MATH 222

Midterm

March 8^{th}, 2013

Time:
50 Minutes

Instructions:
Place each of your answers in the boxes provided

Last Name: KEY
First Name:
ID:

1. Baskerhound had herded Evangeline and her friends into a room containing five caskets. Baskerhound stated that sixty minutes after he leaves the room, four of the five caskets will disintegrate and release a poison gas. The other casket contains the key to the door. They must find the key before the hour is up to escape.

   He gave them a clue: The key is in casket number $3^n + 1$ where $n > 1000$, and it is also in casket number $5^m + 1$ where $m > 1000$. But there is only **one** key.

   You count the caskets as follows: Casket 1 was 1, Casket 2 was 2, and so on, until Casket 5 which was 5. Then the kidnapper started counting backwards: Casket 4 was 6, Casket 3 was 7, Casket 2 was 8, and Casket 1 was 9. Then the count reversed once more, and Casket 2 was 10, Casket 3 was 11, etc.

   Find the casket which contains the key.

   SOLUTION: Evangeline started counting the caskets as shown in the figure below:

1	2	3	4	5
9	8	7	6	
	10	11	12	13
17	16	15	14	
	18	19	20	21
25	24	23	22	
	26	27	28	29
33	32	31	30	
	34	35	36	37
⋮	⋮	⋮	⋮	⋮

   She quickly noticed that

   (1) The numbers labeling casket #1 were all congruent to 1 modulo 8.
   (2) The numbers labeling casket #2 were all congruent to 0 or 2 modulo 8.
   (3) The numbers labeling casket #3 were all congruent to 3 or 7 modulo 8.
   (4) The numbers labeling casket #4 were all congruent to 4 or 6 modulo 8.
   (5) The numbers labeling casket #5 were all congruent to 5 modulo 8.

   Baskerhound had stated that the key was in a casket labeled $3^n + 1$ with $n > 1,000$, and also in a casket labeled $5^m + 1$ with $m > 1,000$, so Evangeline began computing: modulo 8

   (a) For n even, say $n = 2k$, we have $3^{2k} + 1 \equiv 9^k + 1 \equiv 2 \pmod 8$.
   (b) For n odd, say $n = 2k + 1$, we have $3^{2k+1} + 1 \equiv 3 \cdot 9^k + 1 \equiv 4 \pmod 8$.
   (c) For m even, say $m = 2k$, we have $5^{2k} + 1 \equiv 25^k + 1 \equiv 2 \pmod 8$.
   (d) For m odd, say $m = 2k+1$, we have $5^{2k+1} + 1 \equiv 5 \cdot 25^k + 1 \equiv 6 \pmod 8$.

   She immediately deduced that the key was in casket #2.

   Note: If Evangeline had counted modulo 4 instead of 8, then

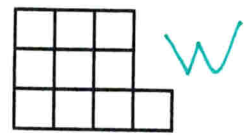
   (1) The numbers labeling casket #1 were all congruent to 1 modulo 4.
   (2) The numbers labeling casket #2 were all congruent to 0 or 2 modulo 4.
   (3) The numbers labeling casket #3 were all congruent to 3 modulo 4.
   (4) The numbers labeling casket #4 were all congruent to 0 or 2 modulo 4.
   (5) The numbers labeling casket #5 were all congruent to 1 modulo 4.

