MATH 300 Fall 2004
Advanced Boundary Value Problems I
Solutions to Assignment 3
Due: Friday October 29, 2004

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Question 1. [p 151, #2]

Solve the problem of heat transfer in a bar of length $L = 1$ with initial heat distribution $f(x) = \cos \pi x$ and no heat loss at either end, where the thermal diffusivity is $c = 1$, that is, solve the initial boundary value problem below:

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}, \qquad 0 < x < 1, \quad t > 0$$

$$\frac{\partial u}{\partial x}(0,t) = \frac{\partial u}{\partial x}(1,t) = 0, \quad t > 0$$

$$u(x,0) = \cos \pi x, \qquad 0 < x < 1.$$

Solution: Since both the partial differential equation and the boundary conditions are linear and homogeneous we may use separation of variables, and we write

$$u(x,t) = X(x) \cdot T(t)$$

where X depends only on x and T depends only on t. Substituting this into the partial differential equation, we have

$$X \cdot T' = X'' \cdot T,$$

and separating variables,

$$\frac{X''}{X} = \frac{T'}{T} = -\lambda$$

which leads to the two ordinary differential equations

$$X'' + \lambda X = 0 \qquad \text{and} \qquad T' + \lambda T = 0.$$

Since

$$\frac{\partial u}{\partial x}(0,t) = X'(0) \cdot T(t) \qquad \text{and} \qquad \frac{\partial u}{\partial x}(1,t) = X'(1) \cdot T(t)$$

we can satisfy the boundary conditions by requiring that $X'(0) = X'(1) = 0$, so that $X(x)$ must satisfy the boundary value problem

$$X'' + \lambda X = 0, \quad 0 < x < 1, \quad t > 0$$
$$X'(0) = 0$$
$$X'(1) = 1.$$

Now we must find those values of λ for which this boundary value problem has a nontrivial solution.

Case 1: $\lambda = 0$

In this case, the differential equation is $X'' = 0$, with general solution

$$X(x) = Ax + B,$$

where A and B are constants. Applying the boundary condition $X'(0) = 0$, we get $B = 0$, so that $X(x) = A$, a constant. In this case, the second boundary condition is automatically fulfilled, and the only nontrivial solution is

$$X_0(x) = 1,$$

the constant solution. The corresponding solution to the equation $T' = 0$ is

$$T_0(t) = 1.$$

Case 2: $\lambda < 0$, say $\lambda = -\mu^2$ where $\mu \neq 0$

In this case, the differential equation becomes $X'' - \mu^2 X = 0$, with general solution

$$X(x) = A\cosh \mu x + B \sinh \mu x$$

where A and B are constants. We will need the derivative

$$X'(x) = \mu A \sinh \mu x + \mu B \cosh \mu x$$

in order to apply the boundary conditions, we have

$$X'(0) = \mu B = 0 \qquad \text{so that} \qquad B = 0,$$

and

$$X'(1) = \mu A \sinh \mu = 0 \qquad \text{so that} \qquad A = 0$$

since $\mu \neq 0$ and $\sinh \mu \neq 0$. Therefore, in this case the only solution is $X(x) = 0$, and there are no nontrivial solutions.

Case 3: $\lambda > 0$, say $\lambda = \mu^2$ where $\mu \neq 0$

In this case, the differential equation becomes $X'' + \mu^2 X = 0$, with general solution

$$X(x) = A\cos \mu x + B \sin \mu x$$

where A and B are constants. Again, we will need the derivative

$$X'(x) = -\mu A \sin \mu x + \mu B \cos \mu x$$

in order to apply the boundary conditions, we have

$$X'(0) = \mu B = 0 \qquad \text{so that} \qquad B = 0.$$

Now however, when we apply the second boundary condition

$$X'(1) = \mu A \sin \mu = 0$$

in order to get a nontrivial solution, we must require that $A \neq 0$, so that $\sin \mu = 0$, and $\mu = n\pi$ for some integer n. In this case, we get a nontrivial solution

$$X_n(x) = \cos n\pi x$$

for each integer $n \geq 1$.

The corresponding solution to the equation $T' + n^2\pi^2 T = 0$ is

$$T_n(t) = e^{-n^2\pi^2 t}$$

for $n \geq 1$.

