $w(x,t)$.

(c) Find the solution $u(x,t)$ to the original problem.

(d) Find the time T_1 such that $u(x,t) < 1$ for every $x \in [0,1]$ and every $t > T_1$.

Solution to Exercise 0.2:

(a) If $u(x,t)$ is the solution to the heat equation above, and $w(x,t) = e^{2t}u(x,t)$, then

$$\begin{aligned}\frac{\partial w}{\partial t} &= e^{2t}\frac{\partial u}{\partial t} + 2e^{2t}u\\ &= e^{2t}\left(\frac{1}{9}\frac{\partial^2 u}{\partial x^2} - 2u\right) + 2e^{2t}u\\ &= e^{2t}\frac{1}{9}\frac{\partial^2 u}{\partial x^2}\end{aligned}$$

so that

$$\frac{\partial w}{\partial t} = \frac{1}{9}\frac{\partial^2 w}{\partial x^2}$$

for $0 \leqslant x \leqslant 1,\ t \geqslant 0$.

Therefore, $w(x,t) = e^{2t}u(x,t)$ satisfies the initial boundary value problem

$$\begin{aligned} \frac{\partial w}{\partial t} &= \frac{1}{9}\frac{\partial^2 w}{\partial x^2} \\ w(0,t) &= 0 \\ w(1,t) &= 0 \\ w(x,0) &= 7\sin 3\pi x \end{aligned}$$

(b) Assuming a solution of the form $w(x,t) = X(x)\cdot T(t)$ and separating variables, we get two ordinary differential equations

$$X'' + \lambda X = 0 \qquad \text{and} \qquad T' + \frac{\lambda}{9}T = 0,$$

where λ is the separation constant. We can satisfy the two boundary conditions by requiring that $X(0) = X(1) = 0$, so that X satisfies the boundary value problem

$$\begin{aligned} X'' + \lambda X &= 0 \\ X(0) &= 0 \\ X(1) &= 0. \end{aligned}$$

The only nontrivial solutions occur when $\lambda > 0$, say $\lambda = \mu^2$, where $\mu \neq 0$. In this case the general solution is

$$X(x) = A\cos\mu x + B\sin\mu x$$

and from the boundary conditions, $X(0) = 0$ implies that $A = 0$, and $X(1) = 0$ implies that $\sin\mu = 0$, so the eigenvalues are $\mu_n = n\pi$, with corresponding eigenfunctions $X_n(x) = \sin n\pi x$ for $n \geqslant 1$.

For $n \geqslant 1$, the corresponding solution to

$$T' + \frac{n^2\pi^2}{9}T = 0$$

is $T_n(t) = e^{-\frac{n^2\pi^2}{9}t}$, and from the superposition principle, we write

$$w(x,t) = \sum_{n=1}^{\infty} b_n \sin n\pi x\, e^{-\frac{n^2\pi^2}{9}t}$$

for $0 \leqslant x \leqslant 1,\ t \geqslant 0$.

From the initial condition, we have

$$7\sin 3\pi x = w(x,0) = \sum_{n=1}^{\infty} b_n \sin n\pi x,$$

so that $b_n = 0$ for $n \neq 3$, while $b_3 = 7$. Therefore,

$$w(x,t) = 7\sin 3\pi x\, e^{-\pi^2 t}$$

for $0 \leqslant x \leqslant 1,\ t \geqslant 0$.

(c) The solution to the original problem is

$$u(x,t) = e^{-2t}w(x,t) = 7\sin 3\pi x\, e^{-(\pi^2+2)t}$$

for $0 \leqslant x \leqslant 1,\ t \geqslant 0$.

