Lecture 3

Warm up problem: The Couriers Problem (6.3 Ecco):

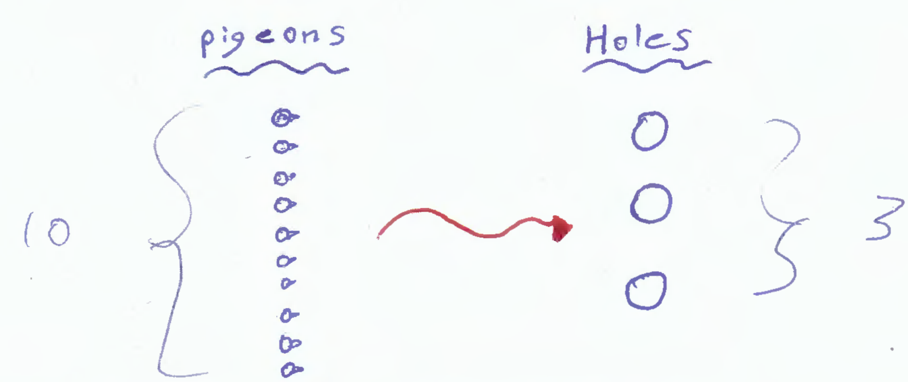
There are five parts to a top secret code: A, B, C, D, and E. The code must be sent across enemy territory amongst 8 couriers safely to our agent. The enemy will attack 2 of the couriers receiving the parts of the code they contain. We can let them intercept any four parts of the code but not all 5 parts. By making 3 copies of the code how should the different parts be split amongst the 8 couriers to ensure a safe delivery.

1) A C
2) A D
3) A D
4) B D
5) B E
6) B E
7) C E
8) C

The Pigeonhole Principle:

If p pigeons enter h pigeonholes and if p is greater than nh for some integer n, then at least one pigeonhole contains more than n pigeons.

Ex. $h = 3$, $p = 10$

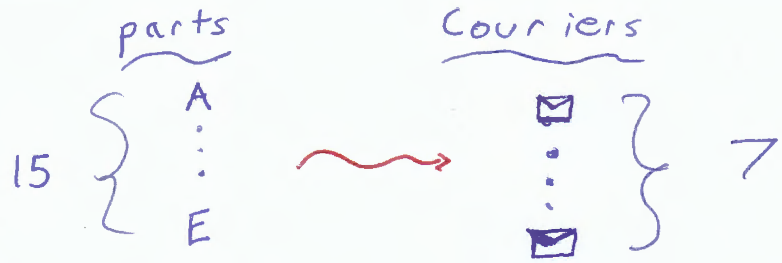


pick $n = 3$ $\Rightarrow$ $p = 10 > 9 = nh$

$\Rightarrow$ One hole has at least 4 pigeons

a) *Does a solution exist with 7 couriers?*
b) *Prove this cannot be done with 6 couriers.*
c) *Suppose we can split the code up into as many parts as we want; then what is the minimum number of couriers needed?*

a) Yes!



$\therefore$ One courier has at least 3 parts:

1) A B C
2) D A
3) D A
4) D B
5) E B
6) E C
7) E C

Now D, E must be separate, but can be paired with one of A, B, C.

b) Again by P.H.P. one courier must have at least 3 parts, but now D, E can not be separated:

1) A B C
2) D
3) D
4) D
5) E
6) E

E ?

c) See Ecco 6.3 (solution)

Example 1: If you pick five different numbers from the integers 1 to 8, show two of them must add up to nine.

Let $x_1, x_2, x_3, x_4, x_5 \in \{1, \dots, 8\}$ so $x_i \neq x_j$ for $i \neq j$.

numbers		Pairs that add to 9
x_1		$\boxed{1 \text{ OR } 8}$
x_2		$\boxed{2 \text{ OR } 7}$
x_3	$\rightsquigarrow$	$\boxed{3 \text{ OR } 6}$
x_4		$\boxed{4 \text{ OR } 5}$
x_5		

$5 \left\{ x_1, \dots, x_5 \right.$ $\qquad$ $\left. \text{4 pairs} \right\} 4$

$\therefore$ One hole has two distinct numbers which add to 9.

Example 2: Prove that at a party, if certain guests shake hands with certain other guests, there will always be a pair of people who shook the same number of hands. (It is possible that someone didn't shake hands with anyone.)

Let there be $n \geq 2$ guests.

