University of Alberta
Mathematical and Statistical Sciences
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Math 525, Differential Equations II
Winter 2015

Assignment 5, due March 27, 2015, 9 AM

Exercise 18: (4)

For $\mu \in [0,1)$ we study the following dynamical system

$$\begin{array}{rcl} \dot{y} & = & -y \\ \dot{x} & = & \begin{cases} \mu(1-x)^2, & 0 \le x < 1 \\ -(1-x)^2, & 1 \le x \end{cases} \end{array}$$

Find the global attractor for each $\mu \in [0,1)$.
Show that the attractors are upper semicontinuous in 0 but not lower semicontinuous.

Exercise 19: (4)

Let $\Lambda(B) := \bigcup_{x\in B} \omega(x)$. Give an example for a dynamical system that satisfies

$$\Lambda(B) \neq \Lambda(\Lambda(B))$$

(Not the exmaple from Robinson, Exercise 10.3, p. 281!!)

Exercise 20: (2)

Show that the solution semigroup of $\dot{u} = u^{2/3}$ is not injective.

Exercise 21: (Corrected version) (Chemotactic blow-up) (10)

Chemotaxis describes the active orientation of moving cells along chemical gradients. The classical Keller-Segel model for chemotaxis reads in its simplest form in two dimensions:

$$\begin{array}{rcl} u_t & = & \nabla \cdot (\nabla u - \chi u \nabla v)) \\ \nabla v & = & -\frac{1}{2\pi} \frac{x}{|x|^2} * u \end{array} \qquad (1)$$

where $\chi > 0$ denotes the chemotactic sensitivity; $u(x,t)$ denotes the cell distribution and $v(x,t)$ is the distribution of the external chemoattractant. The symbol $*$ denotes convolution. We consider the above system (1) on $\mathbb{R}^2$ and we assume that for given initial data

$$u(x,0) = u_0(x) \in L^\infty(\mathbb{R}^2) \cap W^{1,1}(\mathbb{R}^2), \qquad u_0(x) \ge 0$$

there exists a unique non-negative local solution

$$u(x,t) \in C^0([0,\tau), L^\infty(\mathbb{R}^2) \cap W^{1,1}(\mathbb{R}^2)).$$

Prove the following Theorem:

Theorem 0.1 *(Perthame)*
Any solution of (1) with initial conditions satisfying

$$m_2(0) := \int_{\mathbb{R}^2} \frac{|x|^2}{2} u_0(x)dx < \infty$$

and

$$m_0(0) := \int_{\mathbb{R}^2} u_0(x)dx > \frac{8\pi}{\chi}$$

blows up in finite time.

(see next page)

1. For the proof use Nagai's argument by considering the second moment

$$m_2(t) = \int_{\mathbb{R}^2} \frac{|x|^2}{2} u(x,t) dx$$

and show that

$$\frac{d}{dt} m_2(t) = 2m_0 - \frac{\chi}{2\pi} \int_{\mathbb{R}^2 \times \mathbb{R}^2} u(x,t) u(y,t) \frac{x(x-y)}{|x-y|^2} dx dy.$$

2. Show that

$$\int_{\mathbb{R}^2 \times \mathbb{R}^2} u(x,t) u(y,t) \frac{x(x-y)}{|x-y|^2} dx dy = \int_{\mathbb{R}^2 \times \mathbb{R}^2} u(x,t) u(y,t) \frac{-y(x-y)}{|x-y|^2} dx dy$$

and use this equality to show that

$$\frac{d}{dt} m_2(t) = 2m_0 \left(1 - \frac{\chi}{8\pi} m_0\right).$$

3. Proof the Theorem.