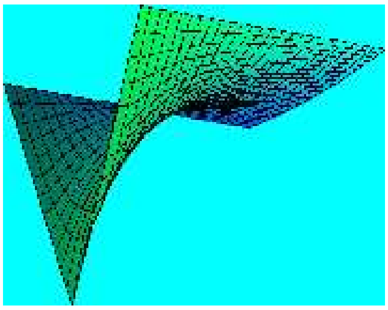

Math 300 Winter 2011
Advanced Boundary Value Problems I

Bessel's Equation and Bessel Functions

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Bessel's Equation and Bessel Functions

We use the following boundary value-initial value problem satisfied by a vibrating circular membrane in the plane to introduce Bessel's equation and its solutions.

$$\frac{\partial^2 u}{\partial t^2} = c^2\left(\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}\right),$$

$$u(r,-\pi,t) = u(r,\pi,t),$$

$$\frac{\partial u}{\partial \theta}(r,-\pi,t) = \frac{\partial u}{\partial \theta}(r,\pi,t),$$

$$u(a,\theta,t) = 0,$$

$$|u(a,\theta,t)| < \infty \text{ as } r \to 0^+$$

$$u(r,\theta,0) = f(r,\theta)$$

$$\frac{\partial u}{\partial t}(r,\theta,0) = g(r,\theta),$$

for $0 < r < a$, $-\pi < \theta < \pi$, and $t > 0$.

We look for a separated solution of the form

$$u(r,\theta,t) = R(r)\cdot\phi(\theta)\cdot G(t),$$

and substituting this into the wave equation, we have

$$R\phi G'' = c^2\left(R'' + \frac{1}{r}R'\right)\phi G + \frac{c^2}{r^2}R\phi'' G,$$

dividing by $c^2 R\phi G$, we have

$$\frac{G''}{c^2 G} = \frac{rR'' + R'}{rR} + \frac{1}{r^2}\frac{\phi''}{\phi} = -\lambda \quad \text{(constant)}$$

since the left-hand side depends only on t and the right-hand side depends only on r and θ. Thus, we have the two differential equations

$$G'' + \lambda c^2 G = 0 \qquad \text{and} \qquad r^2R''\phi + rR'\phi + \lambda r^2 R\phi = -R\phi',$$

and separating variables again in the second equation, we have

$$\frac{r^2R'' + rR' + \lambda r^2 R}{R} = -\frac{\phi'}{\phi} = \mu \quad \text{(constant)}$$

so that

$$r^2R'' + rR' + (\lambda r^2 - \mu)R = 0 \qquad \text{and} \qquad \phi'' + \mu\phi = 0.$$

We can meet the periodicity conditions and the boundary and boundedness conditions by requiring that

$$\phi(-\pi) = \phi(\pi) \qquad \text{and} \qquad \phi'(-\pi) = \phi'(\pi)$$

and

$$R(a) = 0 \qquad \text{and} \qquad |R(r)| < \infty \quad \text{as} \quad r \to 0^+$$

Thus, we have the following ordinary differential equations:

(a) *Angular problem*:

$$\begin{aligned} &\phi'' + \mu\phi = 0, \quad -\pi < \theta < \pi \\ &\phi(-\pi) = \phi(\pi) \\ &\phi'(-\pi) = \phi'(\pi) \end{aligned}$$

with eigenvalues and corresponding eigenfunctions

$$\mu_n = n^2 \qquad \text{and} \qquad \phi_n(\theta) = a_n \cos n\theta + b_n \sin n\theta$$

for $n \geq 0$.

(b) *Radial problem*:

$$\begin{gathered} r^2R'' + rR' + (\lambda r^2 - n^2)R = 0 \\ R(a) = 0 \\ |R(r)| < \infty \quad \text{as} \quad r \to 0^+ \end{gathered}$$

(c) *Temporal problem*:

$$G'' + \lambda c^2 G = 0$$

We can put the radial equation into Sturm-Liouville form:

$$\frac{d}{dr}\left(r\frac{dR}{dr}\right) + \left(\lambda r - \frac{n^2}{r}\right) R = 0,$$

with

$$p(r) = r, \qquad q(r) = -\frac{n^2}{r}, \qquad \sigma(r) = r,$$

and this eigenvalue problem is a singular Sturm-Liouville problem since

$$p(0) = \sigma(0) = 0, \quad \text{and} \quad q(r) \to -\infty \quad \text{as} \quad r \to 0^+,$$

as well, the boundary conditions are not of Sturm-Liouville form. However, we can still find the eigenvalues and eigenfunctions.

If (λ, R) is an eigenpair of the radial equation, the Rayleigh quotient is

$$\lambda = \frac{-rR(r)R'(r)\Big|_0^a + \displaystyle\int_0^a \left[rR'(r)^2 + \frac{n^2}{r}R(r)^2\right] dr}{\displaystyle\int_0^a R(r)^2 r\, dr}.$$

From the boundary condition and the boundedness condition, we have

$$-rR(r)R'(r)\Big|_0^a = -aR(a)R'(a) + \lim_{r \to 0^+} rR(r)R'(r) = 0,$$

and therefore all the eigenvalues are nonnegative.