   And now, modulo 4, we have

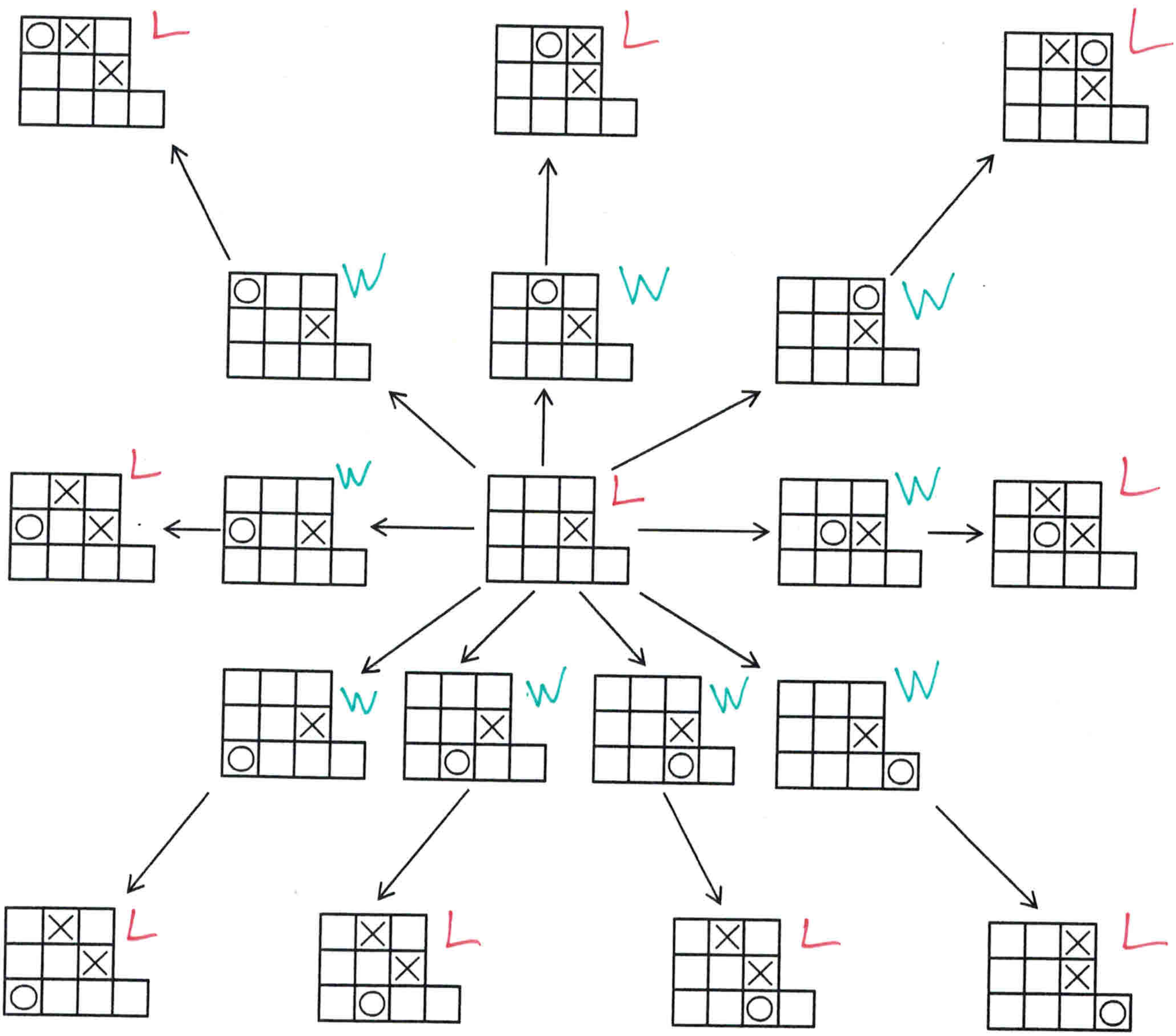
$$3^n + 1 \equiv (-1)^n + 1 \equiv \begin{cases} 0 & \text{for } n \text{ odd} \\ 2 & \text{for } n \text{ even,} \end{cases}$$
$$5^m + 1 \equiv 1^m + 1 \equiv 2 \quad \text{for all m,}$$

   so that n and m must be even, and the label on the casket must be congruent to 2 modulo 4. In this case, all we can say is that the key is in casket #2 or casket #4.

2. The following game is called Modified Tick-Tack-Toe.



How to play: Player X and Player O alternately put their symbol in an empty square on their turn. Player X goes first. Whoever makes a line with 3 X's or 3 O's wins the game. The winning three symbols may be horizontal, vertical, or diagonal. There is one exception: A player can win on the bottom only by taking all four cells. The following portion of the state diagram may help identify the player that can always win. Write a "**W**" next to each winning possition below and an "**L**" next to each lossing position.



The player that can always win is: (*circle one of the two*)	
Player X	Player O

Since the game starts in a winning possition.

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3. A counselor and her campers are at a junction in a hiking trail and they know their campsite is 20 minutes down one of four paths. It will be dark in one hour so the counselor wants to send campers down the paths to see which one leads to camp (the counselor can check a path too). They will then rendezvous in 40 minutes and choose which path to follow.