CASE 1 someone shook no hands

$\Rightarrow$ no one " everyones hand

$\therefore$ a guest shakes either

$\underbrace{0, 1, \dots, n-2}_{n-1}$ others hands

CASE 2 no one shook no hands

$\therefore$ a guest shakes either

$\underbrace{1, 2, \dots, n-1}_{n-1}$ others hands

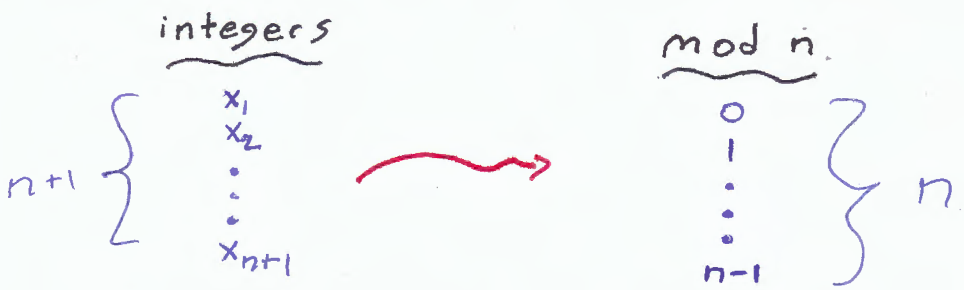
$\therefore$

guests	choices for the # of hands to shake
$n \left\{ \vdots \right.$	$\left. \square \ \vdots \ \square \right\} n-1$

$\therefore$ At least 2 guests shake the same #.

Example 3: Prove that in a collection of $n + 1$ distinct integers, there are distinct integers x and y such that $x - y$ is a multiple of n.

Let $x_1, x_2, \dots, x_{n+1}$ be distinct integers.



$\therefore$ one value in mod n is equal to at least two of the above integers.

$\therefore$ $x_i \equiv x_j \mod n$ for $i \neq j$

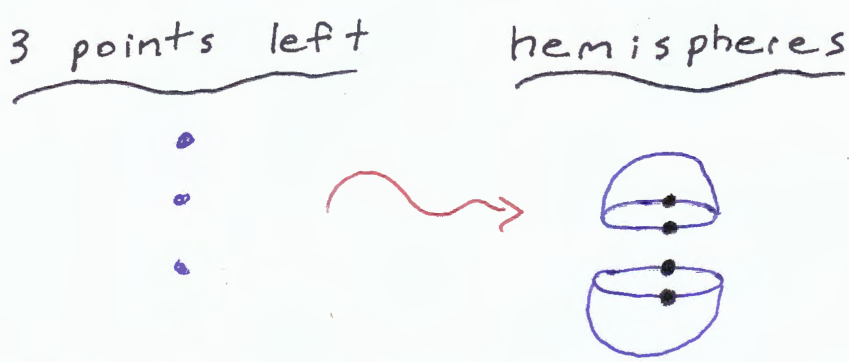
$\Rightarrow x_i - x_j \equiv 0 \mod n$

$\Rightarrow x_i - x_j = mn$ for some integer m.

Example 4: If you pick five points on the surface of an orange, then there is a way to cut the orange so that four of the points will lie on the same hemisphere (suppose a point exactly on the cut belongs to both hemispheres).

pick any two points,

these points make a circle to cut the orange.

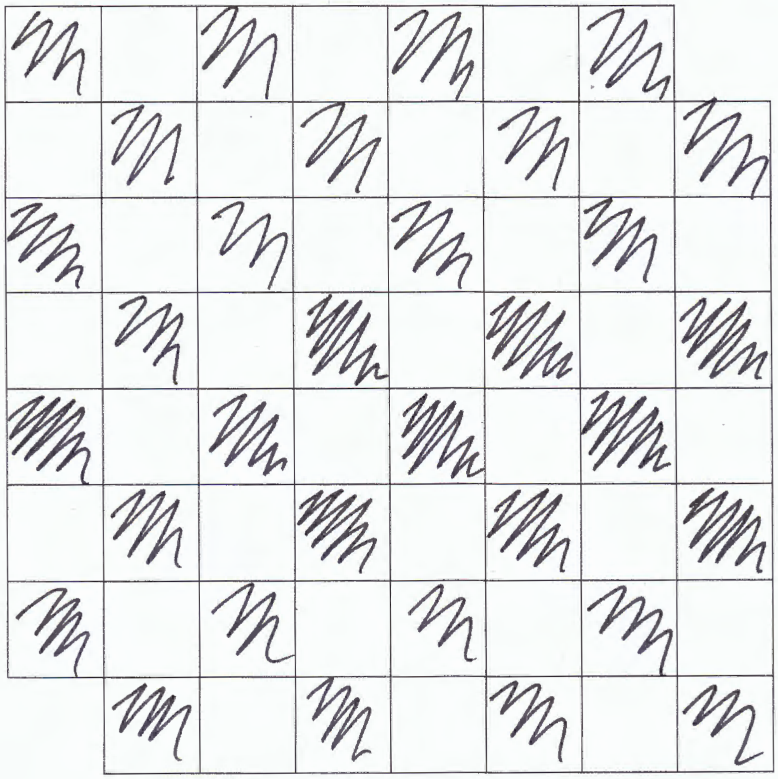


$\therefore$ One hemisphere has two more points

$\therefore$ " " " four ".