For $\lambda = 0$, the radial problem is

$$r^2 R'' + rR' - n^2 R = 0, \quad 0 < r < a$$
$$R(a) = 0,$$
$$|R(r)| < \infty \quad \text{as} \quad r \to 0^+,$$

and the differential equation is a Cauchy-Euler equation or an equidimensional equation with solution

$$R_0(r) = A_0 \log r + B_0, \quad \text{for} \quad n = 0,$$

and

$$R_n(r) = A_n r^n + \frac{B_n}{r^n}, \quad \text{for} \quad n \geq 1.$$

For $\lambda > 0$, we can transform the radial equation into an equation that is independent of λ by letting $x = \sqrt{\lambda}\, r$, then

$$\frac{dR}{dr} = \sqrt{\lambda}\,\frac{dR}{dx} \qquad \text{and} \qquad \frac{d^2R}{dr^2} = \lambda \frac{d^2R}{dx^2},$$

and the equation becomes

$$r^2 \frac{d^2R}{dr^2} + r\frac{dR}{dr} + (\lambda r^2 - n^2)R = x^2 \frac{d^2R}{dx^2} + x\frac{dR}{dx} + (x^2 - n^2)R = 0,$$

that is,

$$x^2 \frac{d^2R}{dx^2} + x\frac{dR}{dx} + (x^2 - n^2)R = 0,$$

Bessel's equation of order n (nonparametric).

We will use a power series method called the **Method of Frobenius**, to find two linearly independent solutions to Bessel's equation.

Bessel Functions

We look for a solution to Bessel's equation

$$x^2 \frac{d^2y}{dx^2} + x\frac{dy}{dx} + (x^2 - n^2)y = 0$$

where n is an integer, of the form

$$y = a_0 x^m + a_1 x^{m+1} + \cdots + a_k x^{m+k} + \cdots,$$

that is,

$$y = \sum_{k=0}^{\infty} a_k x^{m+k}$$

where $a_0 \neq 0$.

Differentiating the series, we have

$$\frac{dy}{dx} = \sum_{k=0}^{\infty} (m+k) a_k x^{m+k-1}$$

and

$$\frac{d^2y}{dx^2} = \sum_{k=0}^{\infty} (m+k)(m+k-1) a_k x^{m+k-2},$$

substituting this into the differential equation we have

$$\sum_{k=0}^{\infty} \left[(m+k)(m+k-1)+(m+k)-n^2\right] a_k x^{m+k} + \sum_{k=0}^{\infty} a_k x^{m+k+2} = 0,$$

and reindexing the last sum, we have

$$\sum_{k=0}^{\infty} \left[(m+k)(m+k-1)+(m+k)-n^2\right] a_k x^{m+k} + \sum_{k=2}^{\infty} a_{k-2} x^{m+k} = 0.$$

Therefore,

$$\sum_{k=0}^{\infty} \left[(m+k)^2-n^2\right] a_k x^{m+k} + \sum_{k=2}^{\infty} a_{k-2} x^{m+k} = 0,$$

that is,

$$(m^2-n^2)a_0x^m + \left[(m+1)^2-n^2\right] a_1 x^{m+1} + \sum_{k=2}^{\infty} \left\{\left[(m+k)^2-n^2\right] a_k + a_{k-2}\right\} x^{m+k} = 0.$$

This must be an identity in x, hence, all coefficients must vanish, so that

$$(m^2-n^2)a_0 = 0$$
$$\left[(m+1)^2-n^2\right] a_1 = 0$$
$$\left[(m+k)^2-n^2\right] a_k + a_{k-2} = 0$$

for $k \geq 2$.

Since $a_0 \neq 0$, we get the **indicial equation** from the first term

$$m^2 - n^2 = 0,$$

so that $m = \pm n$, and for the moment we choose $m = n$.

Also, with this choice of m, $(m+1)^2 - n^2 \neq 0$, we must have $a_1 = 0$.

Finally, we get the **recurrence relation**

$$a_k = -\frac{1}{k(2n+k)} a_{k-2}$$

for $k \geq 2$.

Using the fact that $a_1 = 0$, from the recurrence relation we find

$$a_3 = 0, \quad a_5 = 0, \quad a_7 = 0, \quad a_9 = 0, \quad \ldots,$$

that is, $a_{2k+1} = 0$ for $k \geq 0$.