For each $n \geq 0$, the product

$$u_n(x,t) = X_n(x) \cdot T_n(t) = e^{-n^2\pi^2 t} \cos n\pi x, \quad 0 < x < 1, \quad t > 0$$

satisfies the heat equation and the boundary conditions, and since they are both linear and homogeneous, then any linear combination does also, so we can use the superposition principle to write

$$u(x,t) = a_0 + \sum_{n=1}^{\infty} a_n e^{-n^2\pi^2 t} \cos n\pi x$$

and all we need to do now is find the coefficients a_n for $n \geq 0$, so that the initial condition is also satisfied. Setting $t = 0$ in the series above, we have

$$\cos \pi x = u(x,0) = a_0 + \sum_{n=1}^{\infty} a_n \cos n\pi x,$$

that is, the a_n's are just the coefficients in the Fourier cosine series for $\cos \pi x$ on the interval $[0,1]$.

Since $\cos \pi x$ is its own Fourier cosine series on the interval $[0,1]$, then

$$a_n = \begin{cases} 0 & \text{for} \quad n \neq 1, \\ 1 & \text{for} \quad n = 1. \end{cases}$$

and the solution is

$$u(x,y) = e^{-\pi^2 t} \cos \pi x$$

for $0 < x < 1, \quad t > 0$.

Question 2. [p 151, #6]

Solve the problem of heat transfer in a bar of length $L = \pi$ and thermal diffusivity $c = 1$, with initial heat distribution $u(x, 0) = \sin x$ where one end of the bar is kept at a constant temperature $u(0, t) = 0$, while there is no heat loss at the other end of the bar so that $u_x(\pi, t) = 0$, that is, solve the initial boundary value problem below:

$$
\begin{aligned}
&\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}, && 0 < x < \pi, \quad t > 0 \\
&u(0, t) = 0, && t > 0 \\
&\frac{\partial u}{\partial x}(\pi, t) = 0, && t > 0 \\
&u(x, 0) = \sin x, && 0 < x < \pi.
\end{aligned}
$$

Solution: Assuming $u(x, t) = X(x) \cdot T(t)$ and separating variables, we get the two ordinary differential equations $X'' + \lambda X = 0$ and $T' + \lambda T = 0$, and the boundary conditions lead to the following boundary value problem for X :

$$
\begin{aligned}
X'' + \lambda X = 0, \quad 0 < x < \pi \\
X(0) = 0 \\
X'(\pi) = 0
\end{aligned}
$$

Arguing as in the previous problem, the only nontrivial solutions occur when $\lambda > 0$, say $\lambda = \mu^2$ where $\mu \neq 0$, and the differential equation becomes

$$X'' + \mu^2 X = 0$$

with general solution

$$X(x) = A \cos \mu x + B \sin \mu x$$

and applying the first boundary condition, we have $A = 0$, so that

$$X(x) = B \sin \mu x \qquad \text{and} \qquad X'(x) = \mu B \cos \mu x.$$

Applying the second boundary condition, we have

$$B \cos \mu\pi = 0,$$

and in order to get nontrivial solutions we must have $\mu\pi = \dfrac{(2n-1)\pi}{2}$, so that the eigenvalues are $\mu_n^2 = \dfrac{(2n-1)^2}{4}$ for $n \geq 1$. The corresponding eigenfunctions are

$$X_n(x) = \sin \tfrac{(2n-1)x}{2}$$

for $n \geq 1$. The corresponding solutions to the equation $T' + \mu_n^2 T = 0$ are

$$T_n(t) = e^{-\frac{(2n-1)^2 t}{4}}$$

For each $n \geq 1$, the function

$$u_n(x, t) = X_n(x) \cdot T_n(t) = e^{-\frac{(2n-1)^2 t}{4}} \sin \tfrac{(2n-1)x}{2}$$

satisfies the heat equation and the boundary conditions.

Using the superposition principle, we write

$$u(x,t) = \sum_{n=1}^{\infty} b_n e^{-\frac{(2n-1)^2 t}{4}} \sin \tfrac{(2n-1)x}{2}$$

for $0 < x < \pi,\ t > 0$, and setting $t = 0$, we have

$$\sin x = u(x,0) = \sum_{n=1}^{\infty} b_n \sin \tfrac{(2n-1)x}{2} \qquad (*)$$

for $0 < x < \pi$.

