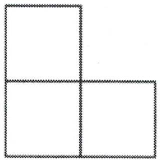
Lecture 16

The Trominoes

Definition: A (*right*) *tromino* is an object made of three unit squares as shown:



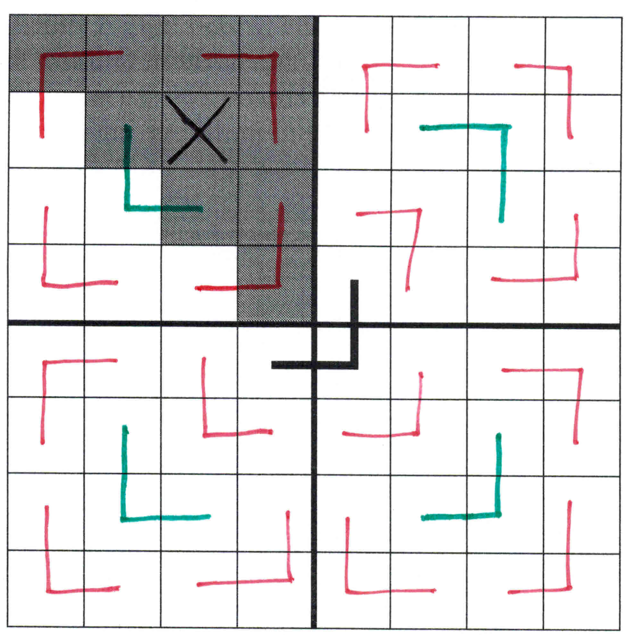
a tromino may appear as above, or it may be rotated through some multiple of 90°.

Definition: A board is called *deficient* if one unit squares is missing.

Definition: A board is said to be *tiled* if you can fit individual tiles together with no gaps or overlaps to fill the board.

Note 1: Suppose we are given a deficient $n \times n$ board. We can rotate the board so the missing square is in the top left quadrant. Further we could reflect the board along its main diagonal. Using this symmetry we only need to look at cases where the missing square above or on the main diagonal (see the example for the 8×8 board below).

Note 2: Given a deficient 8×8 board, by symmetry we need only look at cases where the missing square is in the shaded region below. Notice that by dividing the board into four quadrants and placing one tromino in the center as shown we have four deficient 4×4 boards. Next, divide the board into sixteen 2×2 boards and place four trominoes to make each one deficient. Finally we can complete the tiling of the 8×8 board by tiling the sixteen deficient 2×2 boards.



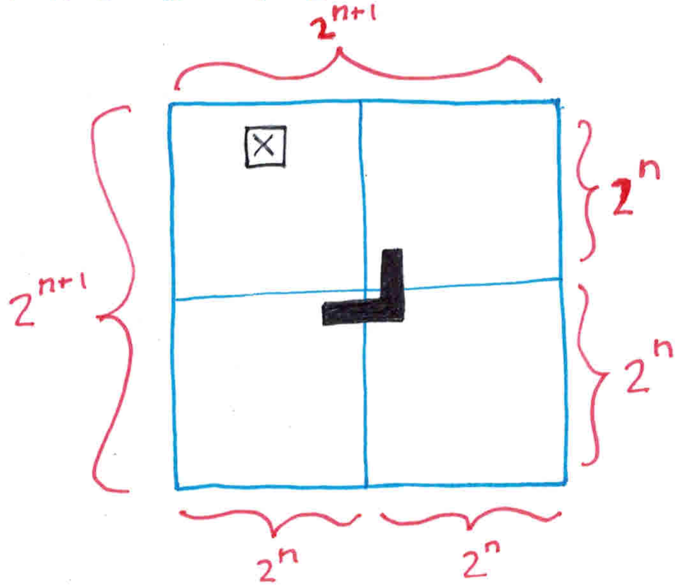
Example 1: For $n \geq 0$, show that any deficient $2^n \times 2^n$ board can be tiled by trominoes.

<u>BC</u> If $n=0$ then the deficient 1×1 board: $\boxed{\times}$ is tiled.

<u>IS</u> Show: any deficient $2^n \times 2^n$ board can be tiled (✱)

$\Rightarrow$ __________ $2^{n+1} \times 2^{n+1}$ __________ .

