Lecture 2

Warm-up Problem: The Keystone Kidnapper

There are 7 caskets. The caskets were counted in the following way: casket 1 was 1, casket 2 was 2, and so on until casket 7 which was 7. Then counting backwards: casket 6 was number 8, casket 5 was number 9, and so on until casket 1 which was number 13. Then the count was reversed again, and casket 2 was 14, casket 3 was 15 and so forth; this pattern continued on and on. Which of the seven caskets will be counted as 54321?

Theorem 1: (Division of Integers)

For every pair of integers a and m with $m > 0$ there exist unique integers q and r such that:

$$a = mq + r$$
$$0 \leq r < m$$

Example 1: Let $a = -10$ and $m = 3$, then $a = -10 = -4 \cdot 3 + 2 = mq + r$.

Definition 1: The number r in this theorem is called the *remainder* of the division of a by m.

Definition 2: $a \equiv b \bmod m$ means: a divided by m has the same remainder as b divided by m. In this case we say "a" is congruent to "b" in mod "m".

Remark: $0 \equiv m \bmod m$, since 0 divided by m has the same remainder as m divided by m.

Remark: In mod m equations, adding, subtracting, and multiplying work the same as with real numbers. In mod m equations we can always reduce to a number r where $0 \leq r < |m|$ before or after adding, subtracting or multiplying.

Example 2: Find the number r where $0 \le r < 7$ in each of the following:

a) $5 + 12 \equiv r \pmod 7$

$5+12 \equiv 17 \equiv \boxed{3} \pmod 7$

b) $50 + 73 \equiv r \pmod 7$

$50+73 \equiv 1+3 \equiv \boxed{4} \pmod 7$

c) $2 - 48 \equiv r \pmod 7$

$2-48 \equiv 2-(-1) \equiv \boxed{3} \pmod 7$

d) $12 \cdot 16 \equiv r \pmod 7$

$12 \cdot 16 \equiv 5 \cdot 2 \equiv 10 \equiv \boxed{3} \pmod 7$

e) $701 \cdot 702 \cdot 703 \cdot 704 \equiv r \pmod 7$

$701 \cdot 702 \cdot 703 \cdot 704 \equiv 1 \cdot 2 \cdot 3 \cdot 4 \equiv 24 \equiv \boxed{3} \pmod 7$

f) $222^{222} \equiv r \pmod 7$

$222^{222} \equiv 5^{222}$

$\equiv 5^{222 \pmod 6}$

$\equiv 5^0$

$\equiv \boxed{1}$

$1 \equiv 5^0 \equiv 5^6 \cdots$

$5 \equiv 5^1 \equiv 5^7 \cdots$

$4 \equiv 5^2 \equiv 5^8 \cdots$

$-1 \equiv 5^3 \equiv 5^9 \cdots$

$2 \equiv 5^4 \equiv 5^{10} \cdots$

$3 \equiv 5^5 \equiv 5^{11} \cdots$

Example 3:

If now is __X__, what time will it be 2500 hours from now?

Let $d = \begin{cases} 1 & \text{Spring daylight savings} \\ 0 & \text{No daylight savings} \\ -1 & \text{Fall daylight savings} \end{cases}$

$X + 2500 + d \equiv X + d + 2400 + 100$

$\equiv X + d + 0 + 4(24) + 4$

$\equiv X + d + 0 + 4$

$\equiv \boxed{X + d + 4} \pmod{24}$

Example 4: When assigning values in mod 7 to days of the week the table below is commonly used:

Day of the Week	Sunday	Monday	Tuesday	Wednesday	Thursday	Friday	Saturday
Mod 7	0	1	2	3	4	5	6

In the current year, using the conversion from the above table, Valentine's Day lands on the X^{th} day of the week. What day will Valentine's Day be in the year 2026?

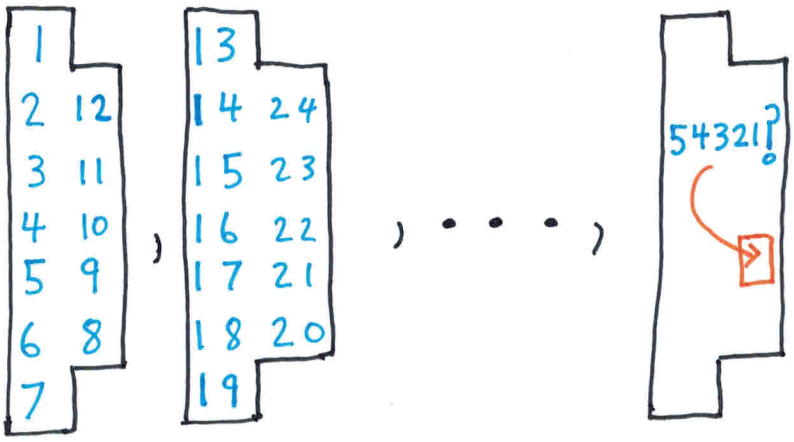
"# of years" = 2026 − "current year"

$X + (\#\text{ of years})\,365 + (\#\text{ of leap years})$

$\equiv X + (\#\text{ of years}) + (\#\text{ of leap years}) \pmod 7$

Note: $365 \equiv 1 \pmod 7$

Example 5: Find a solution to the keystone kidnapper problem.



