1 The Heat Equation and Separation of Variables

The heat equation on a wire of length L is given by the DE

$$\boxed{u_t = \beta u_{xx}} \tag{1}$$

with boundary conditions $u(0,t) = u_0$, $u(L,t) = u_L$. Note that the boundary conditions can depend on time, just as the initial conditions, $u(x,0) = f(x)$ can depend on space.

We solve the homogeneous case where $u_0 = u_L = 0$ using separation of variables. That is, we set

$$\boxed{u(x,t) = \sum T_\alpha(t) X_\alpha(x).} \tag{2}$$

We note that the eigenfunctions of the differential operator $\partial_t + \beta\partial_{xx}$, by the Hilbert-Schmidt theorem, form an orthogonal basis. Thus, $u(x,t)$ has a unique representation of the form (2). Let T and X to be eigenfunctions of ∂_t and ∂_{xx}, respectively. Then, putting (2) into (1), we can look at individual terms:

$$\begin{aligned} \frac{\partial}{\partial t} T_\alpha(t) X_\alpha(x) = \beta \frac{\partial^2}{\partial x^2} T_\alpha(t) X_\alpha(x) &\Rightarrow T_\alpha^{'}(t) X_\alpha(x) = \beta T_\alpha^{'}(t) X_\alpha^{''}(x) \\ &\Rightarrow \frac{T_\alpha^{'}(t)}{\beta T_\alpha(t)} = \frac{X_\alpha^{''}(x)}{X_\alpha(x)} = \alpha. \end{aligned}$$

Since $T_\alpha^{'}(t)/\beta T_\alpha(t)$ does not depend on x, $X_\alpha^{''}(x)/X_\alpha(x)$ does not depend on x either. Thus, α is constant. Thus, we have two differential equations for each α:

$$\boxed{X_\alpha^{''}(x) = \alpha X_\alpha(x)} \tag{3}$$

and

$$\boxed{T_\alpha^{'}(t) = \alpha\beta T_\alpha(t).} \tag{4}$$

We have three cases to deal with in solving equations (3) and (4):

1. $\alpha < 0$.
   The DE for X is $X_\alpha^{''} = \alpha X_\alpha$. Since $\alpha < 0$, so we get sinusoidal behaviour:

   $$X_\alpha = C_1(\alpha)\cos\left(\sqrt{-\alpha}x\right) + C_2(\alpha)\sin\left(\sqrt{-\alpha}x\right)$$

   Similarly, $T_\alpha' = \beta\alpha T_\alpha$ has solution

   $$T_\alpha = A(\alpha)e^{-\beta\alpha t}.$$

   For $\alpha < 0$, define

   $$c_\alpha(x) = \cos\left(\sqrt{-\alpha}x\right), \quad s_\alpha(x) = \sin\left(\sqrt{-\alpha}x\right).$$

2. $\alpha = 0$.
   In this case, $X_0'' = 0$, so $X_0 = C_1(0) + C_2(0)x$. Since $T_0' = 0$, so $T_0 = A(0)$. For $\alpha = 0$, we define

$$c_0(x) = 1, \quad s_0(x) = x.$$

3. $\alpha > 0$.
   $X_\alpha'' = \alpha X_\alpha$, but $\alpha > 0$, so we get exponential behaviour:

$$X_\alpha = C_1(\alpha)\cosh\left(\sqrt{-\alpha}x\right) + C_2(\alpha)\sinh\left(\sqrt{-\alpha}x\right).$$

   and, again, $T_\alpha = A(\alpha)e^{-\beta\alpha t}$. If $\alpha > 0$, then denote

$$c_\alpha(x) = \cosh\left(\sqrt{\alpha}x\right), \quad s_\alpha(x) = \sinh\left(\sqrt{\alpha}x\right).$$

We now have

$$u(x,t) = \int_{\alpha\in\mathbb{R}} T_\alpha(t)X_\alpha(x). = \int_{\alpha=-\infty}^{\infty} e^{-\beta\alpha t}[C_1(\alpha)c_\alpha(x) + C_2(\alpha)s_\alpha(x)], \tag{5}$$

where we have rolled the $A(\alpha)$ into the Cs, and

$$c_\alpha(x) = \begin{cases} \cos(\sqrt{-\alpha}x) & \alpha < 0 \\ 1 & \alpha = 0 \\ \cosh(\sqrt{\alpha}x) & \alpha > 0 \end{cases}, \quad s_\alpha(x) = \begin{cases} \sin(\sqrt{-\alpha}x) & \alpha < 0 \\ x & \alpha = 0 \\ \sinh(\sqrt{\alpha}x) & \alpha > 0 \end{cases}.$$

We now apply the homogeneous boundary condition $u(0,t) = 0$:

Lemma 1: *If $u(0,t) = 0$, then $C_1(\alpha) = 0$ for all α,*

The basic idea of this proof is that, since each $\cos(\sqrt{-\alpha}x)$ or $\cosh(\sqrt{\alpha}x)$ has an $e^{-\beta\alpha t}$ in front of it, the only way to get $u(0,t) = 0$ is for all the coefficients $C_1(\alpha)$ to be zero.

