Useful Formulas for Math 300

1. Method of Characteristics for first order equations:

$$u(x,t) = u(x(t),t)$$

   First find $x(t)$, then find $u(x(t),t)$.

2. D'Alemberts formula for the wave equation on $(-\infty, \infty)$:

$$u(x,t) = \frac{1}{2}(f(x+ct) + f(x-ct) + \frac{1}{2c}\int_{x-ct}^{x+ct} g(s)ds.$$

3. Fourier series on $[-L, L]$:

$$\begin{aligned} f(x) &= a_0 + \sum_{n=1}^{\infty} a_n \cos(\frac{n\pi x}{L}) + b_n \sin(\frac{n\pi x}{L}) \\ a_0 &= \frac{1}{2L}\int_{-L}^{L} f(x)dx \\ a_n &= \frac{1}{L}\int_{-L}^{L} f(x)\cos(\frac{n\pi x}{L})dx \\ b_n &= \frac{1}{L}\int_{-L}^{L} f(x)\sin(\frac{n\pi x}{L})dx \end{aligned}$$

4. Fourier cosine series on $[0, L]$:

$$\begin{aligned} f(x) &= a_0 + \sum_{n=1}^{\infty} a_n \cos(\frac{n\pi x}{L}) \\ a_0 &= \frac{1}{L}\int_{0}^{L} f(x)dx \\ a_n &= \frac{2}{L}\int_{0}^{L} f(x)\cos(\frac{n\pi x}{L})dx \end{aligned}$$

5. Fourier sine series on $[0, L]$:

$$\begin{aligned} f(x) &= \sum_{n=1}^{\infty} b_n \sin(\frac{n\pi x}{L}) \\ b_n &= \frac{2}{L}\int_{0}^{L} f(x)\sin(\frac{n\pi x}{L})dx \end{aligned}$$

6. Separation of variables:
   1) Write $u(x,t) = T(t)X(x)$.
   2) Solve the Sturm-Liouville problem for $X(x)$.
   3) Solve the corresponding time problem for $T(t)$.
   4) Use superposition.
   5) Adjust the initial conditions.

7. Laplacian in polar coordinates:

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}$$

8. Generalized eigenfunction expansion with a weight function $w(x)$:

$$f(x) = \sum_{i=1}^{\infty} \frac{\int_a^b f(x)\phi_i(x)w(x)dx}{\int_a^b \phi_i(x)^2 w(x)dx}$$

9. Fourier-integral formula on $(-\infty, \infty)$:

$$\begin{aligned} f(x) &= \int_0^\infty A(\omega)\cos(\omega x) + B(\omega)\sin(\omega x) d\omega \\ A(\omega) &= \frac{1}{\pi}\int_{-\infty}^\infty f(x)\cos(\omega x) dx \\ B(\omega) &= \frac{1}{\pi}\int_{-\infty}^\infty f(x)\sin(\omega x) dx \end{aligned}$$

10. Fourier-cosine-integral formula on $[0, \infty)$:

$$\begin{aligned} f(x) &= \int_0^\infty A(\omega)\cos(\omega x) \\ A(\omega) &= \frac{2}{\pi}\int_0^\infty f(x)\cos(\omega x) dx \end{aligned}$$

11. Fourier-sine-integral formula on $[0, \infty)$:

$$\begin{aligned} f(x) &= \int_0^\infty B(\omega)\sin(\omega x) d\omega \\ B(\omega) &= \frac{2}{\pi}\int_0^\infty f(x)\sin(\omega x) dx \end{aligned}$$

12. Fourier transform on $(-\infty, \infty)$:

$$\begin{aligned} \mathcal{F}(f)(\omega) = \hat{f}(\omega) &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^\infty f(x) e^{-i\omega x} dx \\ \mathcal{F}^{-1}(\hat{f})(x) = f(x) &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^\infty \hat{f}(\omega) e^{i\omega x} d\omega \end{aligned}$$

13. Learn all the properties of FT.

14. Fourier cosine transform on $[0, \infty)$:

$$\begin{aligned} \mathcal{F}_c(f)(\omega) = \hat{f}_c(\omega) &= \sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\cos(\omega x) dx \\ f(x) &= \sqrt{\frac{2}{\pi}}\int_0^\infty \hat{f}_c(\omega)\cos(\omega x) d\omega \end{aligned}$$

15. Fourier sine transform on $[0, \infty)$:

$$\begin{aligned} \mathcal{F}_s(f)(\omega) = \hat{f}_s(\omega) &= \sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\sin(\omega x) dx \\ f(x) &= \sqrt{\frac{2}{\pi}}\int_0^\infty \hat{f}_s(\omega)\sin(\omega x) d\omega \end{aligned}$$

16. Gauss kernel

$$g(x,t) = \frac{1}{\sqrt{2Dt}} e^{-\frac{x^2}{4Dt}}.$$

17. Error function

$$\mathrm{erf}(z) := \frac{2}{\sqrt{\pi}}\int_0^z e^{-x^2} dx.$$