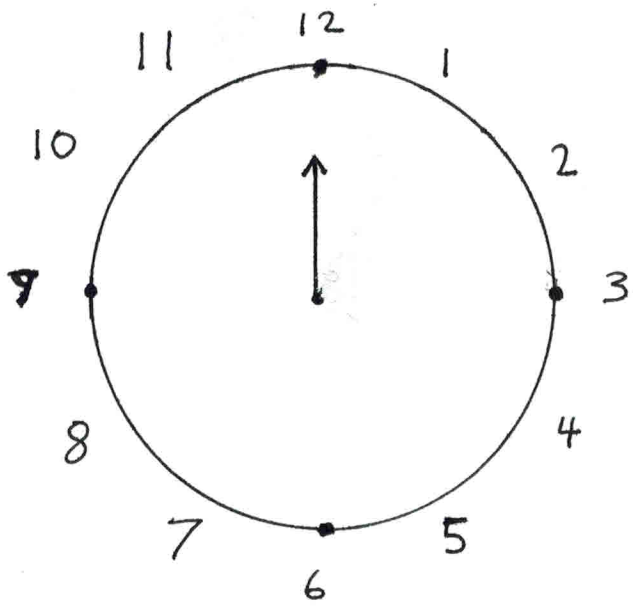
$$\begin{array}{r} 4526 \\ 12\overline{)54321} \\ -48 \\ \hline 63 \\ -60 \\ \hline 32 \\ -24 \\ \hline 81 \\ 72 \\ \hline 9 \end{array}$$

$\therefore 54321 \equiv 9 \pmod{12}$

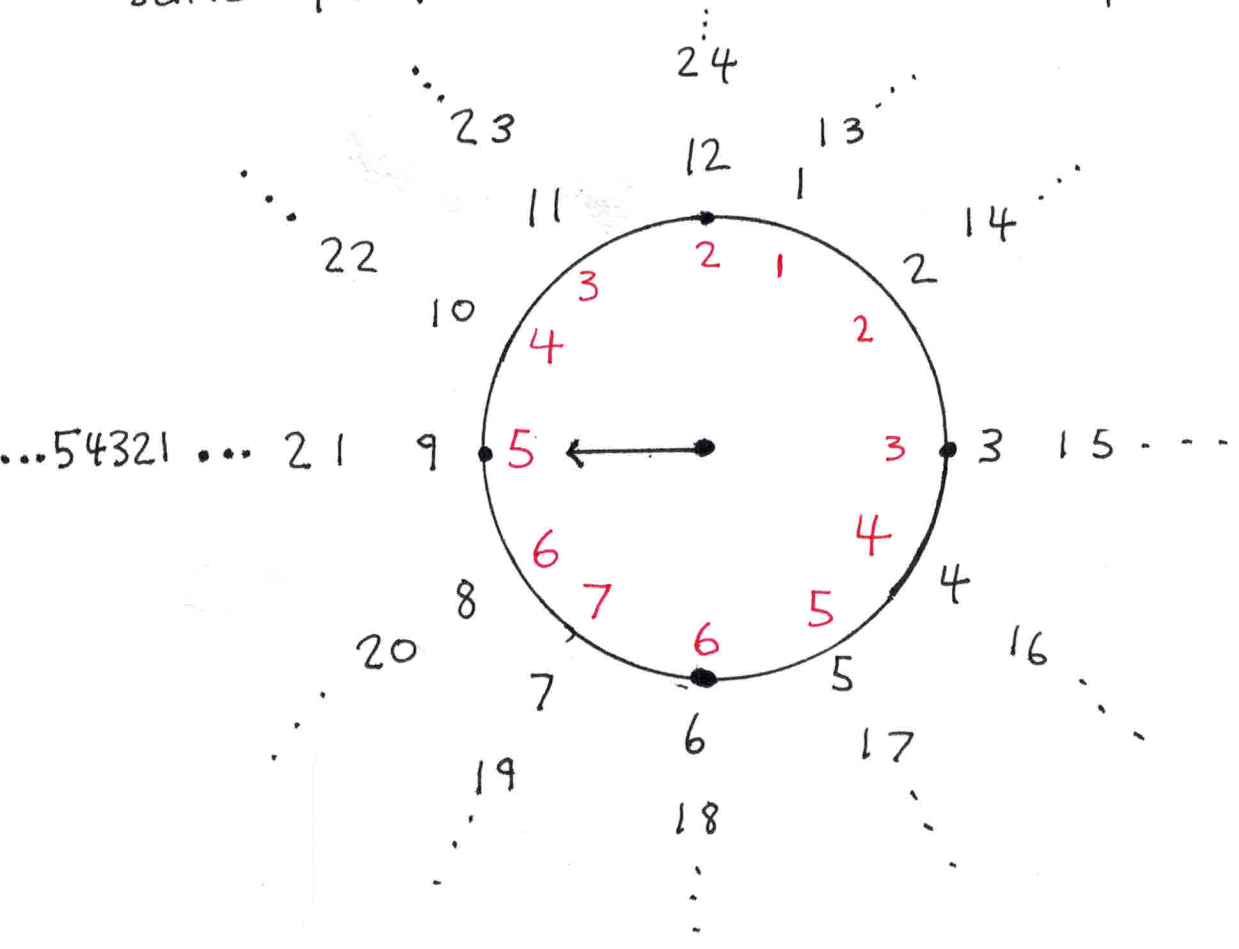
$\therefore$ Casket 5

Another explanation :

