§ Graph Theory

Lecture 17

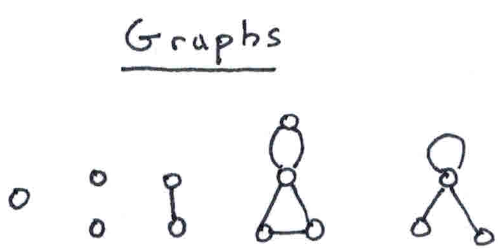
Warm up problem: **The Odd Doors Problem.** (Page 17, section 1.3 Ecco)

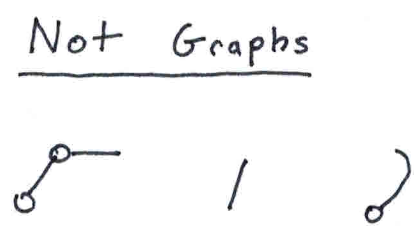
Lawrence Terrence III has a problem. His recently departed father has hidden a cache of jewels in one of two underground labyrinths. Lawrence knows the following facts about the labyrinths:

- The jewels are in a room with an odd number of doors.
- Only one of the labyrinths has a room with an odd number of doors.
- One labyrinth has two doors leading to the outside and the other has three.

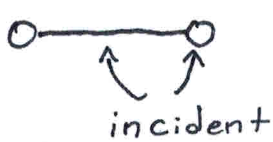
Since it will cost a small fortune to explore each labyrinth, Lawrence Terrence III wants to know which labyrinth he should search.

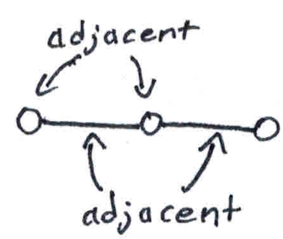
Definition 1: A *graph* is a collection of dots and lines, with every line terminated by a dot at each end. The dots are called *vertices* and the lines are called *edges*. Note we can have an edge that starts and ends at the same vertex; also, we could have multiple edges joining the same two vertices.

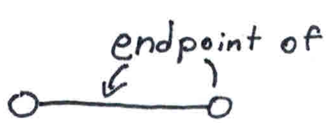




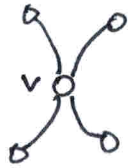
Definition 2: An edge that is joined to a vertex is said to be *incident* to the vertex, and the vertices that are joined to an edge are *incident* to the edge. More formally, the edge that connects the vertices u and v can be written as: $e = \{u, v\}$. Here we also say that u and v are *adjacent* to each other, and that they are the *endpoints* of the edge $\{u, v\}$. Two edges incident with the same vertex are also said to be *adjacent*.







Definition 3: In a graph, the *degree* of a vertex v is the number of edges incident to v, and is denoted $deg(v)$. Each edge contributes 1 to the degree of each of the two vertices incident with it.



$\Rightarrow \deg(v) = 4$

Theorem 1 (The Parity Theorem): The sum of the degrees of all vertices of a graph is equal to twice its number of edges.

Proof. Each edge contributes 2 to the total degree.

$$2E = \sum_{v \in G} \deg(v) \equiv 0 \mod 2$$

The Parity Theorem is also called the Handshaking Lemma and is stated as follows:

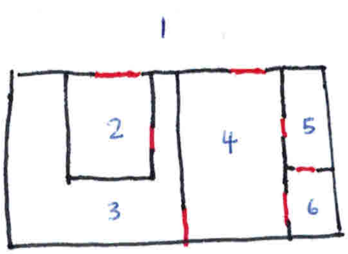
Theorem 2 (The Handshaking Lemma): Every graph has an even number of vertices of odd degree.