In order to determine the coefficients b_n, we use the fact that the functions $\{\sin \frac{(2n-1)x}{2}\}_{n\geq 1}$ are orthogonal on the interval $[0,\pi]$. To see this, note that if $n \neq m$, then

$$\begin{aligned}
\int_0^\pi \sin \mu_m x \sin \mu_n x \, dx &= \frac{1}{2}\int_0^\pi \left[\cos(\mu_m - \mu_n)x - \cos(\mu_m + \mu_n)x\right]\, dx \\
&= \frac{\sin(\mu_m - \mu_n)x}{2(\mu_m - \mu_n)}\bigg|_0^\pi - \frac{\sin(\mu_m + \mu_n)x}{2(\mu_m + \mu_n)}\bigg|_0^\pi \\
&= \frac{\sin(m-n)\pi}{2(m-n)} - \frac{\sin(m+n)\pi}{2(m+n)} \\
&= 0.
\end{aligned}$$

Also, if $m = n$, then

$$\begin{aligned}
\int_0^\pi \sin^2 \mu_m x \, dx &= \int_0^\pi \left(\frac{1}{2} - \frac{1}{2}\cos 2\mu_m x\right)\, dx \\
&= \frac{\pi}{2} - \frac{\sin 2\mu_m x}{4\mu_m}\bigg|_0^\pi \\
&= \frac{\pi}{2} - \frac{\sin(2m-1)\pi}{2(2m-1)} \\
&= \frac{\pi}{2}.
\end{aligned}$$

Multiplying the equation $(*)$ by $\sin \mu_m x$ and integrating from 0 to π, and using the orthogonality result just proven, we have

$$\int_0^\pi \sin x \sin \mu_m x \, dx = b_m \int_0^\pi \sin^2 \mu_m \, dx = \frac{\pi}{2} \cdot b_m,$$

that is,

$$
\begin{aligned}
b_m &= \frac{2}{\pi}\int_0^\pi \sin x \sin \mu_m x \, dx \\
&= \frac{1}{\pi}\int_0^\pi \left[\cos(\mu_m - 1)x - \cos(\mu_n + 1)x\right] \, dx \\
&= \frac{\sin(\mu_m - 1)x}{\pi(\mu_m - 1)}\bigg|_0^\pi - \frac{\sin(\mu_m + 1)x}{\pi(\mu_m + 1)}\bigg|_0^\pi \\
&= \frac{\sin(\mu_m - 1)\pi}{\pi(\mu_m - 1)} - \frac{\sin(\mu_m + 1)\pi}{\pi(\mu_m + 1)} \\
&= \frac{2}{\pi}\left[\frac{\sin\frac{(2m-3)\pi}{2}}{(2m-3)} - \frac{\sin\frac{(2m+1)\pi}{2}}{(2m+1)}\right] \\
&= \frac{2}{\pi}\left[\frac{(-1)^m}{(2m-3)} - \frac{(-1)^m}{(2m+1)}\right] \\
&= \frac{8}{\pi}\,\frac{(-1)^m}{(2m-3)(2m+1)},
\end{aligned}
$$

since $\sin\frac{(2n+1)\pi}{2} = (-1)^n$.

Therefore, the solution is

$$
u(x,t) = \frac{8}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n}{(2n-3)(2n+1)}e^{-\frac{(2n-1)^2 t}{4}}\sin\frac{(2n-1)x}{2}
$$

for $0 < x < \pi,\ t > 0$.

Question 3. [p 152, #8]

In the problem of heat transfer in a bar of length L with initial temperature distribution $f(x)$ and no heat loss at either end, show that the asymptotic temperature is constant and equals the average temperature.

Note: This involves solving the initial boundary value problem

$$\begin{aligned} \frac{\partial u}{\partial t} &= \frac{\partial^2 u}{\partial x^2}, && 0 < x < L, \quad t > 0 \\ \frac{\partial u}{\partial x}(0,t) &= \frac{\partial u}{\partial x}(L,t) = 0, && t > 0 \\ u(x,0) &= f(x), && 0 < x < L, \end{aligned}$$

and finding $\lim_{t\to\infty} u(x,t)$.

Solution: Since both the partial differential equation and the boundary conditions are homogeneous, we write $u(x,t) = X(x) \cdot T(t)$ and separate variables to get the ordinary differential equation $T' + \lambda T = 0$ for T, and the following boundary value problem for X,

$$\begin{aligned} X'' + \lambda X &= 0 \\ X'(0) &= 0 \\ X'(L) &= 0. \end{aligned}$$

The only cases when we get nontrivial solutions are in the case $\lambda = 0$, in which case we get the solutions

$$X_0(x) = 1 \qquad \text{and} \qquad T_0(t) = 1,$$

and in the case when $\lambda > 0$, say $\lambda = \mu^2$, the eigenvalues are

$$\lambda_n = \mu_n^2 = \frac{n^2\pi^2}{L^2}$$

and the eigenfunctions are

$$X_n(x) = \cos \tfrac{n\pi x}{L}$$

for $n \geq 1$. The corresponding solutions for the T equation are

$$T_n(t) = e^{-\frac{n^2\pi^2 t}{L^2}}$$

for $n \geq 1$.

