MATH 300 Fall 2004
Advanced Boundary Value Problems I
Assignment 3
Due: Friday October 22, 2004

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Question 1. [p 151, #2]

Solve the problem of heat transfer in a bar of length $L = 1$ with initial heat distributuion $f(x) = \cos \pi x$ and no heat loss at either end, where the thermal diffusivity is $c = 1$, that is, solve the boundary-value initial-value problem below:

$$\begin{aligned} &\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}, && 0 < x < 1, \quad t > 0 \\ &\frac{\partial u}{\partial x}(0,t) = \frac{\partial u}{\partial x}(1,t) = 0, && t > 0 \\ &u(x,0) = \cos \pi x, && 0 < x < 1. \end{aligned}$$

Question 2. [p 151, #6]

Solve the problem of heat transfer in a bar of length $L = \pi$ and thermal diffusivity $c = 1$, with initial heat distribution $u(x, 0) = \sin x$ where one end of the bar is kept at a constant temperature $u(0, t) = 0$, while there is no heat loss at the other end of the bar so that $u_x(\pi, t) = 0$, that is, solve the boundary-value initial-value problem below:

$$\begin{aligned} &\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}, && 0 < x < \pi, \quad t > 0 \\ &u(0,t) = 0, && t > 0 \\ &\frac{\partial u}{\partial x}(\pi,t) = 0, && t > 0 \\ &u(x,0) = \sin x, && 0 < x < \pi. \end{aligned}$$

Question 3. [p 152, #8]

In the problem of heat transfer in a bar of length L with initial temperature distribution $f(x)$ and no heat loss at either end, show that the asymptotic temperature is constant and equals the average temperature.

Note: This involves solving the boundary-value initial-value problem

$$\begin{aligned} &\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}, && 0 < x < L, \quad t > 0 \\ &\frac{\partial u}{\partial x}(0,t) = \frac{\partial u}{\partial x}(L,t) = 0, && t > 0 \\ &u(x,0) = f(x), && 0 < x < L, \end{aligned}$$

and finding $\lim_{t \to \infty} u(x, t)$.

Question 4. [p 162, #2]

Solve the problem of a thin elastic membrane stretched over a square frame of side 1, where the vibrations are governed by the following two dimensional wave equation:

$$\frac{\partial^2 u}{\partial t^2} = \frac{1}{\pi^2}\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right), \quad 0 < x < 1, \quad 0 < y < 1, \quad t > 0$$

$$u(0, y, t) = u(1, y, t) = 0, \quad 0 \le y \le 1, \quad t \ge 0$$

$$u(x, 0, t) = u(x, 1, t) = 0, \quad 0 \le x \le 1, \quad t \ge 0$$

$$u(x, y, 0) = \sin \pi x \sin \pi y, \quad 0 \le x \le 1, \quad 0 \le y \le 1$$

$$\frac{\partial u}{\partial t}(x, y, 0) = \sin \pi x, \quad 0 \le x \le 1, \quad 0 \le y \le 1.$$

Question 5. [p 163, #12]

Find the temperature distribution in a thin two dimensional plate with thermal diffusivity $c = 1$, in the shape of a unit square, with insulated faces and edges kept at zero temperature with an initial temperature distribution given by $f(x, y) = xy(1-x)(1-y)$ for $0 \le x, y \le 1$, that is, solve the boundary-value initial-value problem given below:

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}, \quad 0 < x < 1, \quad 0 < y < 1, \quad t > 0$$

$$u(0, y, t) = u(1, y, t) = 0, \quad 0 < y < 1, \quad t > 0$$

$$u(x, 0, t) = u(x, 1, t) = 0, \quad 0 < x < 1, \quad t > 0$$

$$u(x, y, 0) = xy(1 - x)(1 - y), \quad 0 < x < 1, \quad 0 < y < 1.$$

Question 6. [p 168, #2]

Solve the Dirichlet problem for the unit square in the plane with the boundary data as given below:

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0, \quad 0 < x < 1, \quad 0 < y < 1,$$

$$u(x, 0) = 0 \quad 0 \le x \le 1,$$

$$u(x, 1) = 100, \quad 0 \le x \le 1,$$

$$u(0, y) = 0 \quad 0 \le y \le 1,$$

$$u(1, y) = 100, \quad 0 \le y \le 1.$$

Question 7. [p 168, #4]

Solve the Dirichlet problem for the unit square in the plane with the boundary data as given below:

$$\begin{aligned}
&\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0, && 0 < x < 1, \quad 0 < y < 1, \\
&u(x,0) = 1 - x && 0 \le x \le 1, \\
&u(x,1) = x, && 0 \le x \le 1, \\
&u(0,y) = 0 && 0 \le y \le 1, \\
&u(1,y) = 0, && 0 \le y \le 1.
\end{aligned}$$

Question 8. [p 169, #8]

Approximate the temperature at the center of the plate in Question 7.

Question 9. [p 198, #2]

Compute the Laplacian of the function

$$u(x,y) = \tan^{-1}\left(\frac{y}{x}\right)$$

in an appropriate coordinate system and decide if the given function satisfies Laplace's equation $\nabla^2 u = 0$.

Question 10. [p 198, #6]

Compute the Laplacian of the function

$$u(x,y) = \ln(x^2 + y^2)$$

in an appropriate coordinate system and decide if the given function satisfies Laplace's equation $\nabla^2 u = 0$.