Also, from the recurrence relation

$$a_2 = -\frac{1}{1\cdot(n+1)} \cdot \frac{1}{2^2}\, a_0$$

$$a_4 = -\frac{1}{2\cdot(n+2)} \cdot \frac{1}{2^2}\, a_2$$

$$a_6 = -\frac{1}{3\cdot(n+3)} \cdot \frac{1}{2^2}\, a_4$$

$$\vdots$$

$$a_{2k} = -\frac{k}{1\cdot(n+k)} \cdot \frac{1}{2^2}\, a_{2k-2}.$$

If we multiply the terms on the left and multiply the terms on the right, and cancel the common factor $a_2 \cdot a_4 \cdot a_6 \cdots a_{2k-2}$, we get

$$a_{2k} = \frac{(-1)^k}{k!(n+1)(n+2)\cdots(n+k)} \cdot \frac{1}{2^{2k}}\, a_0 = \frac{(-1)^k\, n!}{k!(n+k)!} \cdot \frac{1}{2^{2k}}\, a_0$$

for $k \geq 1$.

Therefore, the solution is (recall that $m = n$)

$$y = a_0 x^n + \sum_{k=1}^{\infty} a_{2k} x^{n+2k},$$

and it is an easy exercise to show that this series converges absolutely and uniformly for all real numbers x (use the ratio test).

Note that the constant a_0 can have any nonzero value, and if we chose

$$a_0 = \frac{1}{n! \cdot 2^n},$$

then the solution becomes

$$y = \frac{x^n}{n! \cdot 2^n} + \sum_{k=1}^{\infty} \frac{(-1)^k\, n!}{k!(n+k)!} \left(\frac{x}{2}\right)^{n+2k},$$

and we denote this function by $J_n(x)$, so that

$$J_n(x) = \frac{1}{n!} \left(\frac{x}{2}\right)^n + \sum_{k=1}^{\infty} \frac{(-1)^k}{k!(n+k)!} \left(\frac{x}{2}\right)^{n+2k},$$

that is,

$$J_n(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!(n+k)!} \left(\frac{x}{2}\right)^{n+2k}$$

for $n \geq 0$. The function $J_n(x)$ is called the **Bessel function of the 1st kind of order n**.

Note that

$$J_n(-x) = (-1)^n J_n(x),$$

so that

$J_n(x)$ is an **even** function if $n = 0, 2, 4, \ldots$

$J_n(x)$ is an **odd** function if $n = 1, 3, 5, \ldots$

Also, $J_0(0) = 1$, while $J_n(0) = 0$ for $n \geq 1$.

The Bessel function of order 0 is given by

$$J_0(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{(k!)^2} \left(\frac{x}{2}\right)^{2k},$$

and since $(k!)^2 = 2^2 \cdot 4^2 \cdot 6^2 \cdots (2k)^2$, we can rewrite this as

$$J_0(x) = 1 + \sum_{k=1}^{\infty} \frac{x^2k}{2^2 4^2 6^2 \cdots (2k)^2} = 1 - \frac{x^2}{2^2} + \frac{x^4}{2^2 4^2} - \frac{x^6}{2^2 4^2 6^2} + \cdots,$$

which looks similar to the Maclaurin series for $\cos x$.

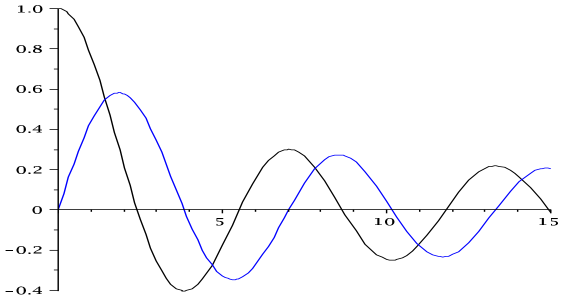
There are also similarities between the derivatives of $J_0(x)$ and $\cos x$; for example,

$$J_0{}'(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{(k!)^2} \frac{(2k)}{2} \left(\frac{x}{2}\right)^{2k-1} = \sum_{k=1}^{\infty} \frac{(-1)^k}{k!(k-1)!} \left(\frac{x}{2}\right)^{2k-1} = \sum_{\ell=0}^{\infty} \frac{(-1)^{\ell+1}}{\ell!(\ell+1)!} \left(\frac{x}{2}\right)^{2\ell+1}$$

that is,

$$J_0{}'(x) = -\sum_{k=0}^{\infty} \frac{(-1)^k}{k!(k+1)!} \left(\frac{x}{2}\right)^{2k+1} = -J_1(x)$$

for all x. Graphs of $J_0(x)$ and $J_1(x)$ are shown below:



There are two linearly independent solutions to Bessel's equation of order m

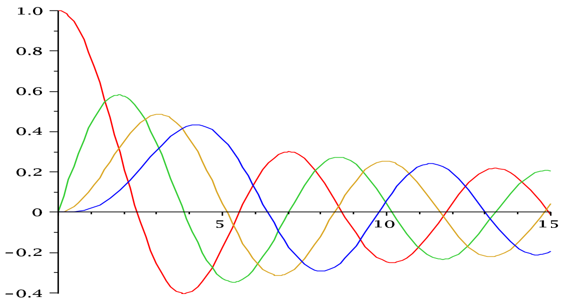
$$x^2 \frac{d^2y}{dx^2} + x\frac{dy}{dx} + (x^2 - m^2)y = 0.$$

One of them we found using series solutions:

$$J_m(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!(m+k)!} \left(\frac{x}{2}\right)^{m+2k}$$

for $m = 0, 1, 2, \ldots$, called a **Bessel function of the 1st kind of order m**.