By symmetry the missing square is in the top left. Divide the board into 4 equal squares and place a tromino in the center:



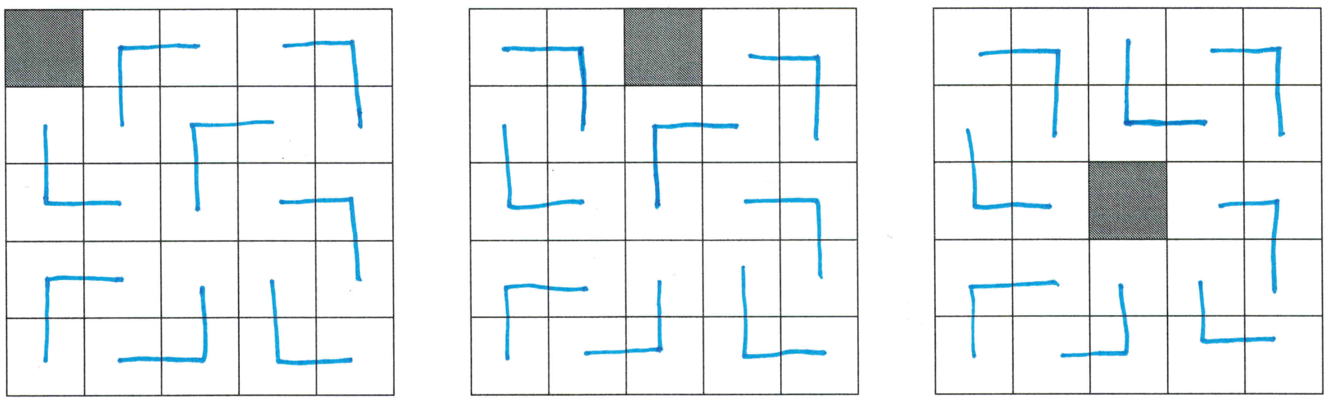
There are now four deficient $2^n \times 2^n$ boards which can be tiled by (✱).

$\therefore$ any deficient $2^{n+1} \times 2^{n+1}$ board can be tiled by trominoes.

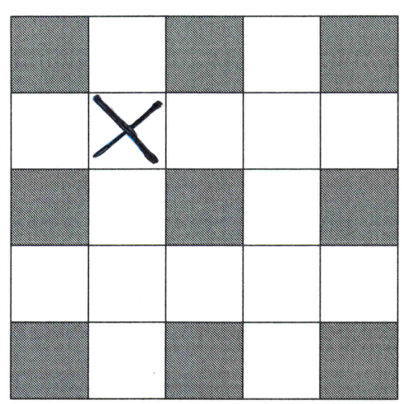
The Deficient 5×5 Boards

Example 2:

a) Tile the following deficient 5×5 boards with trominoes.



b) From the boards above, using symmetry, we know that a deficient 5×5 board made by removing one of the black squares below can be tiled with trominoes. Show that a deficient 5×5 board made by removing one of the white squares below cannot be tiled with trominoes.



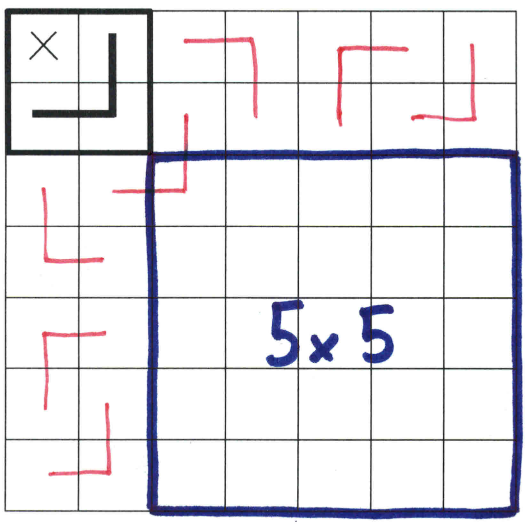
9 trominoes are needed to cover the 9 black squares but there is room for only :

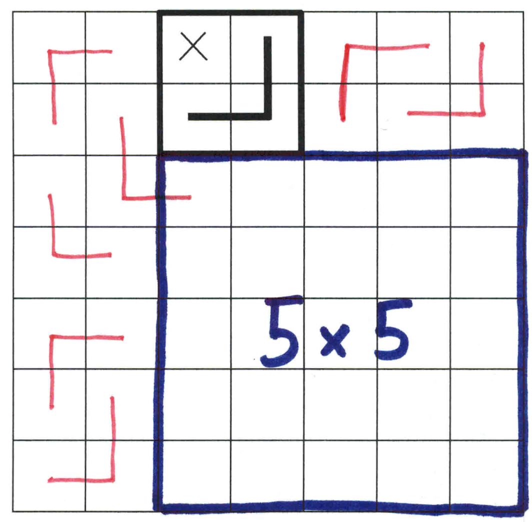
$$\frac{\text{Area of the board}}{\text{Area of a tromino}} = \frac{5^2-1}{3} = \frac{24}{3} = 8$$

