MATH 300 Fall 2004
Advanced Boundary Value Problems I
Solutions to Assignment 4
Due: Friday November 19, 2004

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Question 1. [p 205, #2]

Solve the vibrating membrane problem given below:

$$\begin{aligned}
&\frac{\partial^2 u}{\partial t^2} = 100\left(\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r}\right), && 0 < r < 1, \quad t > 0 \\
&u(a,t) = 0, && t > 0 \\
&u(r,0) = 1 - r^2, && 0 < r < 1 \\
&\frac{\partial u}{\partial t}(r,0) = 1, && 0 < r < 1.
\end{aligned}$$

You may use formula (11) from the text.

SOLUTION: Since $f(r) = 1 - r^2$ and $g(r) = 1$ are radially symmetric, we may assume that the solution does not depend on θ (we can show this by separating variables and applying periodicity conditions in θ). Also, we expect periodic functions in t, and in order to separate variables we write $u(r,t) = R(r)\cdot T(t)$, and obtain the problems

$$\begin{aligned}
rR'' + R' + \lambda^2 rR = 0, &\quad 0 < r < 1 \\
R(a) = 0, & \\
|R(r)| <= M, &\quad 0 \le r \le 1,
\end{aligned}$$

for some constant M, and

$$T'' + 100\lambda^2 T = 0, \quad t > 0.$$

The solutions to the first problem are

$$R(r) = J_0(\lambda r), \quad r > 0,$$

where J_0 is the Bessel function of order 0 of the first kind. The boundary condition $u(1,t) = 0$ for all $t > 0$ can be satisfied by requiring that $R(1) = 0$, that is, $J_0(\lambda) = 0$, so that λ must be a root of the Bessel function J_0. Now, J_0 has infinitely many positive zeros, and we write them as

$$\alpha_1 < \alpha_2 < \alpha_3 < \cdot < \alpha_n < \cdots,$$

and therefore we have nontrivial solutions to the boundary value problem only when

$$\lambda_n = \alpha_n,$$

$n = 1, 2, 3, \ldots$, and these are the eigenvalues of the boundary value problem, the corresponding eigenfunctions are

$$R_n(r) = J_0\left(\alpha_n r\right),$$

for $n = 1, 2, 3 \ldots$.

The solution to the differential equation for T corresponding to $\lambda_n = \alpha_n$ is given by

$$T_n(t) = A_n \cos 10\lambda_n t + B_n \sin 10\lambda_n t,$$

and the functions

$$u_n(r,t) = (A_n \cos 10\lambda_n t + B_n \sin 10\lambda_n t)\, J_0(\lambda_n r)$$

satisfy the wave equation and the boundary condition for each $n = 1, 2, \ldots$.

Using the superposition principle, we write the solution as a Fourier-Bessel expansion

$$u(r,t) = \sum_{n=1}^{\infty} (A_n \cos 10\lambda_n t + B_n \sin 10\lambda_n t)\, J_0(\lambda_n r), \qquad (*)$$

and evaluate the coefficients A_n and B_n from the initial conditions. In order to do this, we need the orthonality conditions

$$\int_0^1 rJ_0(\lambda_n r)J_0(\lambda_m r)\, dr = 0$$

for $n \neq m$. In order to see this, we recall that R_n and R_m satisfy the equations

$$\begin{aligned} (rR_n')' + \lambda_n^2 rR_n &= 0 \\ (rR_m')' + \lambda_m^2 rR_m &= 0 \end{aligned}$$

and multiplying the first equation by R_m and the second equation by R_n and subtracting, we get

$$(rR_n')'\, R_m - (rR_m')'\, R_n = (\lambda_m^2 - \lambda_n^2) rR_nR_m,$$

that is,

$$(r(R_mR_n' - R_nR_m'))' = (\lambda_n^2 - \lambda_m^2) rR_nR_m,$$

and integrating this last equation from 0 to 1 and using the fact that $R_m(1) = R_n(1) = 0$, we have

$$(\lambda_n^2 - \lambda_m^2) \int_0^1 rR_n(r)R_m(r)\, dr = 0$$

for $n \neq m$, and since $\lambda_n \neq \lambda_m$, we have

$$\int_0^1 rJ_0\left(\alpha_n r\right) J_0\left(\alpha_m r\right)\, dr = 0 \qquad (**)$$

for $n \neq m$, and the eigenfunctions are orthogonal with respect to the weight function r on the interval $[0,1]$.