Using the superposition principle, we have

$$u(x,t) = a_0 + \sum_{n=1}^{\infty} a_n e^{-\frac{n^2\pi^2 t}{L^2}} \cos \tfrac{n\pi x}{L}$$

for $0 < x < L,\ t > 0$. We use the initial conditions to evaluate the constants a_n, setting $t = 0$, we get

$$f(x) = u(x,0) = a_0 + \sum_{n=1}^{\infty} a_n \cos \tfrac{n\pi x}{L}$$

which is the Fourier cosine series for $f(x)$ on the interval $[0, L]$, therefore the coefficients are given by

$$a_0 = \frac{1}{L}\int_0^L f(x)\,dx \qquad \text{and} \qquad a_n = \frac{2}{L}\int_0^L f(x)\cos \tfrac{n\pi x}{L}\,dx,\ n \geq 1.$$

Letting $t \to \infty$ in the expression for $u(x,t)$, since the exponential goes to 0, the only term that survives is a_0, therefore,

$$\lim_{t\to\infty} u(x,t) = a_0 = \frac{1}{L}\int_0^L f(x)\,dx,$$

the average initial temperature in the bar.

Question 4. [p 162, #2]

Solve the problem of a thin elastic membrane stretched over a square frame of side 1, where the vibrations are governed by the following two dimensional wave equation:

$$\begin{aligned}
&\frac{\partial^2 u}{\partial t^2} = \frac{1}{\pi^2}\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right), \quad && 0 < x < 1, \quad 0 < y < 1, \quad t > 0\\
&u(0,y,t) = u(1,y,t) = 0, && 0 \le y \le 1, \quad t \ge 0\\
&u(x,0,t) = u(x,1,t) = 0, && 0 \le x \le 1, \quad t \ge 0\\
&u(x,y,0) = \sin \pi x \sin \pi y, && 0 \le x \le 1, \quad 0 \le y \le 1\\
&\frac{\partial u}{\partial t}(x,y,0) = \sin \pi x, && 0 \le x \le 1, \quad 0 \le y \le 1.
\end{aligned}$$

Solutions: Separating variables, we write $u(x,y,t) = \phi(x,y)\cdot T(t)$, and substitute this into the wave equation

$$\pi^2\frac{T''}{T} = \frac{1}{\phi}\left(\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2}\right) = \lambda,$$

this gives the two equations

$$T'' - \frac{\lambda}{\pi^2}T = 0 \qquad \text{and} \qquad \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = \lambda\phi.$$

Separating variables again the second equation, we write $\pi(x,y) = X(x)\cdot Y(y)$, and substituting this into the equation, we get

$$\frac{X''}{X} + \frac{Y''}{Y} = \lambda,$$

that is,

$$\frac{X''}{X} = -\frac{Y''}{Y} + \lambda = -\kappa$$

where κ is a second separation constant. The boundary conditions give rise to two boundary value problems

$$\begin{aligned}
X'' + \kappa X &= 0 \qquad & Y'' - (\kappa+\lambda)Y &= 0\\
X(0) &= 0 & Y(0) &= 0\\
X(1) &= 0 & Y(1) &= 1.
\end{aligned}$$

We find nontrivial solutions to the X equation first, since it involves only one separation constant.

As in previous problems, there are nontrivial solutions only if $\kappa_n = n^2\pi^2$ and the eigenfunctions are

$$X_n(x) = \sin n\pi x$$

for $n \ge 1$.

For each $n \geq 1$, the Y satisfies the boundary value problem

$$\begin{aligned} Y'' - (n^2\pi^2 + \lambda)Y &= 0 \\ Y(0) &= 0 \\ Y(1) &= 0, \end{aligned}$$

and as in previous problems, this has nontrivial solutions only if $\lambda + n^2\pi^2 = -m^2\pi^2$, that is $\lambda = -(n^2+m^2)\pi^2$, and the eigenfunctions are

$$Y_m(y) = \sin m\pi y$$

for $m \geq 1$.

For each $n, m \geq 1$, the function

$$\phi_{n,m}(x,y) = \sin n\pi x \cdot \sin m\pi y$$

satisfies the equation for ϕ, as well as the four boundary conditions.