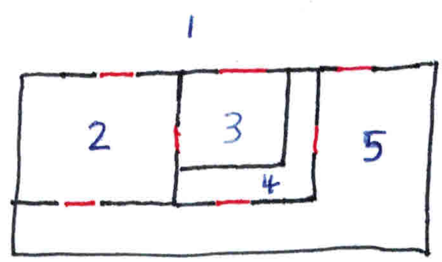
"Even" = "Even" + "Even" + "odd" + "odd" — need an even amount

Example 1: Solve the Odd Doors Problem:

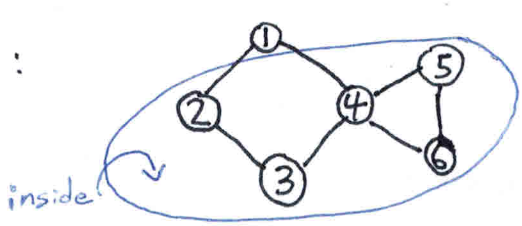
Labyrinth 1 vs Labyrinth 2

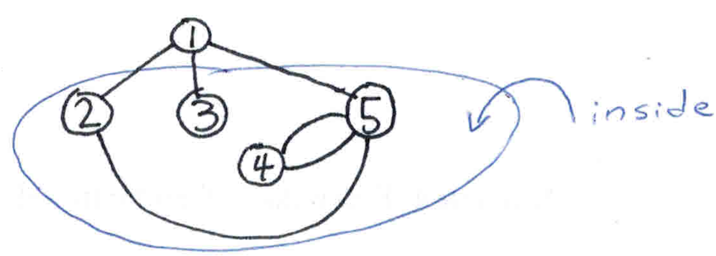
Example:





As a graph:





Result:

This graph could have all of its vertices to be of even degree.

This graph must have at least one vertex of odd degree on the inside (by T2)

$\therefore$ has a room with an odd # of doors.

Example 2: The University of Two Hills has 25 professors each with a telephone. If any professor collaborates over the phone with more than 5 others confusion spreads through the university and nothing gets done. To maximize collaboration without risking confusion they have decided to connect each phone to exactly 5 other phones. Can this be done?

Let
- vertices be phones
- edges represent two phones that are connected

As a graph there would be 25 vertices each of odd (5) degree.

$\therefore$ No! (by T2)

Example 3: In the Kingdom of Glee roads do not intersect nor do they lead to dead ends.

a) If there are 100 cities, and four roads lead out of each city, how many roads are there altogether in the kingdom?

b) If 3 roads lead out of each city can the kingdom have exactly 100 roads?

vertices – cities
edges – roads

a) $2E = \sum_{v \in G} \deg(v) = 100 \cdot 4 \Rightarrow \boxed{E = 200}$

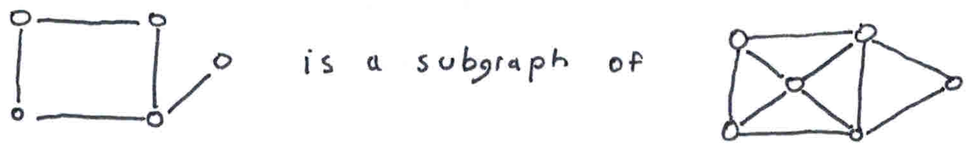
b) If such a graph exists

$\Rightarrow 2 \equiv 200 \equiv 2E \equiv \sum_{v \in G} \deg(v) \equiv 3V \equiv 0 \mod 3$

↯

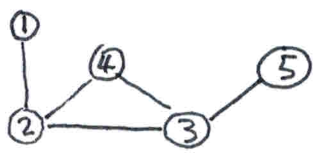
$\therefore$ No!

Definition 4: A *subgraph* of a graph is a subset of its vertices and edges, provided that all vertices incident with edges in the subgraph are included. In other words, a subgraph is a subset of the vertices and edges that itself forms a graph.



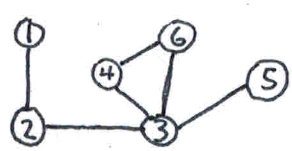
Certain types of subgraphs have specific names:

Definition 5: A *walk* is a subgraph that consists of a sequence of vertices and edges $v_0, e_1, v_1, e_2, v_2, \ldots, e_n, v_n$ such that for $1 \leq i \leq n$ the edge e_i joins vertices v_{i-1} and v_i.