Start with a clock :



Notice the caskets (in red) are counted with the same pattern as hours of a day :



$54321 \equiv 9 \pmod{12}$

∴ after 54321 hour the clock reads 9

∴

Casket 5

Remark: Although division is defined on the non-zero integers it does not make sense to have a fraction in modular arithmetic congruencies. Instead of dividing; solve the following congruence:

$$3x \equiv 2 \pmod 7$$

by multiplying both sides by an integer that isolates x. This motivates the following definition.

Definition 3: Two integers a and b are said to be *multiplicative inverses* of each other in mod m if

$$a \cdot b \equiv 1 \bmod m.$$

Example 6: Find the multiplicative inverse for 1,2,3 and 6 in mod 7.

$1 \cdot 1 \equiv 1$, $2 \cdot 4 \equiv 1$, $3 \cdot 5 \equiv 1$, $6 \cdot 6 \equiv 1 \pmod 7$

Example 7: Solve for x: $3 \cdot (x-2) \equiv -4 \pmod 7$

$\Rightarrow 3x - 6 \equiv -4 \pmod 7$

$\Rightarrow 3x \equiv 2 \pmod 7$

$\Rightarrow (5)\,3x \equiv (5)\,2 \pmod 7$

$\Rightarrow x \equiv 3 \pmod 7$

Theorem 2: (Multiplicative Inverses in mod m)

a is relatively prime with m $\iff$ a has an inverse in mod m.

Example 8: In mod 26 the only elements with multiplicative inverses are:

1, 3, 5, 7, 9, 11, 15, 17, 19, 21, 23, 25

Definition 4: Two integers a and b are said to be *relatively prime* (or "$r.p.$" for short) if they have no common positive factor other than 1.

Theorem 3: (Cancellation Law)

If $a \cdot c \equiv b \cdot c \pmod m$ and c is relatively prime with m $\implies$ $a \equiv b \bmod m$.

Example 9: Can the common number on each of the congruence be cancelled?

a) $1 \cdot 2 \equiv 4 \cdot 2 \pmod 2$ No 2 is not r.p. with 2

b) $1 \cdot 3 \equiv 7 \cdot 3 \pmod 2$ Yes 3 is r.p. with 2

Example 10:

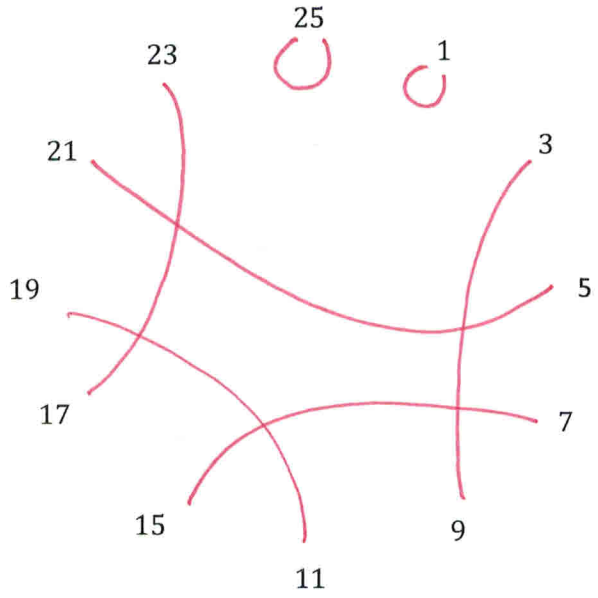
a) Show that the pair a, b in definition 3 is unique in mod m.

Let $ab \equiv 1 \pmod m$ & $ab' \equiv 1 \pmod m$

$\Rightarrow ab \equiv ab' \pmod m$

$\Rightarrow b \equiv b' \pmod m$ (by Thm. 2 & 3)

b) Draw a line between each unique pair of multiplicative inverses in mod 26.



$1 \equiv 27 \equiv 3 \cdot 9 \equiv (-3)(-9) \equiv 23 \cdot 17$

$\equiv 53$

$\equiv 79$

$\equiv 105 \equiv 3 \cdot 5 \cdot 7 \equiv 5 \cdot 21$

$\equiv 7 \cdot 15$

$\equiv (-7)(-15)$

$\equiv 19 \cdot 11$

c) Solve for x: $y \equiv 7x + 6 \pmod{26}$.

$\Rightarrow (15)y \equiv 15 \cdot 7x + 15 \cdot 6 \pmod{26}$

$\Rightarrow 15y \equiv x + 90 \equiv x + 12 \pmod{26}$

$\Rightarrow \boxed{x \equiv 15y - 12} \pmod{26}$