In order to determine the coefficient A_n from the initial condition, we also need to know the value of

$$\int_0^1 rR_n(r)^2\, dr,$$

and we can determine this by considering the differential equation satisfied by R_n, namely,

$$(rR_n')' + \lambda_n^2 rR_n = 0,$$

and multiplying this by $2rR_n'$ to get

$$\frac{d}{dr}\left[(rR_n')^2\right] + 2\lambda_n^2 r^2 R_n R_n' = 0,$$

and integrating both terms we get

$$\left(rR_n'(r)\right)^2\Big|_0^1 + \lambda_n^2\left[r^2R_n(r)^2\Big|_0^1 - \int_0^1 2rR_n(r)^2\,dr\right] = 0,$$

where we integrated by parts in the second integral. Since $R_n(1) = 0$, we get

$$R_n'(1)^2 - \lambda_n^2\int_0^1 2rR_n(r)^2\,dr = 0,$$

that is,

$$\int_0^1 rR_n(r)^2\,dr = \frac{1}{2\lambda_n^2}R_n'(1)^2 = \frac{1}{2}J_0'(\lambda_n)^2 = \frac{1}{2}J_1(\lambda_n)^2 \qquad (***)$$

for $n = 1, 2, 3, \ldots$. Where we have used the identity $J_0'(r) = -J_1(r)$.

Now we can use the initial conditions to determine the coefficients in the solution $(*)$. Setting $t = 0$, multiplying by $rR_m(r)$, and integrating from 0 to 1, we get

$$\int_0^1 rf(r)R_m(r)\,dr = A_m\int_0^1 rR_m(r)^2\,dr = A_m\frac{J_1(\lambda_m)^2}{2},$$

and since $f(r) = 1 - r^2$, we have

$$A_m = \frac{2}{J_1(\lambda_m)^2}\int_0^1 r(1-r^2)R_m(r)\,dr = \frac{2}{J_1(\lambda_m)^2}\int_0^1 r(1-r^2)J_0(\lambda_m r)\,dr$$

for $m = 1, 2, 3, \ldots$.

If we make the substitution $s = \lambda_m r$ in the last integral, we get

$$\int_0^1 r(1-r^2)J_0(\lambda_m r)\,dr = \frac{1}{\lambda_m^4}\int_0^{\lambda_m} s(\lambda_m^2 - s^2)J_0(s)ds,$$

and integrating by parts with $u = \lambda_m^2 - s^2$ and $dv = J_0(s)s\,ds$ so that

$$v = \int sJ_0(s)\,ds = sJ_1(s),$$

we get

$$\int_0^1 r(1-r^2)J_0(\lambda_m r)\,dr = \frac{2}{\lambda_m^4}\int_0^{\lambda_m} J_1(s)s^2\,ds = \frac{2}{\lambda_m^4}s^2J_2(s)\Big|_0^{\lambda_m} = \frac{2}{\lambda_m^2}J_2(\lambda_m),$$

for $m = 1, 2, 3, \ldots$, where we used the identity

$$\int x^{p+1}J_p(x)\,dx = x^{p+1}J_{p+1}(x) + C.$$

Therefore,

$$A_m = \frac{2}{J_1(\lambda_m)^2} \int_0^1 r(1-r^2) J_0(\lambda_m r)\, dr = \frac{4J_2(\lambda_m)}{\lambda_m^2 J_1(\lambda_m)^2},$$

and finally, since λ_m is a zero of J_0, from the identity

$$J_0(x) + J_2(x) = \frac{2}{x} J_1(x),$$

we have

$$A_m = \frac{8}{\lambda_m^3 J_1(\lambda_m)}$$

for $m = 1, 2, 3, \ldots$, and

$$1 - r^2 = f(r) = \sum_{n=1}^{\infty} \frac{8}{\lambda_m^3 J_1(\lambda_m)} J_0(\lambda_n r), \quad 0 < r < 1$$

is the Fourier-Bessel expansion for the initial displacement.

In order to compute the B_n's, we differentiate $(*)$ with respect to t and then set $t = 0$ to get

$$1 = g(r) = \frac{\partial u}{\partial t}(r, 0) = \sum_{n=1}^{\infty} 10\lambda_n B_n J_0(\lambda_n r),$$

and a similar argument to that above shows that

$$B_m = \frac{1}{5\lambda_m^2 J_1(\lambda_m)}$$

for $m = 1, 2, 3, \ldots$, therefore the solution is

$$u(r,t) = \sum_{n=1}^{\infty} \frac{J_0(\lambda_n r)}{5\lambda_n^3 J_1(\lambda_n)} \left[40 \cos(10\lambda_n t) + \lambda_n \sin(10\lambda_n t)\right]$$

for $0 \le r \le 1$, and $t \ge 0$.