The solutions of the equation $T'' - \dfrac{\lambda}{\pi^2}T = 0$ corresponding to the separation constant $\lambda = -(n^2+m^2)\pi^2$ are

$$T_{n,m} = B_{n,m} \cos \sqrt{n^2+m^2}\, t + B^*_{n,m} \sin \sqrt{n^2+m^2}\, t$$

and for each $n, m \geq 1$, the function

$$u_{n,m}(x,y,t) = \phi_{n,m}(x,y) \cdot T_{n,m}(t) = \sin n\pi x \sin m\pi y \left(B_{n,m} \cos \sqrt{n^2+m^2}\, t + B^*_{n,m} \sin \sqrt{n^2+m^2}\, t \right)$$

satisfies the wave equation and all four boundary conditions. Using the superposition principle, we write the solution as

$$u(x,y,t) = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \sin n\pi x \sin m\pi y \left(B_{n,m} \cos \sqrt{n^2+m^2}\, t + B^*_{n,m} \sin \sqrt{n^2+m^2}\, t \right).$$

We evaluate the constants $B_{n,m}$ and $B^*_{n,m}$ using the initial conditions. Setting $t = 0$ in the above expression for $u(x,y,t)$ we see that

$$\sin \pi x \sin \pi y = u(x,y,0) = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} B_{n,m} \sin n\pi x \sin m\pi y,$$

so that

$$B_{n,m} = \begin{cases} 1 & \text{for} \quad n = m = 1, \\ 0 & \text{otherwise.} \end{cases}$$

Differentiating the expression for $u(x,y,t)$ with respect to t, and setting $t = 0$, we see that

$$\sin \pi x = \frac{\partial u}{\partial t}(x,y,0) = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \sqrt{n^2+m^2}\, B^*_{n,m} \sin n\pi x \sin m\pi y,$$

that is,

$$\sin \pi x = \sum_{n=1}^{\infty} \sin n\pi x \left(\sum_{m=1}^{\infty} \sqrt{n^2+m^2}\, B^*_{n,m} \sin m\pi y \right),$$

and we need

$$\sum_{m=1}^{\infty} \sqrt{1+m^2}\, B^*_{1,m} \sin m\pi y = 1, \qquad \text{and} \qquad \sum_{m=1}^{\infty} \sqrt{n^2+m^2}\, B^*_{n,m} \sin m\pi y = 0 \quad \text{if} \quad n \neq 1.$$

Therefore, we may take $B^*_{n,m} = 0$ for all $n \neq 1$, while for $n = 1$, we want $\sqrt{1+m^2}\, B^*_{1,m}$ to be the coefficients in the Fourier sine series of the function $f(x) = 1,\ 0 \le x \le 1$, that is,

$$B^*_{1,m} = \frac{2}{\sqrt{1+m^2}} \int_0^1 \sin m\pi y\, dy = \frac{2}{m\pi\sqrt{1+m^2}} \left[1 - (-1)^m\right]$$

for $m \ge 1$.

Therefore,

$$u(x,y,t) = \sin \pi x \sin \pi y \cos \sqrt{2}\pi t + \sum_{m=1}^{\infty} \frac{2\left[1-(-1)^m\right]}{m\pi\sqrt{1+m^2}} \sin \pi x \cos m\pi y \sin \sqrt{1+m^2}\, t$$

for $0 < x, y < 1,\ t > 0$.

Question 5. [p 163, #12]

Find the temperature distribution in a thin two dimensional plate with thermal diffusivity $c = 1$, in the shape of a unit square, with insulated faces and edges kept at zero temperature with an initial temperature distribution given by $f(x,y) = xy(1-x)(1-y)$ for $0 \le x, y \le 1$, that is, solve the initial boundary value problem given below:

$$\begin{aligned}
&\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}, && 0 < x < 1, \quad 0 < y < 1, \quad t > 0 \\
&u(0,y,t) = u(1,y,t) = 0, && 0 < y < 1, \quad t > 0 \\
&u(x,0,t) = u(x,1,t) = 0, && 0 < x < 1, \quad t > 0 \\
&u(x,y,0) = xy(1-x)(1-y), && 0 < x < 1, \quad 0 < y < 1.
\end{aligned}$$

SOLUTION: After separating variables, using the superposition principle, and applying the boundary conditions, we find the solution has the form

$$u(x,y,t) = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} B_{n,m} \sin n\pi x \sin m\pi y\, e^{-c\pi\sqrt{n^2+m^2}\, t}.$$

We evaluate $B_{n,m}$ using the initial condition

$$\begin{aligned}
B_{n,m} &= 4\int_0^1\int_0^1 x(1-x)y(1-y) \sin n\pi x \sin m\pi y\, dy\, dx \\
&= \left(2\int_0^1 x(1-x)\sin n\pi x\, dx\right) \cdot \left(2\int_0^1 y(1-y)\sin m\pi y\, dy\right) \\
&= \frac{16\left[1-(-1)^n\right] \cdot \left[1-(-1)^m\right]}{n^3 m^3 \pi^6}
\end{aligned}$$

for $n, m \ge 1$, that is,

$$B_{n,m} = \begin{cases} \dfrac{64}{n^3 m^3 \pi^6} & \text{if both } \quad n, m \quad \text{are odd} \\ 0 & \text{otherwise.} \end{cases}$$