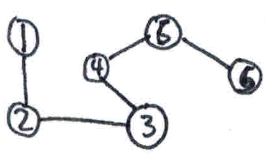
1 -> 2 -> 3 -> 4 -> 2 -> 3 -> 5

Definition 6: A *trail* is a walk in which no edges are repeated.



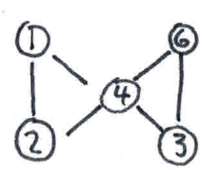
1 -> 2 -> 3 -> 4 -> 6 -> 3 -> 5

Definition 7: A *path* is a trail in which no vertices are repeated except perhaps for the first and last vertex.



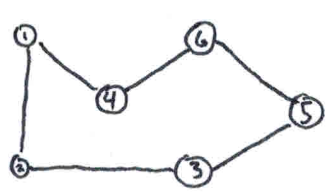
1 -> 2 -> 3 -> 4 -> 6 -> 5

Definition 8: A *circuit* is a trail that's first and last vertices are the same.



1 -> 2 -> 4 -> 3 -> 6 -> 4 -> 1

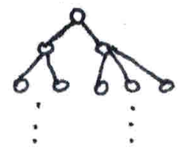
Definition 9: A *cycle* is a path that's first and last vertices are the same.



1 -> 2 -> 3 -> 5 -> 6 -> 4 -> 1

Definition 10: Two vertices of a graph that are joined by a path are said to belong to the same *component* of the graph. If the whole graph is one component, then it is said to be *connected.*

Definition 11: A *tree* is defined as a graph T such that for any two vertices u and v in T, there is exactly one path which joins u and v.



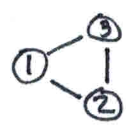
Definition 12: A collection of disjoint trees is called a *forest*.

Theorem 3: (Trees). A tree has the following three properties.
1. It is connected.
2. It has no cycles.
3. It satisfies the Tree Formula $V = E + 1$, where V and E are the numbers of vertices and edges respectively.

Proof.

1) A path joins any two vertices $\therefore$ connected

2) If a tree has a cycle then more than one path joins two vertices.



1 -> 2
1 -> 3 -> 2

$\therefore$ no cycles

3) Pick a vertex ① on a tree. Now for each edge added to the tree we must add a vertex.

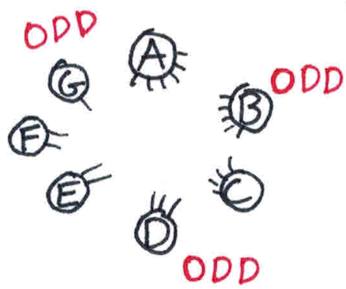
$\therefore V = E + 1$

Theorem 4: (Two out of three is a tree). A graph is a tree if it has any two of the following properties:
1. It is connected.
2. It has no cycles.
3. It satisfies the Tree Formula $V = E + 1$, where V and E are the numbers of vertices and edges respectively.

Example 4: (Ecco 2.1) The police have captured seven criminals Al, Bob, Carl, Dan, Ed, Frank, and Gary. When questioned by the police, Al admitted to having known all of the other six criminals. Bob admitted to having known five, Carl admitted to having known four, Dan to having known three, Ed to having known two, Frank to having known two, and Gary to having known one.

a) Is it possible that all seven criminals are telling the truth? Explain.

vertices — criminals
edges — criminals know each other



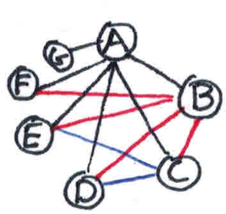
Such a graph would have 3 (an odd number) vertices with odd degree, which is impossible by T2.
∴ someone is lying.

b) Suppose we know that there is only at most one liar and that no criminal would say that they know more people than they actually do. Further we know Frank is not lying. Which of the other criminals could possibly be lying?

