1. Let \( R = \ln(u^2 + v^2 + w^2) \), \( u = x + 2y \), \( v = 2x - y \), and \( w = 2xy \). Use the Chain Rule to find \( \frac{\partial R}{\partial x} \) and \( \frac{\partial R}{\partial y} \) when \( x = y = 1 \).

**Solution:**

The Chain Rule gives

\[
\frac{\partial R}{\partial x} = \frac{\partial R}{\partial u} \frac{\partial u}{\partial x} + \frac{\partial R}{\partial v} \frac{\partial v}{\partial x} + \frac{\partial R}{\partial w} \frac{\partial w}{\partial x}
\]

\[
= \frac{2u}{u^2 + v^2 + w^2} \times 1 + \frac{2v}{u^2 + v^2 + w^2} \times 2 + \frac{2w}{u^2 + v^2 + w^2} \times (2y).
\]

When \( x = y = 1 \), we have \( u = 3 \), \( v = 1 \), and \( w = 2 \), so

\[
\frac{\partial R}{\partial x} = \frac{6}{14} \times 1 + \frac{2}{14} \times 2 + \frac{4}{14} \times 2 = \frac{18}{14} = \frac{9}{7}.
\]

\[
\frac{\partial R}{\partial y} = \frac{\partial R}{\partial u} \frac{\partial u}{\partial y} + \frac{\partial R}{\partial v} \frac{\partial v}{\partial y} + \frac{\partial R}{\partial w} \frac{\partial w}{\partial y}
\]

\[
= \frac{2u}{u^2 + v^2 + w^2} \times 2 + \frac{2v}{u^2 + v^2 + w^2} \times (-1) + \frac{2w}{u^2 + v^2 + w^2} \times (2x).
\]

When \( x = y = 1 \), we have \( u = 3 \), \( v = 1 \), and \( w = 2 \), so

\[
\frac{\partial R}{\partial y} = \frac{6}{14} \times 2 + \frac{2}{14} \times (-1) + \frac{4}{14} \times 2 = \frac{18}{14} = \frac{9}{7}.
\]

2. Find \( \frac{\partial z}{\partial x} \) and \( \frac{\partial z}{\partial y} \) if \( xyz = \sin(x + y + z) \).

**Solution:**

Let \( F(x, y, z) = xyz - \sin(x + y + z) = 0 \). Then, we have

\[
\frac{\partial z}{\partial x} = -\frac{\partial F}{\partial x} = -\frac{yz - \cos(x + y + z)}{xy - \cos(x + y + z)},
\]

\[
\frac{\partial z}{\partial y} = -\frac{\partial F}{\partial y} = -\frac{xz - \cos(x + y + z)}{xy - \cos(x + y + z)}.
\]
3. Let \( f \) and \( g \) be two differentiable real valued functions. Show that any function of the form \( z = f(x + at) + g(x - at) \) is a solution of the wave equation \( \frac{\partial^2 z}{\partial t^2} = a^2 \frac{\partial^2 z}{\partial x^2} \).

**Solution:**

Let \( u = x + at \) and \( v = x - at \). Then \( z = f(u) + g(v) \) and the Chain Rule gives

\[
\frac{\partial z}{\partial x} = \frac{df}{du} \frac{\partial u}{\partial x} + \frac{dg}{dv} \frac{\partial u}{\partial x} = \frac{df}{du} + \frac{dg}{dv}.
\]

Thus

\[
\frac{\partial^2 z}{\partial x^2} = \frac{\partial}{\partial x} \left( \frac{\partial z}{\partial x} \right) = \frac{\partial}{\partial x} \left( \frac{df}{du} + \frac{dg}{dv} \right) = \frac{d^2 f}{du^2} + \frac{d^2 g}{dv^2}.
\]

Similarly

\[
\frac{\partial z}{\partial t} = \frac{df}{du} \frac{\partial u}{\partial t} + \frac{dg}{dv} \frac{\partial v}{\partial t} = a \frac{df}{du} + a \frac{dg}{dv}.
\]

Thus

\[
\frac{\partial^2 z}{\partial t^2} = \frac{\partial}{\partial x} \left( \frac{\partial z}{\partial t} \right) = \frac{\partial}{\partial x} \left( a \frac{df}{du} + a \frac{dg}{dv} \right) = a^2 \frac{d^2 f}{du^2} + a^2 \frac{d^2 g}{dv^2} = a^2 \left( \frac{d^2 f}{du^2} + \frac{d^2 g}{dv^2} \right).
\]