Question 2. [p 206, #4]

Solve the vibrating membrane problem given below:

$$\begin{aligned}
&\frac{\partial^2 u}{\partial t^2} = \left(\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r}\right), && 0 < r < 1, \quad t > 0 \\
&u(a, t) = 0, && t > 0 \\
&u(r, 0) = 0, && 0 < r < 1 \\
&\frac{\partial u}{\partial t}(r, 0) = J_0(\alpha_3 r), && 0 < r < 1.
\end{aligned}$$

You may use formula (11) from the text.

Solution: As in the previous problem, the solution is

$$u(r,t) = \sum_{n=1}^{\infty} (A_n \cos \lambda_n t + B_n \sin \lambda_n t)\, J_0(\lambda_n r),$$

where λ_n is the n^{th} positive root of the Bessel function J_0.

In this case, however, $u(r,0) = f(r) = 0$ for $0 < r < 1$, so that $A_n = 0$ for all $n \geq 1$. We use the initial condition

$$\frac{\partial u}{\partial t}(r,0) = J_0(\alpha_3 r), \quad 0 < r < 1$$

and the orthogonality to determine the B_n's, as in the previous problem, we have

$$B_n = \frac{2}{\lambda_n J_1(\lambda_n)^2} \int_0^1 r J_0(\lambda_3 r) J_0(\lambda_n r)\, dr = 0$$

for all $n \neq 3$, while for $n = 3$, we have

$$B_3 = \frac{2}{\lambda_3 J_1(\lambda_3)^2} \int_0^1 r J_0(\lambda_3 r)^2\, dr = \frac{2}{\lambda_3 J_1(\lambda_3)^2} \cdot \frac{1}{2} J_1(\lambda_3)^2 = \frac{1}{\lambda_3}$$

and the solution is

$$u(r,t) = \frac{1}{\lambda_3} J_0(\lambda_3 r) \sin \lambda_3 t$$

for $0 \leq r \leq 1$, and $t \geq 0$.

Question 3. [p 331, #2]

If $f(x)$ is an even function and $g(x)$ is an odd function, show that the set of functions $\{f(x), g(x)\}$ is orthogonal with respect to the weight function

$$w(x) = 1$$

on any symmetric interval $[-a, a]$ containing 0.

Solution: We have

$$\begin{aligned}
\int_{-a}^{a} f(x)g(x)\, dx &= \underbrace{\int_{-a}^{0} f(x)g(x)\, dx}_{t=-x} + \int_0^a f(x)g(x)\, dx \\
&= \int_0^a f(-t)g(-t)\, dt + \int_0^a f(x)g(x)\, dx \\
&= -\int_0^a f(t)g(t)\, dt + \int_0^a f(x)g(x)\, dx \\
&= 0,
\end{aligned}$$

and therefore f and g are orthogonal on the symmetric interval $[-a, a]$ with respect to the weight function $w(x) = 1$.

Question 4. [p 332, #6]

Show that the set of Laguerre polynomials $\left\{1,\, 1 - x,\, \frac{1}{2}(2 - 4x + x^2)\right\}$ is orthogonal with respect to the weight function

$$w(x) = e^{-x}$$

on the interval $[0, \infty)$.

SOLUTION: Recall that for $n \geq 0$ we have

$$\int_0^\infty x^n e^{-x}\, dx = n!,$$

and therefore

$$< 1, 1 - x >= \int_0^\infty (1 - x)e^{-x}\, dx = 0! - 1! = 0,$$

$$< 1, \frac{1}{2}(2 - 4x + x^2) >= 0! - 2 \cdot 1! + \frac{1}{2} \cdot 2! = 1 - 2 + 1 = 0$$

and finally,

$$\begin{aligned} < 1 - x, \frac{1}{2}(2 - 4x + x^2) > &=< 1, \frac{1}{2}(2 - 4x + x^2) > - < x, \frac{1}{2}(2 - 4x + x^2) > \\ &= 0 - \left(1! - 2 \cdot 2! + \frac{3!}{2}\right) \\ &= -1 + 4 - 3 \\ &= 0, \end{aligned}$$

so the set of functions does form an orthogonal set on the interval $[0, \infty)$ with respect to the weight function $w(x) = e^{-x}$.

