MATH 300 Fall 2004
Advanced Boundary Value Problems I
Solutions to Sample Final Exam
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Department of Mathematical and Statistical Sciences
University of Alberta

Question 1. Given the function

$$f(x) = \cos \tfrac{\pi}{a} x, \quad 0 \le x < a$$

find the Fourier sine series for f.

SOLUTION:

Writing $f(x) = \cos \frac{\pi}{a} x \sim \sum\limits_{n=1}^{\infty} b_n \sin \frac{n\pi}{a} x$, the coefficients b_n in the Fourier sine series are computed as follows:

$$\begin{aligned} b_n &= \frac{2}{a} \int_0^a \cos \tfrac{\pi}{a} x \sin \tfrac{n\pi}{a} x \, dx = \frac{1}{a} \int_0^a \big(\sin \tfrac{(n+1)\pi}{a} x + \sin \tfrac{(n-1)\pi}{a} x \big) \, dx \\ &= \frac{1}{\pi} \Big(-\frac{1}{n+1} \cos \tfrac{(n+1)\pi}{a} x \Big|_0^a \Big) + \frac{1}{\pi} \Big(-\frac{1}{n-1} \cos \tfrac{(n-1)\pi}{a} x \Big|_0^a \Big) \\ &= \frac{1}{\pi(n+1)} \big((-1)^n + 1 \big) + \frac{1}{\pi(n-1)} \big((-1)^n + 1 \big) = \frac{1 + (-1)^n}{\pi} \Big(\frac{1}{n+1} + \frac{1}{n-1} \Big). \end{aligned}$$

Therefore,

$$b_n = \begin{cases} \dfrac{4n}{\pi(n^2 - 1)} & \text{if} \quad n \quad \text{is even} \\ 0 & \text{if} \quad n \quad \text{is odd}, \quad n \ge 3. \end{cases}$$

If $n = 1$,

$$b_1 = \frac{2}{a} \int_0^a \sin \tfrac{\pi}{a} x \, \cos \tfrac{\pi}{a} x \, dx = \frac{1}{a} \sin^2 \tfrac{\pi}{a} x \Big|_0^a = 0.$$

The Fourier sine series for f is therefore

$$\cos \tfrac{\pi}{a} x \sim \frac{8}{\pi} \sum_{n=1}^{\infty} \frac{n}{4n^2 - 1} \sin \tfrac{2n\pi}{a} x.$$

for $0 \le x < a$.

Question 2. Let

$$f(x) = \begin{cases} \cos x & |x| < \pi, \\ 0 & |x| > \pi. \end{cases}$$

(a) Find the Fourier integral of f.

(b) For which values of x does the integral converge to $f(x)$?

(c) Evaluate the integral

$$\int_0^\infty \frac{\lambda \sin \lambda\pi \cos \lambda x}{1-\lambda^2}\, d\lambda$$

for $-\infty < x < \infty$.

SOLUTION:

(a) The function

$$f(x) = \begin{cases} \cos x & |x| < \pi \\ 0 & |x| > \pi \end{cases}$$

is even, piecewise smooth, and is continuous at every $x \in (-\infty, \infty)$ except at $x = \pm\pi$, therefore from Dirichlet's theorem the Fourier integral representation of f converges to $f(x)$ for all $x \neq \pm\pi$, and

$$f(x) \sim \int_0^\infty A(\lambda) \cos \lambda x \, d\lambda,$$

where

$$\begin{aligned} A(\lambda) &= \frac{2}{\pi} \int_0^\infty f(x) \cos \lambda x \, dx = \frac{2}{\pi} \int_0^\pi \cos x \cos \lambda x \, dx \\ &= \frac{1}{\pi} \int_0^\pi \Big\{ \cos(\lambda+1)x + \cos(\lambda-1)x \Big\} \, dx \\ &= \frac{1}{\pi} \left\{ \frac{\sin(\lambda+1)x}{\lambda+1}\Big|_0^\pi + \frac{\sin(\lambda-1)x}{\lambda-1}\Big|_0^\pi \right\} \\ &= \frac{1}{\pi} \frac{\sin(\lambda+1)\pi}{\lambda+1} + \frac{1}{\pi} \frac{\sin(\lambda-1)\pi}{\lambda-1} \\ &= \frac{2\lambda}{\pi} \frac{\sin \lambda\pi}{1-\lambda^2}. \end{aligned}$$

The Fourier integral representation of f is therefore

$$f(x) \sim \frac{2}{\pi} \int_0^\infty \frac{\lambda \sin \lambda\pi \cos \lambda x}{1-\lambda^2} \, d\lambda.$$

(b) From Dirichlet's theorem, the integral converges to $f(x)$ for all $x \neq \pm\pi$, and converges to $-\frac{1}{2}$ for $x = \pm\pi$.