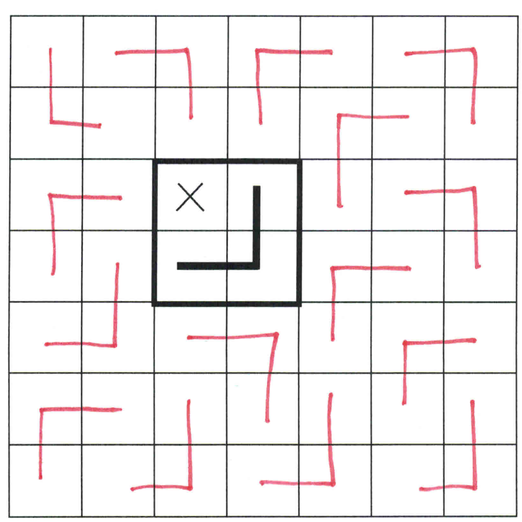
trominoes. Therefore when a white square is missing the board can not be tiled.

The Deficient 7 × 7 Boards

Example 3: By the symmetry described in note 1, the deficiency on a 7 × 7 board is in one of the three bold 2 × 2 sections below. Each deficient 2 × 2 section can be tiled with one tromino. After the deficient 2 × 2 sections are tiled they can be rotated to cover every missing square that needs to be considered. Tile the remaining squares to show any deficient 7 × 7 board can be tiled by trominoes.

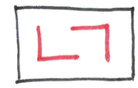






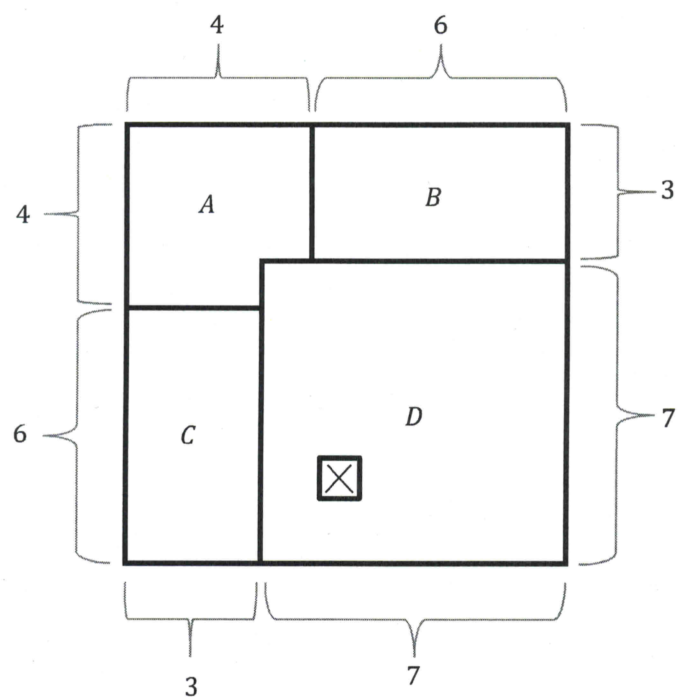
Note: the first two tilings can be completed using Ex. 2.

Proposition 1: Given that $n \equiv 0 \pmod 3$ and $m \equiv 0 \pmod 2$ an $n \times m$ board can be tiled with trominoes.

Proof. This type of board can be tiled by 2×3 blocks made of two trominoes:  .