The solution is therefore

$$u(x,y,t) = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} \frac{64}{\pi^6(2n-1)^3(2m-1)^3} \sin(2n-1)\pi x \sin(2m-1)\pi y\, e^{-\left[(2n-1)^2+(2m-1)^2\right]\pi^2 t}$$

for $0 < x < 1,\ 0 < y < 1,\ t > 0.$

Question 6. [p 168, #2]

Solve the Dirichlet problem for the unit square in the plane with the boundary data as given below:

$$\begin{aligned}
&\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0, && 0 < x < 1, \quad 0 < y < 1,\\
&u(x,0) = 0 && 0 \le x \le 1,\\
&u(x,1) = 100, && 0 \le x \le 1,\\
&u(0,y) = 0 && 0 \le y \le 1,\\
&u(1,y) = 100, && 0 \le y \le 1.
\end{aligned}$$

Solution: We split the original problem into two problems, as below

$$\begin{aligned}
&\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0, && 0 < x < 1, \quad 0 < y < 1,\\
&v(x,0) = 0 && 0 \le x \le 1,\\
&v(x,1) = 100, && 0 \le x \le 1,\\
&v(0,y) = 0 && 0 \le y \le 1,\\
&v(1,y) = 0, && 0 \le y \le 1
\end{aligned}$$

and

$$\begin{aligned}
&\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} = 0, && 0 < x < 1, \quad 0 < y < 1,\\
&w(x,0) = 0 && 0 \le x \le 1,\\
&w(x,1) = 0, && 0 \le x \le 1,\\
&w(0,y) = 0 && 0 \le y \le 1,\\
&w(1,y) = 100, && 0 \le y \le 1
\end{aligned}$$

each with one pair of homogeneous boundary conditions (so we can use separation of variables) and the soluton to the original problem is then $u(x,y) = v(x,y) + w(x,y)$.

Now note that we only have to solve one of these problems, say the first, for $v(x,y)$, since we can get the solution to the second problem by interchanging x and y in the solution to the first problem, that is, $w(x,y) = v(y,x)$, so that the solution to the original problem is $u(x,y) = v(x,y) + v(y,x)$.

Writing $v(x, y) = X(x) \cdot Y(y)$, after substituting this into Laplace's equation and separating variables, we have

$$\frac{X''}{X} = -\frac{Y''}{Y} = -\lambda,$$

so that we get the following boundary value problems for X and Y,

$$\begin{aligned} X'' + \lambda X &= 0 \qquad & Y'' - \lambda Y &= 0 \\ X(0) &= 0 & Y(0) &= 0 \\ X(1) &= 0 \end{aligned}$$

As in previous problems, we have a nontrivial solution for X only if $\lambda = \mu^2 > 0$, and in this case the general solution is

$$X(x) = A \cos \mu x + B \sin \mu x,$$

applying the boundary conditions, we have

$$X(0) = 0 = A, \qquad \text{and} \qquad X(1) = 0 = B \sin \mu.$$

We get a nontrivial solution only when $B \neq 0$, in which case we need $\mu = n\pi$ for some positive integer n, the eigenvalues are $\mu_n^2 = n^2\pi^2$, and the eigenfunctions are

$$X_n(x) = \sin n\pi x$$

for $n \geq 1$. For each $n \geq 1$, the corresponding equation for Y is $Y'' - \mu_n^2 Y = 0$, with general solution

$$Y(y) = A \cosh n\pi y + B \sinh n\pi y,$$

and applying the boundary condition $Y(0) = 0$, we get $A = 0$, so the corresponding solutions are

$$Y_n(y) = \sinh n\pi y, \quad n \geq 1.$$

For each $n \geq 1$, the function

$$v_n(x, y) = X_n(x) \cdot Y_n(y) = \sin n\pi x \sinh n\pi y$$

satisfies Laplace's equation and all of the boundary conditions except $v(x, 1) = 100$.