(c) Therefore, we have

$$\int_0^\infty \frac{\lambda \sin \lambda\pi \cos \lambda x}{1-\lambda^2} \, d\lambda = \begin{cases} \frac{\pi}{2} \cos x & \text{for} \quad |x| < \pi, \\ 0 & \text{for} \quad |x| > \pi, \\ -\frac{\pi}{4} & \text{for} \quad x = \pm\pi. \end{cases}$$

Question 3. Let $\mathcal{F}_c$ denote the Fourier cosine transform and $\mathcal{F}_s$ denote the Fourier sine transform. Assume that $f(x)$ and $xf(x)$ are both integrable.

(a) Show that

$$\mathcal{F}_c(xf(x)) = \frac{d}{d\omega}\mathcal{F}_s(f(x)).$$

(b) Show that

$$\mathcal{F}_s(xf(x)) = -\frac{d}{d\omega}\mathcal{F}_c(f(x)).$$

SOLUTION:

(a) From the definition of the Fourier sine transform, we have

$$\frac{d}{d\omega}\mathcal{F}_s(f(x)) = \frac{d}{d\omega}\left[\sqrt{\frac{2}{\pi}}\int_0^\infty f(t)\sin\omega t\,dt\right],$$

and differentiating under the integral sign,

$$\begin{aligned}\frac{d}{d\omega}\mathcal{F}_s(f(x)) &= \sqrt{\frac{2}{\pi}}\int_0^\infty f(t)\frac{d}{d\omega}\left(\sin\omega t\right)\,dt\\ &= \sqrt{\frac{2}{\pi}}\int_0^\infty tf(t)\cos\omega t\,dt\\ &= \mathcal{F}_c(xf(x)),\end{aligned}$$

and therefore

$$\frac{d}{d\omega}\mathcal{F}_s(f(x)) = \mathcal{F}_c(xf(x))$$

as required.

(b) From the definition of the Fourier cosine transform, we have

$$\frac{d}{d\omega}\mathcal{F}_c(f(x)) = \frac{d}{d\omega}\left[\sqrt{\frac{2}{\pi}}\int_0^\infty f(t)\cos\omega t\,dt\right],$$

and differentiating under the integral sign,

$$\begin{aligned}\frac{d}{d\omega}\mathcal{F}_c(f(x)) &= \sqrt{\frac{2}{\pi}}\int_0^\infty f(t)\frac{d}{d\omega}\left(\cos\omega t\right)\,dt\\ &= -\sqrt{\frac{2}{\pi}}\int_0^\infty tf(t)\sin\omega t\,dt\\ &= -\mathcal{F}_s(xf(x)),\end{aligned}$$

and therefore

$$\frac{d}{d\omega}\mathcal{F}_c(f(x)) = -\mathcal{F}_s(xf(x))$$

as required.

Question 4. Chebyshev's differential equation reads

$$\begin{aligned}(1-x^2)y'' - xy' + \lambda y &= 0, \qquad -1 < x < 1\\ y(1) &= 1,\\ |y'(1)| &< \infty\end{aligned}$$

(a) Divide by $\sqrt{1-x^2}$ and bring the differential equation into Sturm-Liouville form. Decide if the resulting Sturm-Liouville problem is regular or singular.

(b) For $n \geq 0$, the Chebyshev polynomials are defined as follows:

$$T_n(x) = \cos(n \text{ arc } \cos x), \quad -1 \leq x \leq 1.$$

Show that $T_n(x)$ is an eigenfunction of this Sturm-Liouville problem and for each $n \geq 0$ find the corresponding eigenvalue.

Hint: If $v = \text{arc } \cos x$, then $\cos v = x$, and $v' = -\dfrac{1}{\sin v} = -\dfrac{1}{(1-x^2)^{1/2}}$.

(c) Show that

$$\int_{-1}^{1} \frac{T_m(x)T_n(x)}{(1-x^2)^{1/2}}\, dx = 0$$

for $m \neq n$, so that these eigenfunctions are orthogonal on the interval $[-1, 1]$ with respect to the weight function $w(x) = \dfrac{1}{(1-x^2)^{1/2}}$.

SOLUTION:

(a) We can rewrite the differential equation as

$$\left((1-x^2)^{1/2}y'\right)' + \frac{\lambda y}{(1-x^2)^{1/2}} = 0,$$

which is the self-adjoint form of the Sturm-Liouville problem, with

$$p(x) = (1-x^2)^{1/2}, \qquad q(x) = 0, \qquad r(x) = \frac{1}{(1-x^2)^{1/2}}.$$

This is clearly a singular Sturm-Liouville problem since $p(x)$ vanishes at the endpoints $x = \pm 1$, and since $r(x)$ is not defined on the closed interval $[-1, 1]$ let alone continuous there. It also fails to be regular because of the boundary conditions, one of which is a boundedness condition.