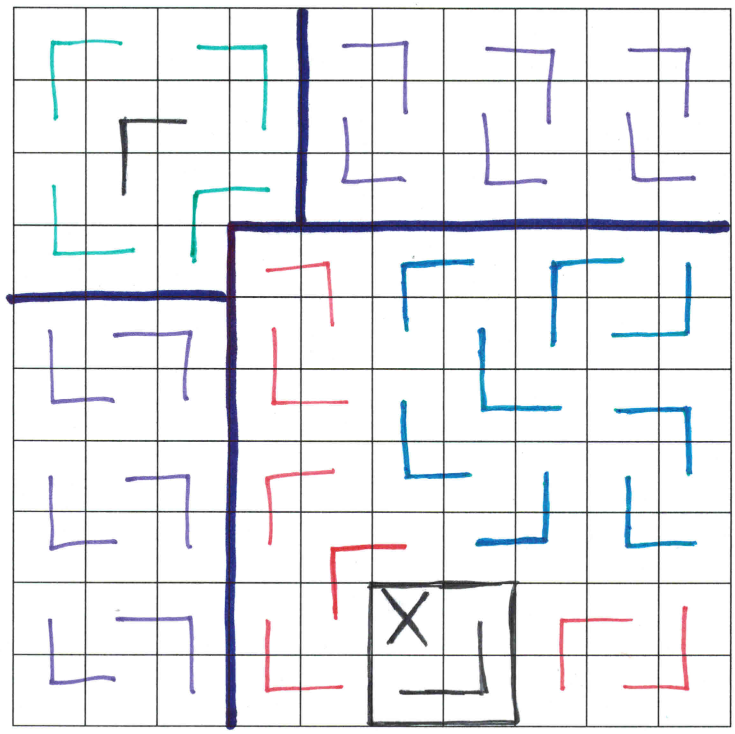
<u>The Deficient 10×10 Boards</u>

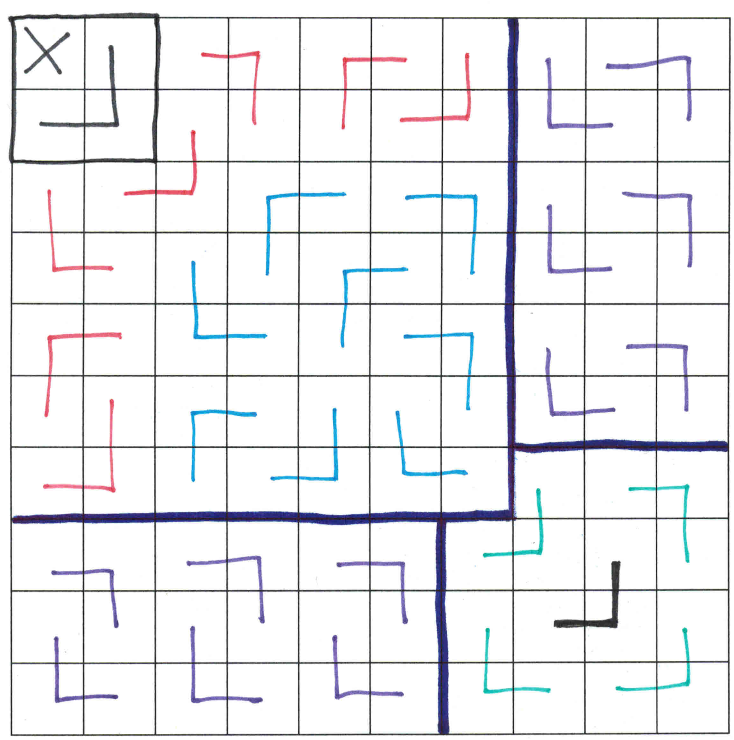
Example 4: Show that any deficient 10×10 board can be tiled by trominoes. To do this, start by splitting the board into the four sections shown below. By symmetry the deficiency is always in section D. Explain why each section can be tiled.



- Section A can be tiled by Ex 1.
- Sections B and C can be tiled by proposition 1.
- Section D can be tiled by Ex 3.

Example 5: Remove any single square from each of the 10×10 boards below to make two deficient 10×10 boards. Use example 4 to tile the deficient boards with trominoes.





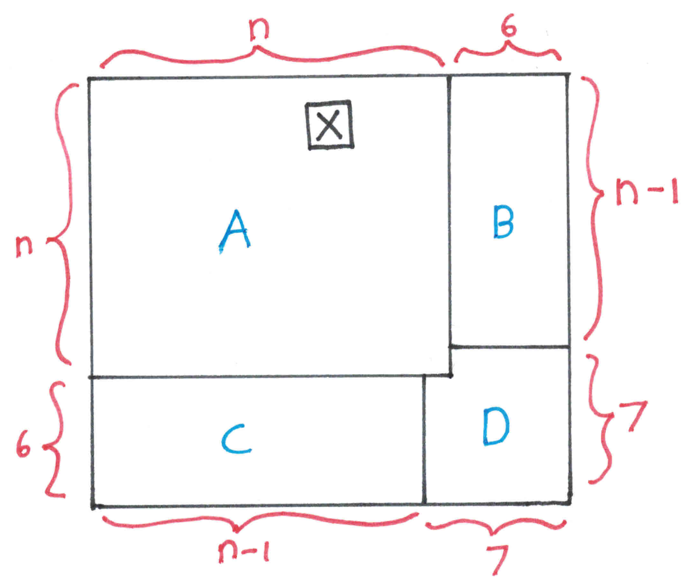
Proposition 2: If $n \equiv 1 \pmod 3$ a deficient $n \times n$ board can be tiled with trominoes.

Proof. Let P_n be the statement for $n \geq 1$.

BC P_1, P_4, P_7, and P_{10} are true by examples 1, 3, and 4.

IS Show: $P_n \Rightarrow P_{n+6}$ for $n \geq 7$.

Let $n \equiv 1 \pmod 3$ and $n \geq 7$. Divide an $n+6 \times n+6$ board as follows:



By symmetry the missing square is in the top left, in section A.

- Section A can be tiled by P_n.
- Sections B and C can be tiled since $n-1 \equiv 0 \pmod 3$, $6 \equiv 0 \pmod 2$
- Section D can be tiled by Example 3.