Question 5. [p 332, #8]

Is the set of functions $\left\{\frac{1}{2}(2 - 4x + x^2),\ -12x + 8x^3\right\}$ orthogonal with respect to the weight function

$$w(x) = e^{-x}$$

on the interval $[0, \infty)$?

SOLUTION: These functions are **not** orthogonal with respect to the weight function $w(x) = e^{-x}$ on the interval $[0, \infty)$, in fact,

$$\begin{aligned} < 8x^3 - 12x, \frac{1}{2}(x^2 - 4x + 2) > &= 2 < 2x^3 - 3x, x^2 - 4x + 2 > \\ &= 2\int_0^\infty (2x^3 - 3x)(x^2 - 4x + 2)e^{-x}\, dx \\ &= 2\int_0^\infty (2x^5 - 8x^4 + x^3 + 12x^2 - 6x)e^{-x}\, dx \\ &= 2\left[2 \cdot 5! - 8 \cdot 4! + 3! + 12 \cdot 2! - 6 \cdot 1!\right] \\ &= 2 \cdot 72 = 144. \end{aligned}$$

As an exercise, show that the first five Laguerre polynomials in the orthogonal basis with respect to this weight function on $[0, \infty)$ are given by

$$L_0(x) = 1, \quad L_1(x) = x - 1, \quad L_2(x) = x^2 - 4x + 2,$$

$$L_3(x) = x^3 - 9x^2 + 18x - 6, \quad L_4(x) = x^4 - 16x^3 + 72x^2 - 96x + 24$$

Question 6. [p 344, #6]

Given the boundary value problem

$$\begin{gathered} y'' + \left(\frac{1+\lambda x}{x}\right) y = 0 \\ y(1) = 0 \\ y(2) = 0, \end{gathered}$$

on the interval $[1, 2]$, put the equation in Sturm-Liouville form and decide whether the problem is regular or singular.

SOLUTION: We can rewrite the boundary value problem in the form

$$\begin{gathered} (xy')' + \lambda xy = 0 \\ y(1) = 0 \\ y(2) = 0 \end{gathered}$$

and here $p(x) = x,\ p'(x) = 1,\ q(x) = 0,\ r(x) = x$ are all continuous on the interval $[1, 2]$, with $p(x) > 0$ and $r(x) > 0$ for all $x \in [1, 2]$. Also, $c_1 = d_1 = 1$ and $c_2 = d_2 = 0$, so this is a **regular** Sturm-Liouville problem on the interval $[1, 2]$.

Question 7. [p 344, #8]

Given the boundary value problem

$$\begin{gathered} (1-x^2)y'' - 2xy' + (1+\lambda x)y = 0 \\ y(-1) = 0 \\ y(1) = 0, \end{gathered}$$

on the interval $[-1, 1]$, put the equation in Sturm-Liouville form and decide whether the problem is regular or singular.

SOLUTION: We can rewrite the boundary value problem in the form

$$\begin{gathered} \big((1-x^2)y'\big)' + (1+\lambda x)y = 0 \\ y(-1) = 0 \\ y(1) = 0 \end{gathered}$$

and here $p(x) = 1 - x^2,\ p'(x) = -2x,\ q(x) = 1,\ r(x) = x$ are all continuous on the interval $[-1, 1]$. Also, $c_1 = d_1 = 1$ and $c_2 = d_2 = 0$.

However, $p(x) = 0$ at the endpoints of the interval $[-1, 1]$, and $r(0) = 0$, so this is a **singular** Sturm-Liouville problem.

Question 8. [p 344, #14]

Find the eigenvalues and eigenfunctions of the Sturm-Liouville problem

$$y'' + \lambda y = 0$$
$$y(-\pi) = y(\pi)$$
$$y'(-\pi) = y'(\pi).$$

SOLUTION:

Case 1: If $\lambda = 0$, then the equation $y'' = 0$ has general solution $y(x) = Ax + B$ with $y' = A$. The first periodicity condition gives

$$-A\pi + B = A\pi + B$$

so that $A = 0$. The second periodicity condition is then automatically satisfied, so there is one nontrivial solution in this case. The eigenvalue is $\lambda = 0$ with corresponding eigenfunction $y_0 = 1$.

