

## Properties of Trees

Given a finite simple graph  $G = (V, E)$ , let  $n = |V|$  be the number of vertices in  $G$ , and let  $m = |E|$  be the number of edges in  $G$ . Given a vertex  $v \in V$ , the *degree* of  $v$ , denoted by  $\deg(v)$  is the number of vertices in  $V$  that are adjacent to  $v$ .

Recall that  $G$  is a *tree* if and only if given any two vertices  $u$  and  $v$  in  $V$ , there is a unique path joining  $u$  and  $v$ .

A *spanning tree* for a finite simple graph  $G = (V, E)$  is a subgraph  $H = (V_H, E_H)$  of  $G$  which is a tree and such that  $V_H = V$ .

The following properties are true:

**Theorem.** If  $G = (V, E)$  is a finite simple graph, then

- (a)  $\sum_{v \in V} \deg(v) = 2m$
- (b)  $G$  is a tree if and only if  $G$  is connected and has no cycles.
- (c)  $G$  is connected if and only if it has a spanning tree.
- (d)  $G$  is a tree if and only if  $G$  is connected and  $n = m + 1$ .
- (e)  $G$  is a tree if and only if  $G$  has no cycles and  $n = m + 1$ .
- (f) If  $G$  is a tree with more than one vertex, then there are at least two vertices of degree 1.

**Proof.**

- (a) As we consider each edge  $\{a, b\} \in E$  in the graph  $G$ , the edge contributes a count of 1 to each of  $\deg(a)$  and  $\deg(b)$ , and therefore, a count of 2 to the sum

$$\sum_{v \in V} \deg(v).$$

Thus,  $2|E|$  accounts for  $\deg(v)$ , for all  $v \in V$ , and

$$\sum_{v \in V} \deg(v) = 2|E|.$$

- (b) If  $G$  is a tree and  $u$  and  $v$  are any two distinct vertices in  $G$ , there is a path joining  $u$  and  $v$ , hence  $G$  is connected.

If  $G$  contained a cycle, and  $u$  and  $v$  were two distinct vertices in the cycle, then there would be two paths joining  $u$  and  $v$ , which is a contradiction. Hence,  $G$  contains no cycles.

Conversely, suppose that  $G$  is connected and contains no cycles. Since  $G$  is connected, then given distinct vertices  $u$  and  $v$  in  $G$ , there is a path joining  $u$  and  $v$ . Also, since  $G$  contains no cycles, this path is unique. Hence,  $G$  is a tree.

- (c) If  $G$  has a spanning tree  $T$ , then  $T$  is a connected subgraph of  $G$  which has no cycles and which contains all the vertices of  $G$ . Therefore,  $G$  must be connected.

Conversely, if  $G$  is connected and  $G$  is not a tree, then  $G$  must contain a cycle. If we remove an edge from this cycle, then the resulting subgraph  $G_1$  is still connected and has the same number of vertices as  $G$ . If  $G_1$  is not a tree, then it must contain a cycle. If we remove an edge from this cycle, then the resulting subgraph  $G_2$  is still connected and has the same number of vertices as  $G$ . If  $G_2$  is not a tree, then it must contain a cycle. If we remove an edge from this cycle, then the resulting subgraph  $G_3$  is still connected and has the same number of vertices as  $G$ . Since there are only finitely many edges in the graph, this process must eventually stop, say with a subgraph  $G_k$  of  $G$  which is still connected and has the same number of vertices as  $G$ , but this last subgraph  $G_k$ , has *no* cycles, and is therefore a spanning tree for  $G$ .

- (d) If  $G$  is a tree, then  $G$  is connected. We will show that  $n = m + 1$  by induction on  $n$ . The result is true when  $n = 1$ , since the only possible tree has no edges in that case, and  $m = 0$ . Now suppose the result is true for all trees with  $k$  vertices,  $1 \leq k \leq n$ , and let  $G = (V, E)$  be a tree with  $|V| = n + 1$ , and let  $\{a, b\}$  be any edge of  $G$ . If  $G_1 = (V_1, E_1)$  and  $G_2 = (V_2, E_2)$  are the trees obtained by removing the edge  $\{a, b\}$  from  $G$ , we have

$$|V_1| + |V_2| = |V| \quad \text{and} \quad |E_1| + |E_2| = |E| - 1.$$

Applying the induction hypothesis to  $G_1$  and  $G_2$  we have

$$|E| = |E_1| + |E_2| + 1 = |V_1| - 1 + |V_2| - 1 + 1 = |V| - 1 = n + 1 - 1 = n,$$

as required. Therefore the result is true for all positive integers  $n$ .

Conversely, suppose that  $G$  is connected and  $|V| = |E| + 1$ . Suppose that  $G$  contains a cycle  $C_r$  with  $r$  vertices and  $r$  edges. Since  $G$  is connected, the remaining vertices of  $G$  can each be connected to a vertex in  $C_r$  by a path in  $G$ . Each such connection requires at least one new edge. Therefore, in  $G$ , we must have  $|E| \geq |V|$ , which contradicts the assumption that  $|V| = |E| + 1$ . Therefore,  $G$  has no cycles, and is connected, thus,  $G$  is a tree.

- (e) If  $G$  is a tree, then  $G$  has no cycles, and from part (d)  $G$  is connected and  $|V| = |E| + 1$ .

Conversely, suppose that  $G$  has no cycles and  $|V| = |E| + 1$ , suppose also that  $G$  is disconnected with components  $G_1, G_2, \dots, G_r$ . For each  $i$ , with  $1 \leq i \leq r$ , select a vertex  $v_i \in G_i$  and add the  $r - 1$  edges

$$\{v_1, v_2\}, \{v_2, v_3\}, \dots, \{v_{r-1}, v_r\}$$

to  $G$  to obtain the graph  $G' = (V, E')$ . Then  $G'$  is a tree, and therefore from part (d) we have  $|V| = |E'| + 1$ . Thus,  $|E'| + 1 = |V| = |E| + 1$ , that is,  $|E'| = |E|$ , and so  $r = 1$ . Therefore,  $G$  is connected and  $|V| = |E| + 1$ , so by part (d),  $G$  is a tree.

- (f) Let  $G = (V, E)$  be a tree with  $|V| \geq 2$ , from part (d), we have  $|E| = |V| - 1$ , while from part (a), we have

$$2(|V| - 1) = 2|V| - 2 = \sum_{v \in V} \deg(v).$$

If  $\deg(v) \geq 2$  for all  $v \in V$ , then

$$\sum_{v \in V} \deg(v) \geq 2|V|,$$

that is,  $2|V| - 2 \geq 2|V|$ , and this gives a contradiction unless there are two vertices  $v$  for which  $\deg(v) < 2$ . Since  $G$  is connected, there are no isolated vertices, that is, with degree 0, and therefore there must be two vertices with degree 1.

□