Example 5:

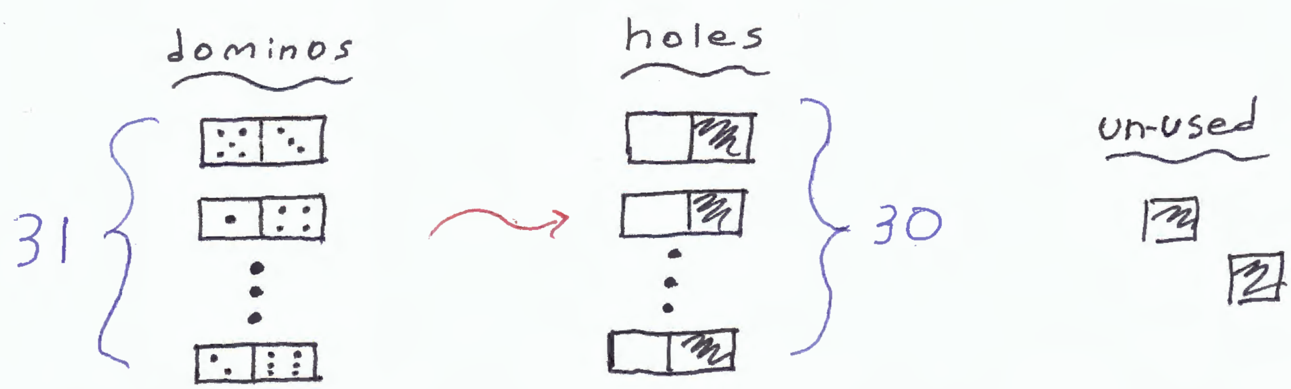
a) Given an 8 by 8 grid of squares, with two opposite corners missing, can you cover the grid with dominos? Why or why not? A domino takes up exactly two board squares.



Checker the grid:

There are 32 black squares and

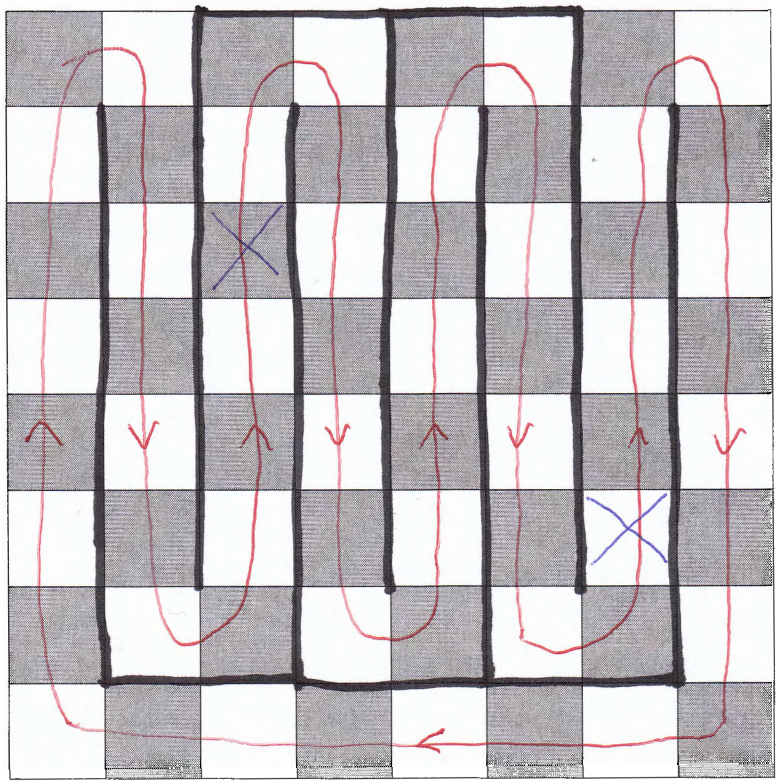
" " 30 white " .



∴ The 31 dominos needed, can cover at most 30 holes.

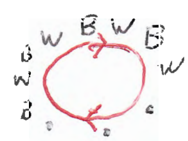
∴ No.

b) Given a checker board (8 by 8) if any two squares of opposite colors are missing can the board be covered with dominos?



STEP 1

Make a maze which cycles through the checker board



STEP 2

Remove any one black square and
" " " white " .