Case 2: If $\lambda < 0$, say $\lambda = -\mu^2$ where $\mu \neq 0$, then the differential equation becomes $y'' - \mu^2 y = 0$, and has general solution $y(x) = A\cosh \mu x + B \sinh \mu x$ with $y' = \mu A \sinh \mu x + \mu B \cosh \mu x$. The first periodicity condition gives

$$A\cosh \mu\pi - B\sinh \mu\pi = A\cosh \mu\pi + B\sinh \mu\pi,$$

since $\cosh \mu x$ is an even function and $\sinh \mu x$ is an odd function. We have $2B\sinh \mu\pi = 0$, and since $\sinh \mu\pi \neq 0$, then $B = 0$. The solution is then $y = A\cosh \mu x$, and the second periodicity condition gives

$$-\mu A \sinh \mu\pi = \mu A \sinh \mu\pi,$$

so that $2\mu A \sinh \mu\pi = 0$, and since $\mu \neq 0$, then $\sinh \mu\pi \neq 0$, so we must have $A = 0$. Therefore, there are no nontrivial solutions in this case.

Case 3: If $\lambda > 0$, say $\lambda = \mu^2$ where $\mu \neq 0$, the differential equation becomes $y'' + \mu^2 y = 0$, and has general solution $y(x) = A\cos \mu x + B \sin \mu x$, with $y'(x) = -A\mu \sin \mu x + B\mu \cos \mu x$.

Applying the first periodicity condition, we have

$$y(-\pi) = A\cos \mu\pi - B\sin \mu\pi = A\cos \mu\pi + B\sin \mu\pi = y(\pi)$$

so that $2B\sin \mu\pi = 0$.

Applying the second periodicity condition, we have

$$y'(-\pi) = A\mu \sin \mu\pi + B\mu \cos \mu\pi = -A\mu \sin \mu\pi + B\mu \cos \mu\pi = y'(\pi)$$

so that $2A\sin \mu\pi = 0$. Therefore, the following equations must hold simultaneously:

$$A\sin \mu\pi = 0$$
$$B\sin \mu\pi = 0$$

In order to get a nontrivial solution, we must have either $A \neq 0$, or $B \neq 0$, and if the equations hold, we must have $\sin \mu\pi = 0$. Therefore, μ must be an integer, so that the eigenvalues are

$$\lambda_n = \mu_n^2 = n^2$$

for $n = 1, 2, 3 \ldots$, and the eigenfunctions corresponding to these eigenvalues are $\sin nx$ and $\cos nx$ for $n = 1, 2, 3 \ldots$.

Question 9. [p 344, #16]

Find the eigenvalues and eigenfunctions of the Sturm-Liouville problem

$$y'' + \lambda y = 0$$
$$y(0) + y'(0) = 0$$
$$y(2\pi) = 0.$$

SOLUTION:

Case 1: If $\lambda = 0$, then the equation $y'' = 0$ has general solution $y(x) = Ax + B$ with $y' = A$. The first boundary condition gives

$$B + A = 0$$

so that $A = -B$. The second boundary condition gives

$$2\pi A + B = 0$$

so that $(2\pi - 1)A = 0$, and $A = -B = 0$, so there are no nontrivial solutions in this case.

Case 2: If $\lambda < 0$, say $\lambda = -\mu^2$ where $\mu \neq 0$, then the differential equation becomes $y'' - \mu^2 y = 0$, and has general solution $y(x) = A\cosh\mu x + B\sinh\mu x$ with $y' = \mu A\sinh\mu x + \mu B\cosh\mu x$. The first boundary condition gives

$$A + \mu B = 0$$

so that $A = -\mu B$. The second boundary condition gives

$$A\cosh 2\pi\mu + B\sinh 2\pi\mu = 0$$

and since $\cosh 2\pi\mu \neq 0$, then

$$B(\tanh 2\pi\mu - \mu) = 0,$$

and in order to get nontrivial solutions we need

$$\tanh 2\pi\mu = \mu.$$

The graphs of $f(\mu) = \tanh 2\pi\mu$ and $g(\mu) = \mu$ intersect at the origin, $\mu = 0$, and since

$$\lim_{\mu\to\infty} \tanh 2\pi\mu = 1 \qquad \text{and} \qquad \lim_{\mu\to-\infty} \tanh 2\pi\mu = -1,$$

and

$$f'(0) = 2\pi > 1 = g'(0),$$

they intersect again in exactly two more points $\mu = \pm\mu_0$, where μ_0 is the positive root of the equation $\tanh 2\pi\mu = \mu$. There is one nontrivial solution in this case, with eigenvalue $\lambda = -(\mu_0)^2$ and the corresponding eigenfunction is