Proof. From equation (5),

$$\begin{aligned} 0 = u(0,t) &= \int_{\alpha\in\mathbb{R}} T_\alpha(t)X_\alpha(0). \\ &= \int_{\alpha=-\infty}^{\infty} e^{-\beta\alpha t}[C_1(\alpha)c_\alpha(0) + C_2(\alpha)s_\alpha(0)]\,d\alpha \\ &= \int_{\alpha=-\infty}^{\infty} e^{-\beta\alpha t}C_1(\alpha)\,d\alpha \end{aligned}$$

Let $a = \beta\alpha$. Then we get

$$\int_{\alpha=-\infty}^{\infty} e^{-\beta\alpha t} C_1(\alpha)\, d\alpha = \int_{a=-\infty}^{\infty} e^{-at} \frac{C_1(a/\beta)}{\beta}\, da$$
$$= \int_{a=-\infty}^{\infty} e^{-at} D(a)\, da$$

where $D(a) = C_1(a/\beta)/b$. This is the *two-sided Laplace transform* of $D(a)$, denoted $\mathcal{B}[D(a)]$. Since $\mathcal{B}[D(a)] = u(0,t) = 0$, the two-sided Laplace transform of $D(a)$ is zero, so $D(a) = 0$. This implies that $C_1(\alpha) = 0$. □

Since $C_1(\alpha) \equiv 0$, we can write equation (5) as

$$u(x,t) = \int_{\alpha=-\infty}^{\infty} e^{-\beta\alpha t} C_2(\alpha) s_\alpha(x)\, d\alpha. \tag{6}$$

We can now apply the boundary condition at $x = L$ to equation (6):

Lemma 2: *If $u(L,t) = 0$, then $C_2(\alpha) = 0$ if $\alpha \neq -n^2\pi^2/L^2$, with $n = 1, 2, 3, \ldots$.*

The idea here is that we have a bunch of exponentials $e^{-\beta\alpha t}$, the only way to get $u(L,t) = 0$ is for all of the terms to be zero, just like before. In this case, however, the sin terms can be zero as well, so $C_2(\alpha)$ can be non-zero for certain values of α.

Proof. Set $x = L$ in equation (6). We then have

$$0 = u(x,L) = \int_{\alpha=-\infty}^{\infty} e^{-\beta\alpha t} C_2(\alpha) s_\alpha(L)\, d\alpha$$

The approach is similar to lemma 1, except, since

$$s_\alpha(L) = \begin{cases} \sin(\sqrt{-\alpha}L) & \alpha < 0 \\ L & \alpha = 0 \\ \sinh(\sqrt{\alpha}L) & \alpha > 0 \end{cases}$$

we must have

$$D(a) = \begin{cases} \frac{C_2(-\sqrt{a/\beta})}{-2\sqrt{a\beta}} \sin(\sqrt{-\alpha}L) & \alpha < 0 \\ \frac{C_2(0)}{-2\sqrt{a\beta}} L & a = 0 \\ \frac{C_2(\sqrt{a/\beta})}{2\sqrt{a\beta}} \sinh(\sqrt{-\alpha}L) & \alpha > 0 \end{cases}$$

Again, we find that $D(a) = 0$ since $\mathcal{B}[D] = 0$. Since $\sinh(\sqrt{\alpha}L) \neq 0$ and $L \neq 0$, this implies that $C(\alpha) \equiv 0$ for $\alpha \geq 0$. However, $\sin(\sqrt{-\alpha}L) = 0$ if $\sqrt{-\alpha}L = n\pi$, so $C(\alpha) = 0$ if $\alpha \neq -n^2\pi^2/L^2$. □

Denote $b_n = \delta_{-n^2\pi^2/L^2}(\alpha)C_2(-n^2\pi^2/L^2)$. The solution to the heat equation then has form

$$\boxed{u(x,t) = \sum_{n=1}^{\infty} b_n e^{-\beta \frac{n^2\pi^2}{L^2} t} \sin\left(\frac{n\pi x}{L}\right).}$$

To get this form, we have used:

1. the differential equation $u_t = \beta u_{xx}$, and
2. the homogeneous boundary conditions $u(0,t) = 0$, $u(L,t) = 0$.

We have not used

1. the initial conditions, $u(x,0) = f(x)$, for some given function $f(x)$.

In order to completely determine the solution to the IVP, we will use the initial conditions to determine the unknowns $b_n, n = 1, 2, 3, \ldots$.