From Equations (1) and (2) we get

\[
\frac{\partial^2 z}{\partial t^2} = a^2 \frac{\partial^2 z}{\partial x^2}.
\]

4. A function \( f \) is called **homogeneous of degree** \( n \) if it is satisfies the equation \( f(tx, ty) = t^n f(x, y) \) for all \( t \), where \( n \) is a positive integer. Show that if \( f \) is homogeneous of degree \( n \), then

\[
x \frac{\partial f}{\partial x} + y \frac{\partial f}{\partial y} = nf(x, y)
\]

**[Hint: Use the Chain Rule to differentiate \( f(tx, ty) \) with respect \( t \).]**

**Solution:**

Let \( u = tx \) and \( v = ty \). Then

\[
\frac{d}{dt} (f(u, v)) = nt^{n-1} f(x, y).
\]

The Chain Rule gives

\[
\frac{\partial f}{\partial u} \frac{du}{dt} + \frac{\partial f}{\partial v} \frac{dv}{dt} = nt^{n-1} f(x, y).
\]

Therefore

\[
x \frac{\partial f}{\partial u} + y \frac{\partial f}{\partial v} = nt^{n-1} f(x, y).
\]

Setting \( t = 1 \) in the Equation (3):

\[
x \frac{\partial f}{\partial x} + y \frac{\partial f}{\partial y} = nf(x, y).
\]
5. Find the directional derivative of the function \( f(x, y, z) = \sqrt{x^2 + y^2 + z^2} \) at the point \((1, 2, -2)\) in the direction of vector \( \mathbf{v} = (-6, 6, -3) \).

**Solution:**

We first compute the gradient vector at \((1, 2, -2)\).

\[
\nabla f(x, y, z) = \left( \frac{x}{\sqrt{x^2 + y^2 + z^2}}, \frac{y}{\sqrt{x^2 + y^2 + z^2}}, \frac{z}{\sqrt{x^2 + y^2 + z^2}} \right)
\]

\[
\nabla f(1, 2, -2) = \left( \frac{1}{3}, \frac{2}{3}, \frac{-2}{3} \right).
\]

Note that \( \mathbf{v} \) is not unit vector, but since \(|\mathbf{v}| = 9\), the unit vector in the direction of \( \mathbf{v} \) is

\[
\mathbf{u} = \frac{\mathbf{v}}{|\mathbf{v}|} = \left( -\frac{2}{3}, \frac{2}{3}, \frac{-1}{3} \right).
\]

Therefore

\[
D_{\mathbf{u}} f(1, 2, -2) = \nabla f(1, 2, -2) \cdot \mathbf{u} = \frac{2}{3}.
\]

6. The temperature at a point \((x, y, z)\) on the surface of a metal is \( T(x, y, z) = 200e^{-x^2-3y^2-9z^2} \) where \( T \) is measured in degree Celsius and \( x, y, z \) in meters.

(a) In which direction does the temperature increase fastest at the point \( P(2, -1, 2) \)?

(b) What is the maximum rate of change at \( P(2, -1, 2) \)?

**Solution:**

We first compute the gradient vector:

\[
\nabla T(x, y, z) = \langle T_x, T_y, T_z \rangle = -e^{-x^2-3y^2-9z^2} \langle 400x, 1200y, 3600z \rangle
\]

\[
\nabla T(2, -1, 2) = -400e^{-43} \langle 2, -3, 18 \rangle.
\]

The temperature increases in the direction of the gradient vector

\[
\nabla T(2, -1, 2) = -400e^{-43} \langle 2, -3, 18 \rangle.
\]

The maximum rate of change is

\[
\left| -400e^{-25} \langle 2, -3, 18 \rangle \right| = 400e^{-43}\sqrt{337}.
\]

7. Find the points on the ellipsoid \( x^2 + 2y^2 + 3z^2 = 1 \) where the tangent plane is parallel to the plane \( 3x - 2y + 3z = 1 \).
8. Find the local maximum and minimum values and saddle point(s) of the function
\[ f(x, y) = 3x^2y + y^3 - 3x^2 - 3y^2 + 2. \]

**Solution:**

The first order partial derivatives are
\[ f_x = 6xy - 6x, \quad f_y = 3x^2 + 3y^2 - 6y. \]

So to find the critical points we need to solve the equations \( f_x = 0 \) and \( f_y = 0 \). \( f_x = 0 \)
implies \( x = 0 \) or \( y = 1 \) and when \( x = 0 \), \( f_y = 0 \) implies \( y = 0 \) or \( y = 2 \); when \( y = 1 \), \( f_y = 0 \)
implies \( x^2 = 1 \) or \( x = \pm 1 \). Thus the critical points are \((0, 0), (0, 2), (\pm 1, 1)\).