(d) Since

$$\sin 3\pi x \leqslant |\sin 3\pi x| \leqslant 1 \qquad \text{and} \qquad e^{-(\pi^2+2)t} > 0,$$

for all $x \in [0,1]$ and all $t \geqslant 0$, then we can make $u(x,t) < 1$ by requiring that

$$\left| 7\sin 3\pi x\, e^{-(\pi^2+2)t} \right| < 1,$$

and this will be true if

$$7e^{-(\pi^2+2)t} < 1,$$

that is, if

$$e^{(\pi^2+2)t} > 7,$$

or equivalently, if

$$t > \frac{\log 7}{\pi^2+2},$$

so we may take

$$T_1 = \frac{\log 7}{\pi^2+2}.$$

Exercise 0.3. ✗

Let $0 < a < \pi$, given the function

$$f(x) = \begin{cases} \dfrac{1}{2a} & \text{if} \quad |x| < a \\ 0 & \text{if} \quad x \in [-\pi,\pi], \quad \text{and} \quad |x| > a \end{cases}$$

find the Fourier series for f and use Dirichlet's convergence theorem to show that

$$\sum_{n=1}^{\infty} \frac{\sin na}{n} = \frac{1}{2}(\pi - a)$$

for $0 < a < \pi$.

Solution to Exercise 0.3: Since $f(x)$ is an even function of the interval $[-\pi, \pi]$, the Fourier series of $f(x)$ is given by

$$f(x) \sim a_0 + \sum_{n=1}^{\infty} a_n \cos nx$$

where

$$a_0 = \frac{1}{\pi} \int_0^{\pi} f(x)\, dx = \frac{1}{\pi} \int_0^{a} \frac{1}{2a}\, dx = \frac{1}{2\pi},$$

and

$$\begin{aligned} a_n &= \frac{2}{\pi} \int_0^{\pi} f(x) \cos nx\, dx \\ &= \frac{2}{\pi} \int_0^{a} \frac{1}{2a} \cos nx\, dx \\ &= \frac{1}{\pi a} \int_0^{a} \cos nx\, dx \\ &= \frac{1}{\pi a} \cdot \frac{1}{n} \sin nx \Big|_0^a \\ &= \frac{1}{\pi a} \cdot \frac{\sin na}{n}, \end{aligned}$$

that is,

$$a_n = \frac{1}{\pi a} \cdot \frac{\sin na}{n}$$

for $n \geqslant 1$, and

$$f(x) \sim \frac{1}{2\pi} + \frac{1}{\pi a} \sum_{n=1}^{\infty} \frac{\sin na \cos nx}{n}$$

for $-\pi < x < \pi$.

Since $f(x)$ is continuous on the interval $-a < x < a$ the Fourier series converges to $f(x)$ for $-a < x < a$, that is,

$$f(x) = \frac{1}{2\pi} + \frac{1}{\pi a} \sum_{n=1}^{\infty} \frac{\sin na \cos nx}{n}$$

for $-a < x < a$, in particular, when $x = 0$, we have

$$\frac{1}{2a} = \frac{1}{2\pi} + \frac{1}{\pi a} \sum_{n=1}^{\infty} \frac{\sin na}{n},$$

so that

$$\sum_{n=1}^{\infty} \frac{\sin na}{n} = \frac{1}{2}(\pi - a)$$

for $0 < a < \pi$.

Exercise 0.4. ✗
Consider the heat equation with a steady source

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + 7\sin 3x$$

subject to the initial and boundary conditions:

$$u(0,t) = 0, \qquad u(\pi,t) = 0, \qquad \text{and} \qquad u(x,0) = 5\sin 3x.$$

Solve this problem using the method of eigenfunction expansions. Show that the solution approaches a steady-state solution as $t \to \infty$.

Solution to Exercise 0.4: Since the problem already has homogeneous boundary conditions, we consider the corresponding homogeneous problem:

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}, \quad 0 \leqslant x \leqslant \pi, \quad t \geqslant 0$$
$$u(0,t) = 0, \quad t \geqslant 0$$
$$u(\pi,t) = 0, \quad t \geqslant 0.$$

The eigenvalues and eigenfunctions for this problem are

$$\lambda_n = n^2 \qquad \text{and} \qquad \phi_n(x) = \sin nx$$

for $n \geqslant 1$.
We write the solution to the nonhomogeneous problem as an expansion in terms of these eigenfunctions:

$$u(x,t) = \sum_{n=1}^{\infty} a_n(t)\sin nx,$$

and determine the coefficients $a_n(t)$ which force this to be a solution to the nonhomogeneous problem.
We will need the eigenfunction expansions for $Q(x) = 7\sin 3x$ and $f(x) = 5\sin 3x$:

$$7\sin 3x = \sum_{n=1}^{\infty} q_n \sin nx, \qquad \text{with} \qquad q_n = 0 \quad \text{for} \quad n \neq 3, \quad q_3 = 7$$

$$5\sin 3x = \sum_{n=1}^{\infty} f_n \sin nx, \qquad \text{with} \qquad f_n = 0 \quad \text{for} \quad n \neq 3, \quad f_3 = 5.$$