The Bessel functions $J_0(x),\ J_1(x),\ J_2(x),$ and $J_3(x)$ are plotted below.



A second linearly independent solution is

$$Y_m(x) = \frac{2}{\pi}\Bigg[J_m(x)\left(\gamma + \log\frac{x}{2}\right) - \frac{1}{2}\sum_{k=0}^{m-1} \frac{(m-k-1)!}{m!} \left(\frac{x}{2}\right)^{2k-m}$$

$$- \frac{1}{2}\sum_{k=1}^{\infty} \frac{(-1)^k \left[\phi(k) + \phi(k+m)\right] \left(\frac{x}{2}\right)^{m+2k}}{k!(m+k)!}\Bigg],$$

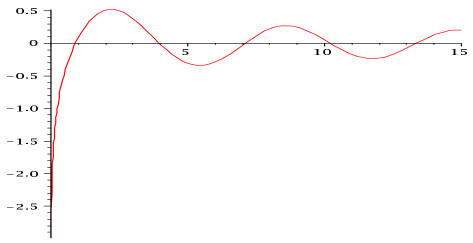
where

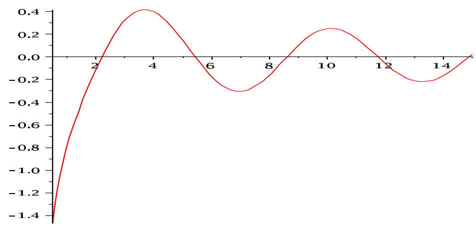
$$\phi(k) = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{k},$$

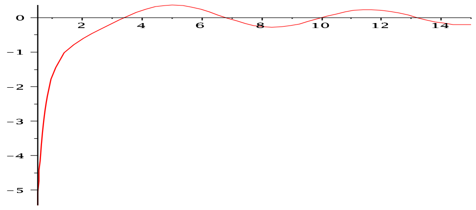
and γ is the **Euler-Mascheroni** constant

$$\gamma = \lim_{k \to \infty} \left(1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{k} - \log k \right).$$

The function $Y_m(x)$ is called a **Bessel function of the 2nd kind of order m**, or a **Neumann function**, or a **Weber function**. The functions $Y_0(x), Y_1(x)$, and $Y_2(x)$ are plotted below.







Note: The functions $J_\nu(x)$ and $Y_\nu(x)$ can also be defined for values of ν which are not integers, and satisfy **Bessel's equation of order** ν :

$$x^2 y'' + xy' + (x^2 - \nu^2)y = 0$$

where ν is a constant (not necessarily an integer).

One solution is the **Bessel function of the 1st kind of order** ν

$$J_\nu(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{\Gamma(n+1)\Gamma(n+\nu+1)} \left(\frac{x}{2}\right)^{2n+\nu},$$

where Γ is the **gamma function** defined by

$$\Gamma(\nu) = \int_0^\infty x^{\nu-1} e^{-x}\, dx$$

for $\nu > 0$.

A second linearly independent solution is

$$Y_\nu(x) = \frac{\cos \pi\nu \, J_\nu(x) - J_{-\nu}(x)}{\sin \pi\nu},$$

the **Bessel function of the 2nd kind of order** ν.

Properties of Bessel Functions

Recall: For a nonnegative integer m,

$$J_m(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!(m+k)!} \left(\frac{x}{2}\right)^{m+2k} \tag{$*$}$$

for $-\infty < x < \infty$.

The following relations hold among Bessel functions and their derivatives, and are true for $J_m(x)$ as well as $Y_m(x)$, whether or not m is an integer.

Theorem 1.

(i) $\dfrac{d}{dx}\left[x^m J_m(x)\right] = x^m J_{m-1}(x) \quad \text{for} \quad m \geq 1$, and

(ii) $\dfrac{d}{dx}\left[x^{-m} J_m(x)\right] = -x^{-m} J_{m+1}(x) \quad \text{for} \quad m \geq 0.$

Proof. We will prove part (i) and leave part (ii) as an exercise. Differentiating $(*)$, we have

$$\begin{aligned}
\frac{d}{dx}\left[x^m J_m(x)\right] &= \frac{d}{dx}\left[\sum_{k=0}^{\infty} \frac{(-1)^k}{k!(m+k)!}\left(\frac{x}{2}\right)^{2m+2k}\right] \\
&= \sum_{k=0}^{\infty} \frac{(-1)^k 2(m+k)}{k!(m+k)!}\left(\frac{x}{2}\right)^{2m+2k-1} 2^{m-1} \\
&= x^m \sum_{k=0}^{\infty} \frac{(-1)^k}{k!(m+k)!}\left(\frac{x}{2}\right)^{2k+m-1} \\
&= x^m J_{m-1}(x).
\end{aligned}$$

$\square$

We also have the following recurrence relations for $J_m(x)$ and ${J_m}'(x)$:

Theorem 2.

