University of Alberta
Mathematical and Statistical Sciences
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**Math 525, Ordinary Differential Equations II**
Winter 2015

**Assignment 2, due January 30, 2015, 9 AM**

**Exercise 6:** (Mollifier) (4)

1. The mollification of a function can be written in two ways. Show that
$$\frac{1}{h^n}\int_{\mathbb{R}^n}\rho\left(\frac{x-z}{h}\right)u(z)dz = \frac{1}{h^n}\int_{\mathbb{R}^n}\rho\left(\frac{z}{h}\right)u(x-z)dz$$
2. Assume a Lipschitz continuous function $u \in C^{0,1}(\mathbb{R}^n)$ is uniformly Lipschitz continuous with constant $K$:
$$|u(x)-u(y)| \le K|x-y|.$$
Show that each mollification $u_h = \rho_h * u$ is uniformly Lipschitz continuous with the same constant $K$.

**Exercise 7:** (Interpolation Inequality) (4)
Use Hölder's inequality to show the *interpolation inequality*: Assume $1 \le p \le q \le r < \infty$ and consider $\lambda \in (0,1)$ such that $\frac{1}{q} = \lambda\frac{1}{p} + (1-\lambda)\frac{1}{r}$. Show
$$\|u\|_{L^q} \le \|u\|_{L^p}^{\lambda}\|u\|_{L^r}^{(1-\lambda)}.$$

**Exercise 8:** (Weak convergence) (2)
If $x_n \in C^0([a,b])$ and $x_n \rightharpoonup x$ in $C^0([a,b])$, show that $\{x_n\}$ is pointwise convergent on $[a,b]$, i.e. that $x_n(t)$ converges for all $t \in [a,b]$.

**Exercise 9:** (Weak convergence in a Hilbert space) (2)
Let $H$ be a Hilbert space. Show that if $x_n \rightharpoonup x$ in $H$, and $\|x_n\| \to \|x\|$, then $x_n \to x$.

**Exercise 10:** (Energy Method) (8)
We use the *energy method* to show that all solutions of the following reaction-diffusion equation approach 0 as $t \to \infty$: On $\Omega = [0,1]$ we consider
$$\begin{aligned} u_t &= 2u_{xx} - 3u \\ u_x(0,t) = 0, &\qquad u_x(1,t) = 0, \end{aligned}$$
where lower case indices denote the partial derivative with respect to that variable, e.g. $u_t = \frac{\partial}{\partial t}u(x,t)$.

1. Use Young's inequality (with $p = q = 2$) to show that the *energy*
$$E[u,u_x](t) := \frac{1}{2}\int_0^1 (|u|^2 + |u_x|^2)dx$$
satsfies the differential inequality
$$\frac{\partial}{\partial t}E(t) \le -2\int_0^1 |u_{xx}|^2 dx - 3E(t).$$
2. Use Gronwall's inequality to show that $E(t) \to 0$ as $t \to \infty$.
3. Argue that $u(t)$ converges in $H^1([0,1])$ to 0 as $t \to \infty$. Does $u(t)$ also converge in $C^0([0,1])$ ?