Substituting these expansions into the nonhomogeneous equation

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + 7\sin 3x,$$

we obtain

$$\frac{d\,a_3(t)}{dt}\sin 3x = -9\,a_3(t)\sin 3x + 7\sin 3x,$$

and the coefficient $a_3(t)$ satisfies the initial value problem

$$\frac{d\,a_3(t)}{dt} + 9\,a_3(t) = 7, \quad t \geqslant 0$$

$$a_3(0) = 5.$$

The solution to this initial value problem is

$$a_3(t) = 5e^{-9t} + 7\int_0^t e^{-9(t-s)}\,ds,$$

that is,

$$a_3(t) = \frac{7}{9} + \left(5 - \frac{7}{9}\right)e^{-9t}, \quad t \geqslant 0.$$

Note that $\lim_{t\to\infty} a_3(t) = \frac{7}{9}$.
The solution to the heat equation with a steady source is therefore

$$u(x,t) = \left[\frac{7}{9} + \left(5 - \frac{7}{9}\right)e^{-9t}\right]\sin 3x$$

for $0 \leqslant x \leqslant \pi$ and $t \geqslant 0$.
For large value of t, this solution approaches $r(x)$ where

$$r(x) = \lim_{t\to\infty} u(x,t) = \frac{7}{9}\sin 3x$$

for $0 \leqslant x \leqslant \pi$. where
Differentiating this twice with respect to x, we see that

$$r''(x) = -7\sin 3x,$$

and since $r(0) = r(\pi) = 0$, then the function $r(x)$ satisfies the boundary value problem

$$\frac{d^2r}{dx^2} + 7\sin 3x = 0, \quad 0 \leqslant x \leqslant \pi$$

$$r(0) = 0$$

$$r(\pi) = 0,$$

which is exactly the boundary value problem for the steady-state solution, that is, $r(x)$ is the steady-state or equilibrium solution to the original heat flow problem.

Exercise 0.5. ✗

(a) Using the method of characteristics, solve

$$\frac{\partial w}{\partial t} + c\frac{\partial w}{\partial x} = e^{2x}, \quad -\infty < x < \infty, \quad t \geqslant 0$$

$$w(x,0) = \frac{1}{2}\,e^{2x}, \quad -\infty < x < \infty.$$

(b) For which values of c does this initial value problem have a time-independent solution?

Solution to Exercise 0.5:

(a) Let $\dfrac{dx}{dt} = c$, then along the characteristic curve $x(t) = ct + a$, where $a = x(0)$, the partial differential equation becomes

$$\frac{dw}{dt} = \frac{\partial w}{\partial t} + \frac{\partial w}{\partial x}\frac{dx}{dt} = e^{2x(t)} = e^{2(ct+a)},$$

so that

$$w(x(t),t) = \frac{1}{2c}e^{2(ct+a)} + K = \frac{1}{2c}e^{2x(t)} + K$$

where K is a constant, and $K = w(x(0),0) - \dfrac{1}{2c}e^{2x(0)}$ so that

$$w(x(t),t) = \frac{1}{2c}e^{2x(t)} + w(x(0),0) - \frac{1}{2c}e^{2x(0)},$$

that is,

$$w(x(t),t) = \frac{1}{2c}e^{2x(t)} + \frac{1}{2}e^{2(x(t)-ct)} - \frac{1}{2c}e^{2(x(t)-ct)}.$$

Given the point (x,t), let $x = ct+a$ be the unique characteristic curve passing through this point, then

$$w(x,t) = \frac{1}{2c}\,e^{2x} + \frac{1}{2}\,e^{2(x-ct)} - \frac{1}{2c}\,e^{2(x-ct)}$$

for $-\infty < x < \infty$ and $t > 0$.

(b) Note that if $c = 1$, then the solution is

$$w(x,t) = \frac{1}{2}\,e^{2x}, \quad -\infty < x < \infty$$

which is time-independent.