(i) ${J_m}'(x) + \dfrac{m}{x} J_m(x) = J_{m-1}(x)$

(ii) ${J_m}'(x) - \dfrac{m}{x} J_m(x) = -J_{m+1}(x)$

(iii) $2{J_m}'(x) = J_{m-1}(x) - J_{m+1}(x)$

(iv) $\dfrac{2m}{x} J_m(x) = J_{m-1}(x) + J_{m+1}(x)$

Proof. From the previous theorem we have

$$\frac{d}{dx}\left[x^m J_m(x)\right] = x^m {J_m}'(x) + m x^{m-1} J_m(x) = x^m J_{m-1}(x),$$

and

$$\frac{d}{dx}\left[x^{-m} J_m(x)\right] = x^{-m} {J_m}'(x) - m x^{-m-1} J_m(x) = x^{-m} J_{m+1}(x).$$

Dividing the first equation by x^m and multiplying the second equation by x^m, we get (i) and (ii).

Adding and subtracting (i) and (ii), we get (iii) and (iv).

$\square$

Note: From the above, it is clear that every Bessel function $J_n(x)$ with n an integer can be expressed in terms of $J_0(x)$ and $J_1(x)$, for example, taking $m = 1$ in (iv), we have

$$J_2(x) = \frac{2}{x} J_1(x) - J_0(x),$$

and taking $m = 2$ in (iv), we have

$$J_3(x) = -\frac{4}{x} J_0(x) - \left(1 - \frac{8}{x^2}\right) J_1(x).$$

Also, we can write the differentiation formulas

$$\frac{d}{dx} \left[x^m J_m(x)\right] = x^m J_{m-1}(x) \quad \text{for} \quad m \geq 1,$$

and

$$\frac{d}{dx} \left[x^{-m} J_m(x)\right] = -x^{-m} J_{m+1}(x) \quad \text{for} \quad \geq 0,$$

as integral formulas

$$\int x^m J_{m-1}(x)\, dx = x^m J_m(x) + C,$$

and

$$\int x^{-m} J_{m+1}(x)\, dx = -x^{-m} J_m(x) + C$$

where C is constant.

For example, when $m = 1$, the first equation yields

$$\int x\, J_0(x)\, dx = x\, J_1(x) + C.$$

Fourier - Bessel Series

Given a fixed nonnegative integer m, the function $J_m(x)$ has an infinite number of positive zeros: z_{nm}, $n = 1, 2, 3, \ldots$, so that

$$J_m(z_{nm}) = 0$$

for $n = 1, 2, 3, \ldots$.

Suppose that we want to expand a given function $f(x)$ in terms of a fixed Bessel function, that is,

$$f(x) = \sum_{n=1}^{\infty} a_n J_m(z_{nm} x) = a_1 J_m(z_{1m} x) + a_2 J_m(z_{2m} x) + \cdots a_n J_m(z_{nm} x) + \cdots,$$

where $f(x)$ is defined for $0 \leq x \leq 1$ and the $z_{nm}{}'s$ are the positive zeros of $J_m(x)$.

In order to determine the coefficients a_n, we need an **orthogonality relation** just as we did for Fourier series. In this case, however, we are expanding the function $f(x)$ in terms of a fixed Bessel function $J_m(x)$ and the series is summed over the positive zeros of $J_m(x)$.

Recall that the Bessel function $J_m(x)$ was a solution to a singular Sturm-Liouville problem, and we will show that the eigenfunctions $J_m(z_{nm} x)$, for $n = 1, 2, 3, \ldots$, are orthogonal on the interval $[0, 1]$ with respect to the weight function $\sigma(x) = x$.

Theorem 3. For a fixed integer $m \geq 0$,

$$\int_0^1 x J_m(z_{nm}x) J_m(z_{km}x)\, dx = 0$$

for $n \neq k$, and

$$\int_0^1 x J_m(z_{nm}x)^2\, dx = \frac{1}{2}\, J_{m+1}(z_{nm})^2$$

where z_{nm} is the n^{th} positive zero of $J_m(x)$.

Thus, the functions $\sqrt{x}\, J_m(z_{nm}x)$ are orthogonal on the interval $0 \leq x \leq 1$.