Now we use the superposition principle to write

$$v(x, y) = \sum_{n=1}^{\infty} b_n \sin n\pi x \sinh n\pi y$$

and determine the constants b_n using this last boundary condition, that is,

$$100 = v(x, 1) = \sum_{n=1}^{\infty} b_n \sin n\pi x \sinh n\pi = \sum_{n=1}^{\infty} (b_n \sinh n\pi) \sin n\pi x,$$

and we recognize the constant $b_n \sinh n\pi$ as the Fourier sine series coefficient of the constant function 100 on the interval $[0, 1]$, therefore

$$b_n \sinh n\pi = 2 \int_0^1 100 \sin n\pi x \, dx = \frac{200}{n\pi} [1 - (-1)^n] = \begin{cases} \dfrac{400}{n\pi} & \text{if } n \text{ is odd} \\ 0 & \text{if } n \text{ is even.} \end{cases}$$

The solution to the first problem is therefore

$$v(x,y) = \frac{400}{\pi}\sum_{n=1}^{\infty}\frac{1}{(2n-1)\sinh(2n-1)\pi}\sin n\pi x\sinh n\pi y$$

for $0 \le x, y \le 1$. Interchanging x and y in this solution, we get the solution to the second problem,

$$w(x,y) = \frac{400}{\pi}\sum_{n=1}^{\infty}\frac{1}{(2n-1)\sinh(2n-1)\pi}\sin n\pi y\sinh n\pi x,$$

and the solution to the original problem is therefore

$$u(x,y) = \frac{400}{\pi}\sum_{n=1}^{\infty}\frac{1}{(2n-1)\sinh(2n-1)\pi}\left[\sin n\pi x\sinh n\pi y + \sin n\pi y\sinh n\pi x\right]$$

for $0 \le x, y \le 1$.

Question 7. [p 168, #4]

Solve the Dirichlet problem for the unit square in the plane with the boundary data as given below:

$$\begin{aligned}
&\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0, && 0 < x < 1, \quad 0 < y < 1,\\
&u(x,0) = 1 - x && 0 \le x \le 1,\\
&u(x,1) = x, && 0 \le x \le 1,\\
&u(0,y) = 0 && 0 \le y \le 1,\\
&u(1,y) = 0, && 0 \le y \le 1.
\end{aligned}$$

SOLUTION: Again, we divide the problem into two problems:

$$\begin{aligned}
\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} &= 0, \quad 0 < x, y < 1, & \frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} &= 0, \quad 0 < x, y < 1,\\
v(x,0) &= 0, \quad 0 \le x \le 1, & w(x,0) &= 1 - x, \quad 0 \le x \le 1,\\
v(x,1) &= x, \quad 0 \le x \le 1, & w(x,1) &= 0, \quad 0 \le x \le 1,\\
v(0,y) &= 0, \quad 0 \le y \le 1, & w(0,y) &= 0, \quad 0 \le y \le 1,\\
v(1,y) &= 0, \quad 0 \le y \le 1, & w(1,y) &= 0, \quad 0 \le y \le 1
\end{aligned}$$

each with one pair of homogeneous boundary conditions (so we can use separation of variables) and the soluton to the original problem is then $u(x,y) = v(x,y) + w(x,y)$.

Note that if we find the solution $v(x,y)$ to the first problem, then the solution to the second problem is

$$w(x,y) = v(1-x, 1-y).$$

We leave it to you to check that $w(x, y)$ satisfies Laplace's equation, and for the boundary conditions, note that

$$\begin{aligned}
w(x,0) &= v(1-x,1) = 1-x \\
w(x,1) &= v(1-x,0) = 0 \\
w(0,y) &= v(1,1-y) = 0 \\
w(1,y) &= v(0,1-y) = 0
\end{aligned}$$

so that $w(x, y)$ is a solution to the second problem.

We can use separation of variables as in the previous problem to find the solution $v(x, y)$, and the result is

$$v(x,y) = \sum_{n=1}^{\infty} b_n \sin n\pi x \sinh n\pi y$$

and the constants b_n are determined from the last boundary condition

$$x = v(x,1) = \sum_{n=1}^{\infty} b_n \sinh n\pi \sin n\pi x$$

so that

$$b_n \sinh n\pi = 2\int_0^1 x \sin n\pi x \, dx = \frac{2(-1)^{n+1}}{n\pi},$$

so that

$$b_n = \frac{2(-1)^{n+1}}{n\pi \sinh n\pi}$$

for $n \geq 1$.

The solution to the first problem is

$$v(x,y) = \frac{2}{\pi}\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n \sinh n\pi} \sin n\pi x \sinh n\pi y,$$

and the solution to the second problem is

$$w(x,y) = \frac{2}{\pi}\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n \sinh n\pi} \sin n\pi(1-x) \sinh n\pi(1-y).$$

The solution to the original problem is given by

$$u(x,y) = \frac{2}{\pi}\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n \sinh n\pi} \left[\sin n\pi x \sinh n\pi y + \sin n\pi(1-x) \sinh n\pi(1-y)\right]$$

for $0 < x < 1,\ 0 < y < 1$.