Now \( f_{xx} = 6y - 6, f_{yy} = 6y - 6 \) and \( f_{xy} = 6x \). So \( D = f_{xx}f_{yy} - f_{xy}^2 = (6y - 6)^2 - 36x^2 \).

<table>
<thead>
<tr>
<th>Critical point</th>
<th>Value of ( f )</th>
<th>( f_{xx} )</th>
<th>( D )</th>
<th>Conclusion</th>
</tr>
</thead>
<tbody>
<tr>
<td>((0, 0))</td>
<td>2</td>
<td>-6</td>
<td>36</td>
<td>local maximum</td>
</tr>
<tr>
<td>((0, 2))</td>
<td>-2</td>
<td>6</td>
<td>36</td>
<td>local minimum</td>
</tr>
<tr>
<td>((1, 1))</td>
<td>0</td>
<td>0</td>
<td>-36</td>
<td>saddle point</td>
</tr>
<tr>
<td>((-1, 1))</td>
<td>0</td>
<td>0</td>
<td>-36</td>
<td>saddle point</td>
</tr>
</tbody>
</table>

9. Find the points on surface \( x^2y^2z = 1 \) that are closest to the origin.

**Solution:**

The distance from any point \((x, y, z)\) to the origin is
\[ d = \sqrt{x^2 + y^2 + z^2} \]

but if \((x, y, z)\) lies on the surface \( x^2y^2z = 1 \), then \( z = \frac{1}{x^2y^2} \) and so we have
\[ d = \sqrt{x^2 + y^2 + x^{-4}y^{-4}}. \]

We can minimize \( d \) by minimizing the simpler expression
\[ d^2 = x^2 + y^2 + x^{-4}y^{-4} = f(x, y). \]

\( f_x = 2x - \frac{4}{x^3y^2}, \quad f_y = 2y - \frac{4}{x^2y^3}, \) so the critical points occur when \( 2x = \frac{4}{x^3y^2} \) and \( 2y = \frac{4}{x^2y^3} \)
or \( x^6y^4 = x^4y^6 \) so, \( x^2 = y^2 \) and \( x^{10} = 2 \Rightarrow x = \pm 2^{\frac{1}{10}}, y = \pm 2^{\frac{1}{10}} \). The four critical points \((\pm 2^{\frac{1}{10}}, \pm 2^{\frac{1}{10}})\). Thus the points on the surface closes to origin are \((\pm 2^{\frac{1}{10}}, \pm 2^{\frac{1}{10}})\). There is no maximum since the surface is infinite in extent.
10. Find the extreme values of \( f(x, y) = 2x^2 + 3y^2 - 4x - 5 \) on the region

\[ D = \{(x, y) \mid x^2 + y^2 \leq 16\}. \]

**Solution:**

We first need to find the critical points. These occur when

\[
\begin{align*}
f_x &= 4x - 4 = 0, \\
f_y &= 6y = 0
\end{align*}
\]

so the only critical point of \( f \) is \((1, 0)\) and it lies in the region \( x^2 + y^2 \leq 16 \).

On the circle \( x^2 + y^2 = 16 \), we have \( y^2 = 16 - x^2 \) and

\[
g(x) = f(x, \sqrt{16 - x^2}) = 2x^2 + 3(16 - x^2) - 4x - 5 = -x^2 - 4x + 43.
\]

\[
g'(x) = 0 \Rightarrow -2x - 4 = 0 \Rightarrow x = -2
\]

\[
y^2 = 16 - x^2 = 16 - 4 = 12 \Rightarrow y = \pm 2\sqrt{3}.
\]

Now \( f(1, 0) = -7 \) and \( f(-2, \pm 2\sqrt{3}) = 47 \). Thus the maximum value of \( f(x, y) \) on the disc \( x^2 + y^2 \leq 16 \) is \( f(-2, \pm 2\sqrt{3}) = 47 \), and the minimum value is \( f(1, 0) = -7 \).