First note that: $\deg(A) \geq 6$, $\deg(B) \geq 5$, $\deg(C) \geq 4$, $\deg(D) \geq 3$, $\deg(E) \geq 2$, $\deg(F) = 2$, $\deg(G) \geq 1$.

- A is not lying (Since $6 \leq \deg(A) \leq 6$)
- B is not lying (if B lies ⇒ G lies but there is only one liar)

CASE 1 G is not lying

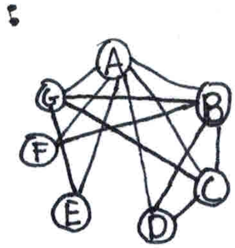


⇒ B knows everyone but G
⇒ C is not lying (if C lies ⇒ F or G lies)
⇒ C knows everyone but F & G
⇒ $\deg(E) \geq 3$
⇒ E is the liar

CASE 2 G lies

⇒ A, B, C, D, E are not lying

For example:



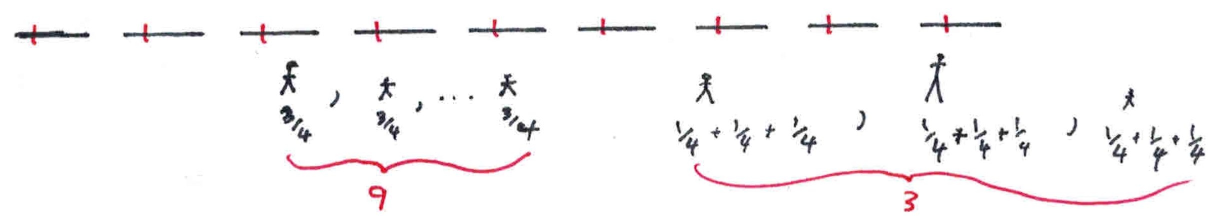
∴ E or G could be lying

Example 5: Kids and Chocolate bars. What are the possible values of $n > 9$ such that n children can equally share 9 identical chocolate bars, with the restriction that no bar be cut into more than two pieces.

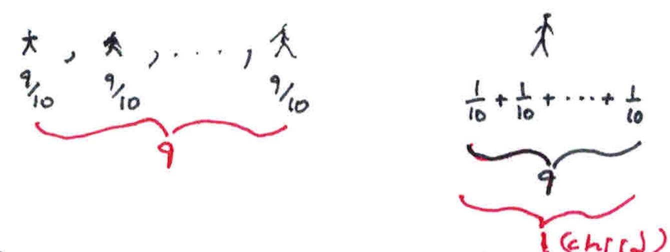
Step 1. Think about different possibilities for n. Note: Each child gets $\frac{9}{n}$ of a bar.

$n = 18$

$n = 12$



$n = 10$



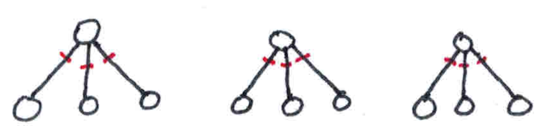
Step 2. Represent the problem as a graph.

Let
- vertices be children
- edges be bars
- ①—② represent child ① sharing with child ②

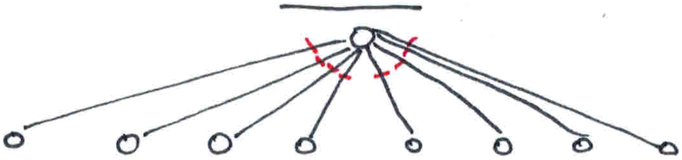
Step 1 =>

$n = 18$

$n = 12$



$n = 10$



Step 3. Explain why this graph has no cycles.

On a cyle at least one child gets too much chocolate.

Ex.  => 3 bars for 3 children => All 3 children cannot get less than one whole bar. => one child gets at least a whole bar.

But each child gets $\frac{9}{n} < 1$ of a bar.