What is the smallest number of campers the counselor will need if 10 of them sometimes lie? If the counselor checks one path there are three remaining paths. How does the counselor initially divide the campers amongst the three remaining paths?

The smallest number of campers is given by $3n+2$ where n of them can lie.

When $n=6$

$$3n+2=20.$$

The campers are divided into 3 groups of $n+1$, $n+1$, and n.

The **smallest** number of campers the counselor will need is $\boxed{20}$.

These campers will be divided so that:

- The counselor will send: $\boxed{7}$ campers down the first path.
- The counselor will send: $\boxed{7}$ campers down the second path.
- The counselor will send: $\boxed{6}$ campers down the third path.

4. The telephone numbers in town run from 000000 to 999999: The automated switchboard may switch two digits which have exactly **two** other digits between them, at most one such switch is made and there are no other kinds of errors.

It has been decided that a seventh digit X will be added to the end of each phone number $abcdef$. There are three different proposals for the choice of X:

Code 1: $a + 2b + c + 2d + e + 2f + X \equiv 0 \pmod{10}$

Code 2: $a + 3b + c + 3d + e + 3f + X \equiv 0 \pmod{10}$

Code 3: $a + b + c + X \equiv 0 \pmod{10}$

Out of the 3 codes given, choose one that can detect the described switch and one that cannot.

Code 1 suppose $a \leftrightarrow d$.

- CASE 1: $d + 2b + c + 2a + e + 2f + X \not\equiv 0 \pmod{10}$.

  $\therefore$ report the error.

- CASE 2: $d + 2b + c + 2a + e + 2f + X \equiv 0 \pmod{10}$

$$
\begin{array}{r}
d + 2b + c + 2a + e + 2f + X \equiv 0 \pmod{10} \\
- \quad a + 2b + c + 2d + e + 2f + X \equiv 0 \pmod{10} \\
\hline
a - d \equiv 0 \pmod{10}
\end{array}
$$

$\implies a \equiv d \pmod{10}$

$\implies a = d$ since a, d are digits

$\therefore$ NO ERROR

Code 2 Both!

5000005 & 0005005

are valid codewords but the code does not detect the switch.

Code $\boxed{2}$ **cannot** detect this switching error.

Code $\boxed{1}$ **can** detect this switching error.

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Code 3 Suppose $a \leftrightarrow d$.

- CASE 1: $d + b + c + X \not\equiv 0 \pmod{10}$

  $\therefore$ report an error

- CASE 2: $d + b + c + X \equiv 0 \pmod{10}$

$$
\begin{aligned}
&\; d + b + c + X \equiv 0 \pmod{10} \\
&-\; a + b + c + X \equiv 0 \pmod{10} \\
\hline
&d - a \equiv 0 \pmod{10} \\
\Rightarrow\;& d \equiv a \pmod{10} \\
\Rightarrow\;& d = a \text{ since } d, a \text{ are digits.}
\end{aligned}
$$

$\therefore$ there was no error

Code $\boxed{}$ **cannot** detect this switching error.

Code $\boxed{3}$ **can** detect this switching error.

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5. The "two-out-of-five" code consists of all possible binary words (words made from the symbols 0 and 1) of length 5 containing exactly two 1's.

X: 1 1 0 0 0
y: 1 0 1 0 0
1 0 0 1 0
1 0 0 0 1
0 1 1 0 0
0 1 0 1 0
0 1 0 0 1
0 0 1 1 0
0 0 1 0 1
0 0 0 1 1

a) What is the minimum Hamming distance between codewords?
b) How many corrupted digits of a codeword can be detected by this code?
c) How many corrupted digits of a codeword can be corrected by this code?

a) $H(x,y) = 2 \quad \therefore \quad \min H(x,y) \leq 2.$

Two codewords can not have minimum Hamming distance of 1 since each codeword has exactly two 1's and three 0's.

$\therefore \quad \min H(x,y) = 2$

b) $n+1 = 2 \quad \Rightarrow \quad n = 1$ $\therefore$ 1 error can be detected

c) $2n+1 = 2 \quad \Rightarrow \quad n = \frac{1}{2}$ $\therefore \lfloor \frac{1}{2} \rfloor = 0$ errors can be corrected

The **minimum** Hamming distance between codewords is $\boxed{2}$

The code can **detect** up to $\boxed{1}$ corrupted digits.

The code can **correct** up to $\boxed{0}$ corrupted digits.