Proof. Note that $y = J_m(x)$ is a solution to the differential equation

$$y'' + \frac{1}{x} y' + \left(1 - \frac{m^2}{x^2}\right) y = 0,$$

and if a and b are distinct positive constants, then the functions

$$u(x) = J_m(ax) \qquad \text{and} \qquad v(x) = J_m(bx)$$

satisfy the differential equations

$$u'' + \frac{1}{x} u' + \left(a^2 - \frac{m^2}{x^2}\right) u = 0$$

and

$$v'' + \frac{1}{x} v' + \left(b^2 - \frac{m^2}{x^2}\right) v = 0,$$

respectively. Multiplying the first of these equations by v and the second by u, and subtracting, we have

$$\frac{d}{dx}\left[u'v - v'u\right] + \frac{1}{x}\left(u'v - v'u\right) + (a^2 - b^2)uv = 0,$$

and multiplying by x, we have

$$\frac{d}{dx}\left[x\left(u'v - v'u\right)\right] = (b^2 - a^2)xuv.$$

Integrating from 0 to 1,

$$(b^2 - a^2)\int_0^1 x u(x) v(x)\, dx = \left[x(u'v - v'u)\right]\Big|_0^1,$$

and since

$$u(1) = J_m(a) \qquad \text{and} \qquad v(v) = J_m(b),$$

if a and b are distinct positive zeros of $J_m(x)$, say z_{nm} and z_{km}, then

$$\left(z_{nm}^2 - z_{km}^2\right)\int_0^1 x J_m(z_{nm}x) J_m(z_{km}x)\, dx = 0,$$

and since $z_{nm} \neq z_{km}$, then

$$\int_0^1 J_m(z_{nm}x) J_m(z_{km}x), x\, dx = 0,$$

that is, the functions $\sqrt{x}\, J_m(z_{nm}x)$ are orthogonal on the interval $0 \leq x \leq 1$.

To find the normalization constant, we note again that for $a > 0$, the function $J_m(ax)$ satisfies the differential equation

$$u'' + \frac{1}{x} u' + \left(a^2 - \frac{m^2}{x^2}\right) u = 0,$$

and multiplying this equation by $2x^2u'$, we get

$$2x^2u'u'' + 2xu'u' + 2a^2x^2uu' - 2m^2uu' = 0,$$

that is,

$$\frac{d}{dx}\left[x^2{u'}^2\right] + \frac{d}{dx}\left[a^2x^2u^2\right] - 2a^2xu^2 - \frac{d}{dx}\left[m^2u^2\right] = 0.$$

Integrating this equation from 0 to 1, we have

$$2a^2\int_0^1 xu^2\,dx = \left[x^2{u'}^2 + (a^2x^2 - m^2)u^2\right]\Big|_0^1,$$

and since $u(0) = J_m(0)$ for $m \geq 1$, then at $x = 0$,

$$x^2{u'}^2 + (a^2x^2 - m^2)u^2 = 0$$

for all $m \geq 0$.

Also, at $x = 1$,

$$u'(1) = \frac{d}{dx}\left[J_m(ax)\right]\Big|_{x=1} = a{J_m}'(a),$$

so that

$$\int_0^1 J_m(ax)^2x\,dx = \frac{1}{2}\,{J_m}'(a)^2 + \frac{1}{2}\left(1 - \frac{m^2}{a^2}\right)\,J_m(a)^2$$

for $m \geq 0$. Now put $a = z_{nm}$, the m^{th} positive zero of $J_m(x)$, then

$$\int_0^1 J_m(z_{nm}x)^2x\,dx = \frac{1}{2}\,{J_m}'(z_{nm})^2 = \frac{1}{2}J_{m+1}(z_{nm}))^2$$

for $m \geq 0$.

□

Now suppose that we have a function f defined on the interval $0 \leq x \leq 1$, and that it has a **Fourier-Bessel series expansion** given by

$$f(x) = \sum_{k=1}^{\infty} a_k J_m(z_{km}x),$$

we can find the coefficients in this expansion by multiplying the equation by $xJ_m(z_{nm}x)$ and integrating, to get

$$\int_0^1 f(x)J_m(z_{nm}x)x\,dx = \frac{a_n}{2}\,J_{m+1}(z_{nm})^2.$$

Therefore, if

$$f(x) = \sum_{n=1}^{\infty} a_n J_m(z_{nm}x),$$

then

$$a_n = \frac{2}{J_{m+1}(z_{nm})^2}\int_0^1 f(x)J_m(z_{nm}x)\,x\,dx$$

for $n \geq 1$.

We have a convergence theorem for Fourier-Bessel series similar to Dirichlet's theorem:

Theorem 4. (Fourier-Bessel Expansion Theorem)

If f and f' are piecewise continuous on the interval $0 \leq x \leq 1$, then for $0 < x_0 < 1$, the series

$$\sum_{n=1}^{\infty} a_n J_m(z_{nm} x_0)$$

converges to $\dfrac{1}{2}\left[f(x_0^+) + f(x_0^-)\right]$.

At $x_0 = 1$, the series converges to 0, since every $J_m(z_{nm}) = 0$.

At $x_0 = 0$, the series converges to 0 if $m \geq 1$, and to $f(0^+)$ if $m = 0$.

