MATH 222

Final Exam

Sample

Time: 3 hours

No Calculators

Instructions:
Show all your work and place your answer in the box when provided.

Last Name: KEY
First Name:
ID:

1. Dr. Ecco concluded that the aliens from whom a message was received also had an "alphabet" consisting of 26 distinct characters. Another moment's thought convinced him that the code was a linear code with encoding function $E(X) \equiv 5x \pmod{26}$. Your mission, should you decide to accept it, is to help Dr. Ecco find the decoding function $D(x)$, that is, find an integer a such that:

$$D(x) \equiv ax \pmod{26}$$

and decode the following message from the aliens:

IH FUAN, KADD JSIU.

For your convenience, the correspondence between the letters of the alphabet and the integers 0 to 25 is given below.

Note $5 \cdot 21 = 105 \equiv 1 \pmod{26}$ $\therefore$ $5^{-1} \equiv 21 \equiv -5 \pmod{26}$

$\therefore$ $D(x) \equiv -5x \pmod{26}$

$D(I) \equiv D(8) \equiv -40 \equiv 12$
$D(H) \equiv D(7) \equiv -35 \equiv 17$
$D(F) \equiv D(5) \equiv -25 \equiv 1$
$D(U) \equiv D(-6) \equiv 30 \equiv 4$
$D(A) \equiv D(0) \equiv 0$
$D(N) \equiv D(13) \equiv -65 \equiv 13$
$D(K) \equiv D(10) \equiv -50 \equiv 2$
$D(D) \equiv D(3) \equiv -15 \equiv 11$
$D(J) \equiv D(9) \equiv -45 \equiv 7$
$D(S) \equiv D(-8) \equiv 40 \equiv 14$ $\pmod{26}$

A	B	C	D	E	F	G	H	I	J	K	L	M	N	O	P	Q	R	S	T	U	V	W	X	Y	Z
0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25

The plain text is: MR. BEAN, CALL HOME.

10

2. Suppose a general wants to send a message to his troops behind enemy lines. He has several couriers that he can use to carry the message; however, up to three of them may be caught by the enemy. To ensure the message gets across, the general makes 4 copies of the message. To ensure the enemy doesn't intercept the entire message, the general cuts each of the four messages into five pieces and sends each courier with some combination of the different pieces. What is the minimum number of couriers the general needs to send so the message gets across and the enemy cannot intercept the entire message?

Messages : A B C D E
A B C D E
A B C D E
A B C D E

Let the first courier have 2 pieces :

1) AB

Then C, D, E must be sent on separate couriers:

1) A B
2) C A
3) C B
4) C B
5) C B
6) D
7) D
8) D
9) D
10) E
11) E
12) E
13) E

14 couriers works:

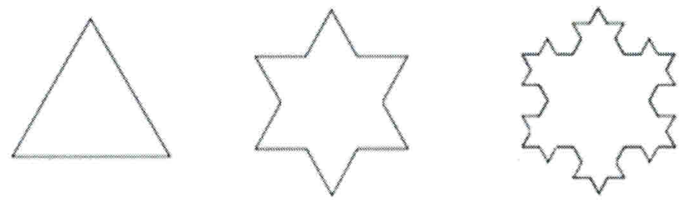
1) A B
2) A B
3) C A
4) C A
5) C B
6) C B
7) D
8) D
9) D
10) D
11) E
12) E
13) E
14) E

Let the 2nd courier carry CA, then B must go with C (since B, D, E must be sent on separate couriers). Now none of the 13 couriers can take another A.

∴ we can not solve this problem with only 13 couriers.

The minimum number of couriers needed is: $\boxed{14}$

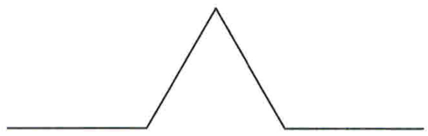
3. The limit of the following sequence of shapes is known as the Koch snowflake:



The sequence of shapes is defined as follows:

- K_1 is an equilateral triangle with a perimeter of 3
- for $n > 1$, K_n is formed by replacing each line segment

of K_{n-1} with the shape



according to the following three steps:

1) The line segment was divided into three segments of equal length.
2) An equilateral triangle was drawn pointing outward that has its middle segment from step 1 as its base.
3) The line segment that is the base of the triangle from step 2 was removed.

