MATH 300 Fall 2004
Advanced Boundary Value Problems I
Sample Midterm Problems
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MATHEMATICS 300 - SAMPLE MIDTERM

1. Solve the following differential equation for u

$$e^{-x}\frac{d}{dx}\left(e^{x}\frac{du}{dx}\right) = -x \qquad 0 < x < a$$

$$u(0) = 0, \quad u(a) = 0.$$

2. Given the function

$$f(x) = \begin{cases} \cos x & 0 \le x \le \pi \\ 0 & -\pi < x < 0 \end{cases}$$

and $f(x + 2\pi) = f(x)$ otherwise.

(a) Find the Fourier series of f.

(b) For which values of $x \in [-\pi, \pi]$ does the Fourier series converge to f?

3. Find all functions w for which $u(x,t) = w(x - ct)$ is a solution of the first order partial differential equation

$$x\frac{\partial u}{\partial x} + t\frac{\partial u}{\partial t} = Au$$

where A and c are constants.

Table of Integrals

$$\int \sin\lambda x \sin\mu x\, dx = \frac{\sin(\mu-\lambda)x}{2(\mu-\lambda)} - \frac{\sin(\mu+\lambda)x}{2(\mu+\lambda)} \quad (\lambda \ne \mu)$$

$$\int \sin\lambda x \cos\mu x\, dx = \frac{\cos(\mu-\lambda)x}{2(\mu-\lambda)} - \frac{\cos(\mu+\lambda)x}{2(\mu+\lambda)} \quad (\lambda \ne \mu)$$

$$\int \cos\lambda x \cos\mu x\, dx = \frac{\sin(\mu-\lambda)x}{2(\mu-\lambda)} + \frac{\sin(\mu+\lambda)x}{2(\mu+\lambda)} \quad (\lambda \ne \mu)$$

$$\int \sin^2\lambda x\, dx = \frac{x}{2} - \frac{\sin 2\lambda x}{4\lambda}$$

$$\int \sin\lambda x \cos\lambda x\, dx = \frac{\sin^2\lambda x}{2\lambda}$$

$$\int \cos^2\lambda x\, dx = \frac{x}{2} + \frac{\sin 2\lambda x}{4\lambda}$$

$$\int x\sin\lambda x\, dx = \frac{\sin\lambda x}{\lambda^2} - \frac{x\cos\lambda x}{\lambda}$$

$$\int x\cos\lambda x\, dx = \frac{\cos\lambda x}{\lambda^2} + \frac{x\sin\lambda x}{\lambda}$$

$$\int e^{kx}\sin\lambda x\, dx = \frac{e^{kx}(k\sin\lambda x - \lambda\cos\lambda x)}{k^2+\lambda^2}$$

$$\int e^{kx}\cos\lambda x\, dx = \frac{e^{kx}(k\cos\lambda x + \lambda\sin\lambda x)}{k^2+\lambda^2}$$

Solutions:

1. Since $\frac{d}{dx}\left(e^x \frac{du}{dx}\right) = -xe^x$, integrating we get

$$e^x \frac{du}{dx} = -\int xe^x\,dx + c_1 = -[xe^x - \int e^x\,dx] + c_1$$

therefore $e^x \frac{du}{dx} = -xe^x + e^x + c_1$, and so $\frac{du}{dx} = -x + 1 + c_1 e^{-x}$. Integrating again,

$$u(x) = -\frac{1}{2}x^2 + x - c_1 e^{-x} + c_2$$

and $u(0) = 0 \implies c_1 = c_2$, while $u(a) = 0 \implies c_1 = \frac{a^2-2a}{2(1-e^{-a})}$. The solution is

$$u(x) = -\frac{1}{2}x^2 + x + \frac{1}{2}(a^2 - 2a)\left(\frac{1-e^{-x}}{1-e^{-a}}\right).$$

2. (a) Writing $f(x) = a_0 + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx)$, the coefficients in the Fourier series are given by