(b) If $y = T_n(x)$, then

$$y = \cos nk$$

where $k = k(x) = \text{arc } \cos x$, so that $x = \cos k$ and using the chain rule, we have

$$y' = -n \sin nk \cdot k' = -n \sin nk \cdot \left(-\frac{1}{\sin k}\right) = \frac{n \sin nk}{\sin k},$$

and

$$y'' = -\frac{-n^2 \cos nk + n \sin nk \cot k}{\sin^2 k} = \frac{-n^2 y}{1-x^2} + \frac{xy'}{1-x^2},$$

and $y = T_n(x)$ satisfies the differential equation $(1-x^2)y'' - xy' + n^2 y = 0, \quad -1 < x < 1$, for each $n \geq 0$. Therefore, $T_n(x)$ is an eigenfunction of this Sturm-Liouville problem with eigenvalue n^2 for $n = 0, 1, 2 \ldots.$

(c) In the integral

$$\int_{-1}^{1} \frac{T_m(x)T_n(x)}{(1-x^2)^{1/2}}\,dx$$

make the substitution $x = \cos t$, so that

$$dx = -\sin t\,dt = -(1-cos^2 t)^{1/2}\,dt = -(1-x^2)^{1/2}\,dt$$

that is,

$$dt = -\frac{1}{(1-x^2)^{1/2}}\,dx.$$

Therefore,

$$\int_{-1}^{1} \frac{T_m(x)T_n(x)}{(1-x^2)^{1/2}}\,dx = \int_0^{\pi} \cos mt \cos nt\,dt = 0$$

if $m \neq n$, and the Chebyshev polynomials are orthogonal on the interval $[-1,1]$ with respect to the weight function $w(x) = \dfrac{1}{(1-x^2)^{1/2}}$.

Question 5. Solve the following initial value problem for the damped wave equation

$$\frac{\partial^2 u}{\partial t^2} + 2\frac{\partial u}{\partial t} + u = \frac{\partial^2 u}{\partial x^2}$$

$$u(x,0) = \frac{1}{1+x^2},$$

$$\frac{\partial u}{\partial t}(x,0) = 1.$$

Hint: Do not use separation, instead consider $w(x,t) = e^t \cdot u(x,t)$.

Solution: Note that $u(x,t) = e^{-t} \cdot w(x,t)$, so that

$$\frac{\partial^2 u}{\partial x^2} = e^{-t}\frac{\partial^2 w}{\partial x^2}$$

and

$$\frac{\partial u}{\partial t} = -e^{-t}w + e^{-t}\frac{\partial w}{\partial t}$$

and

$$\frac{\partial^2 u}{\partial t^2} = e^{-t}w - 2e^{-t}\frac{\partial w}{\partial t} + e^{-t}\frac{\partial^2 w}{\partial t^2}.$$

Therefore,

$$\frac{\partial^2 u}{\partial t^2} + 2\frac{\partial u}{\partial t} + u = e^{-t}\frac{\partial^2 w}{\partial t^2},$$

while

$$\frac{\partial^2 u}{\partial x^2} = e^{-t}\frac{\partial^2 w}{\partial x^2}$$

and if u is a solution to the original partial differential equation, then w is a solution to the equation

$$e^{-t}\left[\frac{\partial^2 w}{\partial t^2} - \frac{\partial^2 w}{\partial x^2}\right] = 0,$$

and since $e^{-t} \neq 0$, then w satisfies the initial value problem

$$\frac{\partial^2 w}{\partial t^2} = \frac{\partial^2 w}{\partial x^2}, \quad -\infty < x < \infty, \quad t > 0,$$
$$w(x,0) = \frac{1}{1+x^2},$$
$$\frac{\partial w}{\partial t}(x,0) = 1.$$

From D'Alembert's equation to the wave equation, we have (since $c = 1$)

$$w(x,t) = \frac{1}{2}\left[\frac{1}{1+(x+t)^2} + \frac{1}{1+(x-t)^2}\right] + \frac{1}{2}\int_{x-t}^{x+t} 1\,ds,$$

so that

$$u(x,t) = \frac{e^{-t}}{2}\left[\frac{1}{1+(x+t)^2} + \frac{1}{1+(x-t)^2}\right] + te^{-t},$$

for $-\infty < x < \infty, \quad t \geq 0.$