Let P_n be the perimeter of K_n. Write down a recurrence relation for P_n and solve the recurrence relation.

$P_1 = 3$

$P_n = \left(\frac{4}{3}\right) P_{n-1}$ is the recurrence relation.

$P_1 = 3$

$P_2 = 3\left(\frac{4}{3}\right)$

$P_3 = 3\left(\frac{4}{3}\right)^2$

$\vdots$

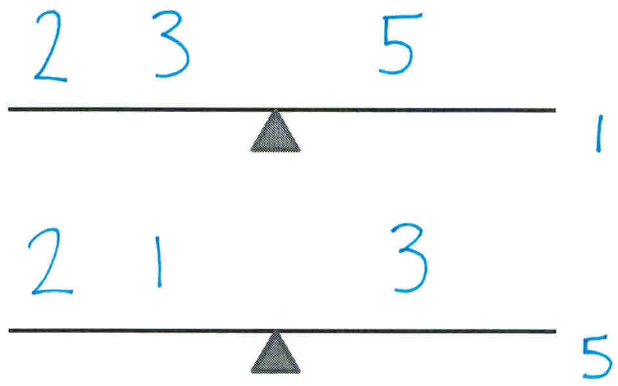
$P_n = 3\left(\frac{4}{3}\right)^{n-1}$

The solved form is: $P_n = 3\left(\frac{4}{3}\right)^{n-1}$

4. A certain country uses gold coins called buckazoids. The coins weigh the same as their denominations, that is, a 1-buckazoid coin weighs 1 gram, a 2-buckazoid coin weighs 2 grams, and a 3-buckazoid coin weighs 3 grams, etc.

   Someone has given you one 1-buckazoid coin, one 2-buckazoid coin, one 3-buckazoid coin, and one 5-buckazoid coin. One of the four coins is fake, and you don't know whether it is too heavy or too light. The other three coins are real and are the correct weight. Find the fake coin by using only two weighings on a pan balance. A non-adaptive solution is required.

Weigh the four coins in the following pattern:



LL, RR => 2

LR, RL => 3

BL, BR => 1

RB, LB => 5

5. Find the integer equal to:

$$S = \sum_{i=-99}^{299} i$$

Recall the formula:

$$m + (m+1) + \cdots + n = \frac{(n+m)(n-m+1)}{2}$$

$$\therefore S = \frac{(299-99)(299+99+1)}{2}$$

$$= \frac{200(399)}{2}$$

$$= 100(399)$$

$$= 39900$$

The integer is: $\boxed{\sum_{i=-99}^{299} i = 39900}$

6. Recall the Fibonacci numbers defined by:

$$F_1 = 1,$$
$$F_2 = 1,$$
$$F_n = F_{n-1} + F_{n-2}.$$

Let x be a real number such that $x^2 = 1 - x$, and show that:

$$x^{2n} = F_{2n-1} - xF_{2n}$$

for all $n \geq 1$.

BC $(n=1)$

$$x^2 = x^{2(1)} = F_{2(1)-1} - x F_{2(1)} = F_1 - xF_2 = 1 - x$$

IS

Show $x^{2n} \overset{*}{=} F_{2n-1} - x F_{2n}$

$\Rightarrow x^{2n+2} = F_{2n+1} - x F_{2n+2}$.

$$x^{2n+2} = x^{2n} x^2$$
$$\overset{*}{=} (F_{2n-1} - x F_{2n})(1-x)$$
$$= F_{2n-1} - x F_{2n-1} - x F_{2n} + x^2 F_{2n}$$
$$= F_{2n-1} - x F_{2n+1} + (1-x) F_{2n}$$
$$= F_{2n-1} + F_{2n} - x F_{2n+1} - x F_{2n}$$
$$= F_{2n+1} - x F_{2n+2}$$