Question 8. [p 169, #8]

Approximate the temperature at the center of the plate in Question 7.

SOLUTION: Note that at the center of the plate $x = y = \frac{1}{2}$, and from the previous problem

$$u(\tfrac{1}{2},\tfrac{1}{2}) = \frac{4}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n\sinh n\pi}\sin\tfrac{n\pi}{2}\sinh\tfrac{n\pi}{2}.$$

Now,

$$\sinh n\pi = 2\sinh\tfrac{n\pi}{2}\cosh\tfrac{n\pi}{2},$$

and

$$\sin\tfrac{n\pi}{2} = 0$$

if n is even, while

$$\sin\tfrac{(2k+1)\pi}{2} = (-1)^k$$

if $n = 2k+1$ is odd.

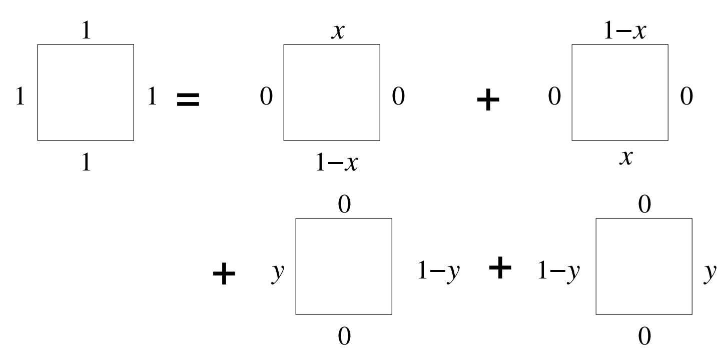
Therefore,

$$u(\tfrac{1}{2},\tfrac{1}{2}) = \frac{2}{\pi}\sum_{k=0}^{\infty}\frac{(-1)^k}{(2k+1)\cosh\frac{(2k+1)\pi}{2}},$$

and a simple symmetry argument as in the text shows that this series converges to $\frac{1}{4}$, that is,

$$u(\tfrac{1}{2},\tfrac{1}{2}) = \tfrac{1}{4}.$$

To see this, note that we can decompose the problem when the solution is identically 1 on the boundary of the square into four separate problems as shown in the figure:



By symmetry, each of the four problems has exactly the same value of the solution at the center $(\frac{1}{2},\frac{1}{2})$, and since the solution to the original problem is identically 1 on the square, then

$$4u(\tfrac{1}{2},\tfrac{1}{2}) = 1,$$

that is, $u(\frac{1}{2},\frac{1}{2}) = \frac{1}{4}$.

Question 9. [p 198, #2]

Compute the Laplacian of the function

$$u(x, y) = \tan^{-1}\left(\frac{y}{x}\right)$$

in an appropriate coordinate system and decide if the given function satisfies Laplace's equation $\nabla^2 u = 0$.

SOLUTION: Note that in polar coordinates $\theta = \tan^{-1}\left(\frac{y}{x}\right)$, so that

$$u(r, \theta) = \theta,$$

and since

$$\frac{\partial u}{\partial r} = \frac{\partial^2 u}{\partial r^2} = 0,$$

then Laplace's equation becomes

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{\partial^2 u}{\partial \theta^2} = \frac{\partial^2 \theta}{\partial \theta^2} = 0,$$

and $u(x, y) = \tan^{-1}\left(\frac{y}{x}\right)$ does satisfy Laplace's equation.

Question 10. [p 198, #6]

Compute the Laplacian of the function

$$u(x, y) = \ln(x^2 + y^2)$$

in an appropriate coordinate system and decide if the given function satisfies Laplace's equation $\nabla^2 u = 0$.

SOLUTION: Note that in polar coordinates, $r^2 = x^2 + y^2$, so that

$$u(r, \theta) = \ln r^2 = 2 \ln r,$$

and

$$\frac{1}{r}\frac{\partial u}{\partial r} = \frac{2}{r^2} \qquad \text{and} \qquad \frac{\partial^2 u}{\partial r^2} = -\frac{2}{r^2},$$

and since $\frac{\partial^2 u}{\partial \theta^2} = 0$, then

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{\partial^2 u}{\partial \theta^2} = -\frac{2}{r^2} + \frac{2}{r^2} = 0$$

and $u(x, y) = \ln(x^2 + y^2)$ does satisfy Laplace's equation.