Vibrating Circular Membrane

We now return to the vibrating circular membrane problem we started with:

$$\frac{\partial^2 u}{\partial t^2} = c^2 \left(\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} \right),$$

$$u(r, -\pi, t) = u(r, \pi, t),$$

$$\frac{\partial u}{\partial \theta}(r, -\pi, t) = \frac{\partial u}{\partial \theta}(r, \pi, t),$$

$$u(a, \theta, t) = 0,$$

$$|u(a, \theta, t)| < \infty \text{ as } r \to 0^+,$$

$$u(r, \theta, 0) = f(r, \theta),$$

$$\frac{\partial u}{\partial t}(r, \theta, 0) = g(r, \theta),$$

for $0 < r < a$, $-\pi < \theta < \pi$, and $t > 0$. Separating variables, we assumed that $u(r, \theta, t) = R(r)\phi(\theta)G(t)$, and obtained three ordinary differential equations

Temporal equation:

$$G'' + \lambda c^2 G = 0 \quad t > 0;$$

Angular equation:

$$\begin{aligned} &\phi'' + \mu\phi = 0, \quad -\pi < \theta < \pi \\ &\phi(-\pi) = \phi(\pi), \\ &\phi'(-\pi) = \phi'(\pi); \end{aligned}$$

Radial equation:

$$\begin{aligned} &r^2 R'' + rR' + (\lambda r^2 - \mu)R = 0, \quad 0 < r < a \\ &R(a) = 0, \\ &|R(r)| < \infty \quad \text{as} \quad r \to 0^+; \end{aligned}$$

where λ and μ are separation constants.

For the angular equation, the eigenvalues and eigenfunctions are

$$\mu_m = m^2 \qquad \text{and} \qquad \phi_m(\theta) = a_m \cos m\theta + b_m \sin m\theta$$

for $m \geq 0$.

We saw earlier using the Rayleigh quotient, that all the eigenvalues of the radial problem are nonnegative.

For $\lambda = 0$, the radial equation is

$$r^2 R'' + rR' - m^2 R = 0, \quad 0 < r < a$$
$$R(a) = 0,$$
$$|R(r)| < \infty \quad \text{as} \quad r \to 0^+,$$

and the differential equation is a Cauchy-Euler equation or an equidimensional equation with solution

$$R_0(r) = A_0 \log r + B_0, \quad \text{for} \quad m = 0,$$

and

$$R_m(r) = A_m r^m + \frac{B_m}{r^m}, \quad \text{for} \quad m \geq 1.$$

Since the solutions must be bounded at $r = 0$, then we have $A_0 = 0$ and $B_m = 0$ for all $m \geq 1$, so that

$$R_0(r) = B_0 \qquad \text{and} \qquad R_m(r) = A_m r^m,$$

and applying the boundary condition $R(a) = 0$, we see that $B_0 = 0$ and $A_m = 0$ for all $m \geq 1$. Thus, $\lambda = 0$ is not an eigenvalue.

For $\lambda > 0$, the radial equation is

$$r^2 R'' + rR' + (\lambda r^2 - m^2) R = 0$$
$$R(a) = 0$$
$$|R(r)| < \infty \quad \text{as} \quad r \to 0^+,$$

with general solution

$$R(r) = A J_m(\sqrt{\lambda}\, r) + B Y_m(\sqrt{\lambda}\, r),$$

and since Y_m is not bounded as $r \to 0^+$, then we must have $B = 0$, and the solution is

$$R(r) = A J_m(\sqrt{\lambda}\, r).$$

The boundary condition $R(a) = 0$ gives us

$$A J_m(\sqrt{\lambda}\, a) = 0,$$

and in order to get a nontrivial solution we must have

$$J_m(\sqrt{\lambda}\, a) = 0.$$

The eigenvalues are those values of λ for which $J_m(\sqrt{\lambda}\, a) = 0$, that is, for which $\sqrt{\lambda}\, a = z_{nm}$, where z_{nm} is the n^{th} positive zero of J_m :

$$\lambda_{nm} = \left(\frac{z_{nm}}{a} \right)^2$$

for $n \geq 1$ and $m \geq 0$.

The corresponding eigenfunctions are

$$R_{nm}(r) = J_m\left(\frac{z_{nm}}{a}\, r \right)$$

for $n \geq 1$ and $m \geq 0$.

Note: For a fixed $m \geq 0$, these eigenfunctions form an orthogonal basis for the linear space $PWS[0, a]$ of piecewise smooth functions on $[0, a]$ with weight function $\sigma(r) = r$, so that

$$\int_0^a J_m\left(\frac{z_{nm}}{a}\, r \right) J_m\left(\frac{z_{km}}{a}\, r \right) r\, dr = 0$$

for $n \neq k$.

The generalized Fourier series is now called a **Fourier-Bessel series**, and for $w \in PWS[0,a]$ we have

$$w(r) = \sum_{n=1}^{\infty} a_n J_m\left(\frac{z_{nm}}{a}\, r\right),$$

where

$$a_n = \frac{\displaystyle\int_0^a f(x) J_m\left(\frac{z_{nm}}{a}\, r\right) r\, dr}{\displaystyle\int_0^a J_m\left(\frac{z_{nm}}{a}\, r\right)^2 r\, dr}$$

for $n \geq 1$.