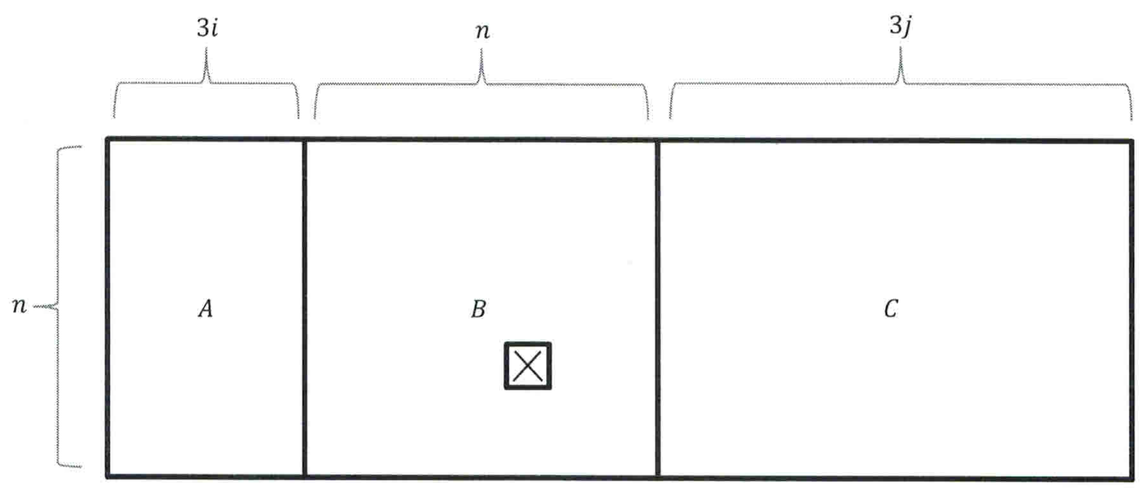
Therefore the above equality is true.

7. Proposition: any deficient $n \times m$ board can be tiled with right trominoes where:

- $n \equiv 2,4 \pmod{6}$
- $n \equiv m \pmod{3}$
- $m > n \geq 4$

Complete the proof of the Proposition.

Proof. Let i and j be non-negitive integers so that $m = 3i + n + 3j$ and so that the board can be sectioned as follows:



• $n \equiv 2, 4 \pmod{6} \Rightarrow n$ is even

$\Rightarrow n \equiv 0 \pmod{2}$

Also $3j \equiv 3i \equiv 0 \pmod{3}$ $\Rightarrow$ Section A, C can be tiled

• $n \equiv 2, 4 \pmod{6}$

$\Rightarrow n = 2 + n_1 6$ OR $4 + n_2 6$

$\Rightarrow n \equiv 2$ OR $4 \pmod{3}$

$\Rightarrow n \not\equiv 0 \pmod{3}$

Also $n \neq 5$

$\therefore$ Section B can be tiled.

8. Baskerhound herded Evangeline and her friends into a room containing eleven caskets. Baskerhound stated that sixty minutes after he leaves the room, ten of the eleven caskets will disintegrate and release a poisonous gas. The other casket contains the key to the door. To escape unharmed, they must find the key before the hour is up.

He gave them clues: The key is in casket number $2^n + 2$ where $n > 1000$, in casket number $7^m + 7$ where $m > 1000$, and in casket number l where the last (rightmost) digit of the number l is not a zero. The three numbers $2^n + 2$, $7^m + 7$, and l might be different numbers from each other, but there is still only one key because the caskets are counted in a funny way:

Casket 1 was 1, Casket 2 was 2, and so on, until Casket 11 which was 11. Then the kidnapper started counting backwards: Casket 10 was 12, Casket 9 was 13, and so on, until Casket 1 which was 21. Then the count reversed once more, and Casket 2 was 22, Casket 3 was 23, etc.

Find the casket which contains the key.