$$\sinh\mu_0 x - \mu_0\cosh\mu_0 x.$$

Case 3: If $\lambda > 0$, say $\lambda = \mu^2$ where $\mu \neq 0$, then the differential equation becomes $y'' + \mu^2 y = 0$, and has general solution $y(x) = A\cos\mu x + B\sin\mu x$ with $y' = -\mu A\sin\mu x + \mu B\cos\mu x$. The first boundary condition gives

$$y(0) + y'(0) = A + \mu B = 0$$

so that $A = -\mu B$. The second boundary condition gives

$$y(2\pi) = A\cos 2\pi\mu + B\sin 2\pi\mu = 0,$$

and so

$$B\left[\sin 2\pi\mu - \mu\cos 2\pi\mu\right] = 0,$$

and the eigenvalues are $\lambda_n = \mu_n^2$, where μ_n is the n^{th} positive root of the equation $\tan 2\pi\mu = \mu$. The corresponding eigenfunctions are

$$y_n = \sin\mu_n x - \mu_n\cos\mu_n x$$

for $n = 1, 2, 3, \ldots$.

Question 10. [p 344, #22]

Show that the boundary value problem

$$y'' - \lambda y = 0$$
$$y(0) + y'(0) = 0$$
$$y(1) + y'(1) = 0$$

has one positive eigenvalue. Does this contradict Theorem 1?

Solution:

Case 1: If $\lambda = 0$, the differential equation $y'' = 0$ has general solution $y = Ax + B$, with $y' = A$. Applying the first boundary condition, we have

$$B + A = 0,$$

so that $B = -A$. Applying the second boundary condition, we have

$$A + B + A = 0,$$

so that $B = -2A$, and therefore $B = 2B$, and $B = A = 0$. Therefore, there are no nontrivial solutions in this case.

Case 2: If $\lambda < 0$, say $\lambda = -\mu^2$ where $\mu \neq 0$, the differential equation becomes $y'' + \mu^2 y = 0$ and has general solution $y = A\cos\mu x + B\sin\mu x$, with $y' = -\mu A\sin\mu x + \mu B\cos\mu x$. The first boundary condition gives

$$y(0) + y'(0) = A + \mu B = 0$$

so that $A = -\mu B$.

The second boundary condition gives

$$y(1) + y'(1) = A\cos\mu + B\sin\mu - \mu A\sin\mu + \mu B\cos\mu = 0,$$

that is,

$$(\cos\mu - \mu\sin\mu)A + (\sin\mu + \mu\cos\mu)B = 0.$$

The system of linear equations for A and B

$$\begin{aligned} A + \mu B &= 0 \\ (\cos\mu - \mu\sin\mu)A + (\sin\mu + \mu\cos\mu)B &= 0 \end{aligned}$$

has nontrivial solutions if and only if

$$(1+\mu^2)\sin\mu = 0,$$

that is if and only if $\sin\mu = 0$. The eigenvalues are $\lambda_n = -(\mu_n)^2 = -n^2$, with corresponding eigenfunctions

$$y_n = \sin nx - n\cos nx$$

for $n = 1, 2, 3, \ldots$.

Case 3: If $\lambda > 0$, say $\lambda = \mu^2$, the differential equation becomes $y'' - \mu^2 y = 0$ and has general solution $y = A\cosh\mu x + B\sinh\mu x$, with $y' = \mu A\sinh\mu x + \mu B\cosh\mu x$. The first boundary condition gives

$$y(0) + y'(0) = A + \mu B = 0$$

The second boundary condition gives

$$y(1) + y'(1) = A\cosh\mu + B\sinh\mu + \mu A\sinh\mu + \mu B\cosh\mu = 0,$$

that is,

$$(\cosh\mu + \mu\sinh\mu)A + (\sinh\mu + \mu\cosh\mu)B = 0.$$

The system of linear equations for A and B

$$\begin{aligned} A + \mu B &= 0 \\ (\cosh\mu + \mu\sinh\mu)A + (\sinh\mu + \mu\cosh\mu)B &= 0 \end{aligned}$$

has nontrivial solutions if and only if

$$(1-\mu^2)\sinh\mu = 0,$$

and since $\sinh\mu \neq 0$, if and only if $1 - \mu^2 = 0$, that is, if and only if $\mu = \pm 1$.

Therefore, there is only one positive eigenvalue, namely

$$\lambda = (\pm 1)^2 = 1,$$

with corresponding eigenfunction

$$y = \sinh x - \cosh x.$$

Note: If $r(x) = -1 < 0$, then the problem is not a regular Sturm-Liouville problem, and so this does not contradict Theorem 1, since Theorem 1 does not apply.