$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\,dx = \frac{1}{2\pi}\int_0^{\pi} \cos x\,dx = \frac{1}{2\pi}\sin x\Big|_0^{\pi} = 0$$

and for $n > 1$,

$$\begin{aligned}
a_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx = \frac{1}{\pi}\int_0^{\pi} \cos x \cos nx\,dx \\
&= \frac{1}{2\pi}\int_0^{\pi}\big(\cos(n+1)x + \cos(n-1)x\big)\,dx = \frac{1}{2\pi}\frac{\sin(n+1)x}{n+1}\Big|_0^{\pi} + \frac{1}{2\pi}\frac{\sin(n-1)x}{n-1}\Big|_0^{\pi} = 0,
\end{aligned}$$

while for $n = 1$,

$$a_1 = \frac{1}{\pi}\int_0^{\pi}\cos^2 x\,dx = \frac{1}{2\pi}\int_0^{\pi}\big(1 + \cos 2x\big)\,dx = \frac{1}{2}.$$

Also, for $n = 1$,

$$b_1 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin x\,dx = \frac{1}{\pi}\int_0^{\pi}\sin x\cos x\,dx = \frac{1}{2\pi}\sin^2 x\Big|_0^{\pi} = 0,$$

while for $n > 1$,

$$\begin{aligned}
b_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx = \frac{1}{\pi}\int_0^{\pi}\cos x \sin nx\,dx \\
&= \frac{1}{2\pi}\int_0^{\pi}\big(\sin(n+1)x + \sin(n-1)x\big)\,dx = -\frac{1}{2\pi}\frac{\cos(n+1)x}{n+1}\Big|_0^{\pi} - \frac{1}{2\pi}\frac{\cos(n-1)x}{n-1}\Big|_0^{\pi} \\
&= -\frac{1}{2\pi}\big((-1)^{n+1} - 1\big)\left\{\frac{1}{n+1} + \frac{1}{n-1}\right\} = -\frac{\big((-1)^{n+1} - 1\big)}{2\pi}\frac{2n}{n^2-1}
\end{aligned}$$

and

$$b_n = \begin{cases} 0 & \text{if } n \text{ is odd} \\ \dfrac{2n}{\pi(n^2-1)} & \text{if } n \text{ is even.} \end{cases}$$

The Fourier series for f is therefore

$$f(x) \sim \frac{1}{2}\cos x + \frac{4}{\pi}\sum_{n=1}^{\infty} \frac{n}{4n^2-1}\sin 2nx.$$

(b) The Fourier series converges to $f(x)$ for all x in $[-\pi, \pi]$, except at $x = 0$, $x = \pi$ and $x = -\pi$. From Dirichlet's theorem, the series converges to $\dfrac{1}{2}$ at $x = 0$ and converges to $-\dfrac{1}{2}$ at $x = \pi$ and $x = -\pi$.

5. Suppose that $u(x,t) = w(x-ct)$ is a solution to the first order partial differential equation

$$x\frac{\partial u}{\partial x} + t\frac{\partial u}{\partial t} = Au. \qquad (*)$$

Let $\xi = x - ct$, so that

$$\frac{\partial w}{\partial x} = \frac{dw}{d\xi}\frac{\partial \xi}{\partial x} = \frac{dw}{d\xi} \quad \text{and} \quad \frac{\partial w}{\partial t} = \frac{dw}{d\xi}\frac{\partial \xi}{\partial t} = -c\frac{dw}{d\xi},$$

then $w = w(\xi)$ satisfies the equation

$$(\xi + ct)\frac{dw}{d\xi} - ct\frac{dw}{d\xi} = Aw \quad \text{that is} \quad \xi\frac{dw}{d\xi} = Aw.$$

This is a first order linear ordinary differential equation for w, which we can write as $\frac{dw}{d\xi} - \frac{A}{\xi}w = 0$ and which has as integrating factor $e^{-A\log|\xi|}$,
so that $\frac{d}{d\xi}\left(e^{-A\log|\xi|}\, w\right) = 0$.
Integrating, we have $e^{-A\log|\xi|}\, w(\xi) = K$, where K is a constant, therefore if $u(x,t) = w(x-ct)$ is a solution to $(*)$, then

$$u(x,t) = K\, e^{A\log|x-ct|}$$

for some constant K.