The caskets are counted in a loop of 20:

Casket:	1	2	3	4	5	6	7	8	9	10	11	10	9	8	7	6	5	4	3	2
mod 20:	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	0

$$2^1 \equiv 2$$
$$2^2 \equiv 4 \equiv 2^6 \equiv \cdots$$
$$2^3 \equiv 8 \equiv 2^7 \equiv \cdots$$
$$2^4 \equiv 16 \equiv 2^8 \equiv \cdots$$
$$2^5 \equiv 12 \equiv 2^9 \equiv \cdots$$

$$7^1 \equiv 7 \equiv 7^5 \equiv \cdots$$
$$7^2 \equiv 9 \equiv 7^6 \equiv \cdots$$
$$7^3 \equiv 3 \equiv 7^7 \equiv \cdots$$
$$7^4 \equiv 1 \equiv 7^8 \equiv \cdots \pmod{20}$$

$$\therefore \quad KEY \equiv 2^n + 2 \equiv 7^m + 7 \equiv l \pmod{20} \quad \text{for } n, m > 1000$$

$$\Rightarrow KEY \equiv \begin{cases} 4+2 \\ 8+2 \\ 16+2 \\ 12+2 \end{cases} \equiv \begin{cases} 7+7 \\ 9+7 \\ 3+7 \\ 1+7 \end{cases} \not\equiv \begin{cases} 10 \\ 0 \end{cases} \pmod{20}$$

$$\Rightarrow KEY \equiv \begin{cases} 6 \\ 10 \\ 18 \\ 14 \end{cases} \equiv \begin{cases} 14 \\ 16 \\ 10 \\ 8 \end{cases} \not\equiv \begin{cases} 10 \\ 0 \end{cases} \pmod{20}$$

$$\Rightarrow KEY \equiv 14 \pmod{20}$$

The casket that contains the key is: (*circle one of the eleven numbers*)

1 2 3 4 5 6 7 8 9 10 11

10

9. Let G be a graph with no self-loops (which are edges beginning and ending at the same vertex), or repeated edges (which means only one edge is between every pair of vertices). If G is connected and the average degree of G is equal to 2, show that G contains a cycle.

The average degree of G is 2:

$$\frac{1}{V} \sum_{v \in G} \deg(v) = 2$$

H.L.

$$\Longrightarrow \quad \frac{2E}{V} = 2$$

$$\Longrightarrow \quad E = V$$

If there are no cycles then:

$$\left.\begin{array}{l} \bullet \text{ no cycles} \\ \bullet \text{ connected} \end{array}\right\} \Longrightarrow G \text{ is a tree}$$

$$\therefore \quad E + 1 = V$$

$$\lightning$$

$\therefore$ there is at least one cycle

10. Professor Scarlett has attempted an inductive proof of the following statement:

P_n: n can be written as a sum of distinct powers of two.

for integers $n \geq 1$. Unfortunately Professor Scarlett forgot to include the base case(s). Help Professor Scarlett complete the proof by completing the base case(s). You must use the smallest number of base cases necessary to complete the proof.

Base Case(s):

$(n=1)$ $\quad 1 = 2^0 \quad \therefore P_1$ is true

Inductive Step:

Show: $P_n \Rightarrow (P_{2n} \,\&\, P_{2n+1})$ for $n \geq 1$.

Since P_n is true, we can write:

$$2n = 2\underbrace{(2^i + \cdots + 2^j)}_{\text{distinct powers of 2}} = \underbrace{2^{i+1} + \cdots + 2^{j+1}}_{\text{distinct powers of 2}}$$

Adding 1 to both sides gives:

$$2n + 1 = \overbrace{2^{i+1} + \cdots + 2^{j+1}}^{\text{distinct powers of 2}} + 1 = \overbrace{2^{i+1} + \cdots + 2^{j+1} + 2^0}^{\text{distinct powers of 2}}$$

Therefore P_{2n} & P_{2n+1} are true.

The **minimum** number of base cases needed to complete the proof is $\boxed{1}$.