STEP 3

Now we have 2 pieces (one could have length 0)
which can be covered with dominos

BW... BW

WB... WB

Example 6: Trevor is trying to fatten up to fit into his tuxedo before an important day. Over a 30 day period, he pledges to eat bacon at least once per day, and 45 times in all. Show there will be a period of consecutive days where he eats bacon exactly 14 times.

Let S_i be the sum of bacon ate on day i.

$$\Rightarrow \quad 0 < S_1 < S_2 < \cdots < S_{30} = 45$$

$$\Rightarrow \quad 14 < S_1+14 < S_2+14 < \cdots < S_{30}+14 = 59$$

$$\begin{array}{ccc} \underline{S_i \,/\, S_i+14} & & \underline{\text{range}} \\[6pt] 60\left\{\begin{array}{l} \text{distinct}\left\{\begin{array}{l} S_1 \\ S_2 \\ \vdots \\ S_{30} \end{array}\right. \\ \text{distinct}\left\{\begin{array}{l} S_1+14 \\ S_2+14 \\ \vdots \\ S_{30}+14 \end{array}\right. \end{array}\right. & \rightsquigarrow & \begin{array}{c} 1 \\ 2 \\ \vdots \\ 59 \end{array} \end{array}$$

$\therefore$ One value in the range is equal to at least 2 of : $S_1, S_2, \ldots, S_{30}, S_1+14, S_2+14, \ldots, S_{30}+14$.

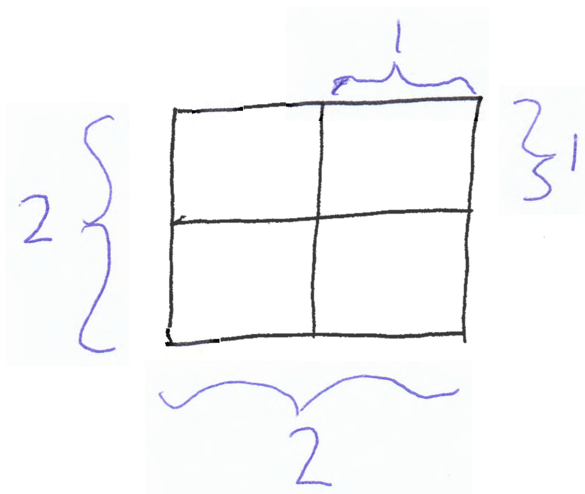
Since $S_1, S_2, \ldots, S_{30}$ are all different and $S_1+14, S_2+14, \ldots, S_{30}+14$ " " " we have:

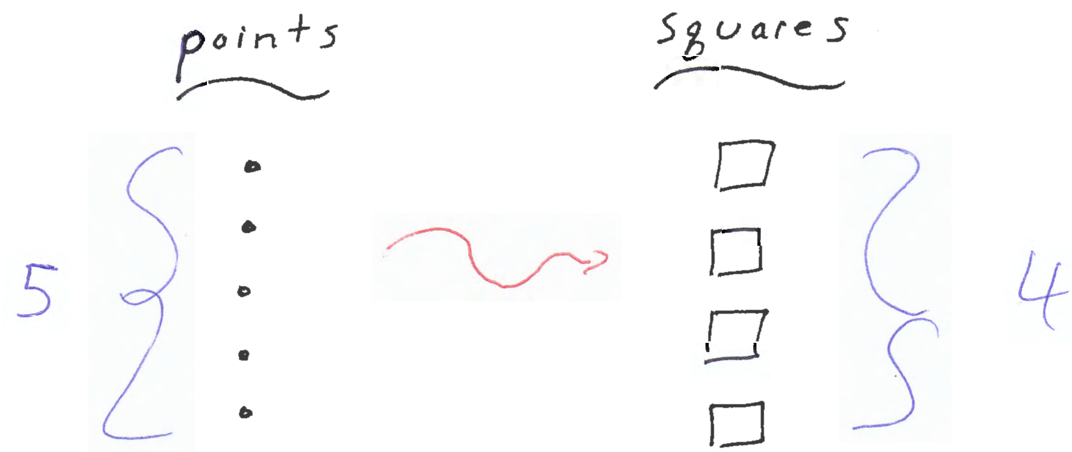
$$S_i = S_j + 14 \quad \text{for some } i, j.$$

$\therefore$ from day j to day i Trevor eats bacon exactly 14 times.

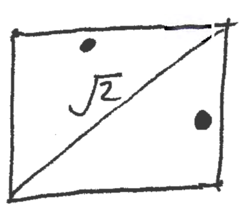
Example 7: Show that by placing 5 points anywhere within a square of side length 2, there will surely be a pair of points that are distance $\sqrt{2}$ or less from each other.

Disect the square into 4 1 by 1 squares:





$\therefore$ one square contains at least 2 points:



which are $\leq \sqrt{2}$ apart.