We only need to solve the time equation for those values of λ for which we have a nontrivial solution to the radial equation, that is, $\lambda = \lambda_{nm}$, and the time equation is

$$G'' + c^2 \lambda_{nm} G = 0,$$

with general solution

$$G_{nm}(t) = A\cos(c\sqrt{\lambda_{nm}}\, t) + B\sin(c\sqrt{\lambda_{nm}}\, t)$$

for $n \geq 1$ and $m \geq 0$.

For each $m \geq 0,\ n \geq 1$, the product solution

$$u_{nm}(r,\theta,t) = R_{nm}(r) \cdot \phi_m(\theta) \cdot G_{nm}(t)$$

satisfies the PDE, the boundary condition, and the boundedness condition. In order to satisfy the initial conditions we use the superposition principle to write

$$u(r,\theta,t) = \sum_{m=0}^{\infty}\sum_{n=1}^{\infty} \Big\{ J_m\big(\sqrt{\lambda_{nm}}\, r\big) \Big[a_{nm}\cos m\theta \cos\big(c\sqrt{\lambda_{nm}}\, t\big) + b_{nm}\cos m\theta \sin\big(c\sqrt{\lambda_{nm}}\, t\big)$$

$$+ c_{nm}\sin m\theta \cos\big(c\sqrt{\lambda_{nm}}\, t\big) + d_{nm}\sin m\theta \sin\big(c\sqrt{\lambda_{nm}}\, t\big) \Big] \Big\}$$

for $0 < r < a,\ -\pi < \theta < \pi,\ t > 0$.

Now we apply the initial conditions to determine the constants a_{nm}, b_{nm}, c_{nm}, and d_{nm}. For simplicity, we assume that

$$\frac{\partial u}{\partial t}(r,\theta,0) = g(r,\theta) = 0,$$

and this implies, after differentiating term-by-term and setting $t = 0$, that

$$b_{nm} = 0 \qquad \text{and} \qquad d_{nm} = 0$$

for $m \geq 0$ and $n \geq 1$, so that the solution becomes

$$u(r,\theta,t) = \sum_{m=0}^{\infty}\sum_{n=1}^{\infty} J_m\big(\sqrt{\lambda_{nm}}\, r\big)\cos\big(c\sqrt{\lambda_{nm}}\, t\big)\big[a_{nm}\cos m\theta + c_{nm}\sin m\theta\big].$$

Setting $t = 0$, we have

$$f(r,\theta) = u(r,\theta,0) = \sum_{m=0}^{\infty}\sum_{n=1}^{\infty} J_m\big(\sqrt{\lambda_{nm}}\, r\big)\big[a_{nm}\cos m\theta + c_{nm}\sin m\theta\big],$$

which is a Fourier series for $f(r,\theta)$ on the interval $[-\pi,\pi]$ holding r fixed.

Therefore,

$$\sum_{n=1}^{\infty} a_{n0} J_0\big(\sqrt{\lambda_{n0}}\, r\big) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(r,\theta)\, d\theta, \quad \text{for } m = 0$$

$$\sum_{n=1}^{\infty} a_{nm} J_m\big(\sqrt{\lambda_{nm}}\, r\big) = \frac{1}{\pi} \int_{-\pi}^{\pi} f(r,\theta) \cos m\theta\, d\theta, \quad \text{for } m \geq 1$$

$$\sum_{n=1}^{\infty} c_{nm} J_m\big(\sqrt{\lambda_{nm}}\, r\big) = \frac{1}{\pi} \int_{-\pi}^{\pi} f(r,\theta) \sin m\theta\, d\theta, \quad \text{for } m \geq 1.$$

Note: These Fourier series coefficients are actually Fourier-Bessel series, so that

$$a_{n0} = \frac{\dfrac{1}{2\pi} \displaystyle\int_{-\pi}^{\pi} \int_0^a f(r,\theta) J_0\big(\sqrt{\lambda_{n0}}\, r\big)\, r\, dr\, d\theta}{\displaystyle\int_0^a J_0\big(\sqrt{\lambda_{n0}}\, r\big)^2 r\, dr}, \qquad \text{for } m = 0,\ n \geq 1$$

$$a_{nm} = \frac{\dfrac{1}{\pi} \displaystyle\int_{-\pi}^{\pi} \int_0^a f(r,\theta) J_m\big(\sqrt{\lambda_{nm}}\, r\big) \cos m\theta\, r\, dr\, d\theta}{\displaystyle\int_0^a J_m\big(\sqrt{\lambda_{nm}}\, r\big)^2 r\, dr}, \quad \text{for } m \geq 1,\ n \geq 1$$

$$c_{nm} = \frac{\dfrac{1}{\pi} \displaystyle\int_{-\pi}^{\pi} \int_0^a f(r,\theta) J_m\big(\sqrt{\lambda_{nm}}\, r\big) \sin m\theta\, r\, dr\, d\theta}{\displaystyle\int_0^a J_m\big(\sqrt{\lambda_{nm}}\, r\big)^2 r\, dr}, \quad \text{for } m \geq 1,\ n \geq 1$$