# Lecture 15

The **Principle of Mathematical Induction**: to show a statement $P_n$ is true for all positive integers greater or equal to $n_0$ there are two steps:

- *Base Case*: Show $P_{n_0}$ is true

- *Inductive Step:* Show: $P_n \Rightarrow P_{n+1}$ (for all $n \geq n_0$)

**Example 1:** Prove:

$$1 + 2 + 3 + \cdots + n = \frac{(n+1) \cdot n}{2}$$

for $n \geq 1$ by using mathematical induction.

Here the statement $P_n$ is: $1 + 2 + \cdots + n = \frac{(n+1)n}{2}$ for $n \geq 1$

BC (Base Case)

if $n = 1 \Rightarrow 1 + 2 + \cdots + n = 1 = \frac{(1+1) \cdot 1}{2} = \frac{(n+1) \cdot n}{2}$

($P_1$ is true)

IS (Inductive Step)

Show: $\underbrace{1 + 2 + \cdots + n \overset{*}{=} \frac{(n+1)n}{2}}_{P_n} \Longrightarrow \underbrace{1 + 2 + \cdots + (n+1) = \frac{(n+2)(n+1)}{2}}_{P_{n+1}}$ for $n \geq 1$.

$$(1 + 2 + \cdots + n) + n + 1 \overset{*}{=} \frac{(n+1)\, n}{2} + (n+1) \cdot \left(\frac{2}{2}\right)$$

$$= \frac{n+1}{2}(n+2) = \frac{(n+2)(n+1)}{2}$$

$\therefore P_1 \Rightarrow P_2 \Rightarrow P_3 \Rightarrow \cdots$

**Example 2:** Prove:

$$\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\cdots+\frac{1}{n\cdot(n+1)}=\frac{n}{n+1}$$

for $n \geq 1$ by using mathematical induction.

Base Case $\quad n=1 \Rightarrow \frac{1}{1\cdot 2} = \frac{1}{2} = \frac{1}{1+1}$

IS

Show: $\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\cdots+\frac{1}{n(n+1)} \overset{*}{=} \frac{n}{n+1}$

$\Rightarrow \frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\cdots+\frac{1}{(n+1)(n+2)} = \frac{n+1}{n+2}$ for $n \geq 1$

$$\left(\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\cdots+\frac{1}{n(n+1)}\right)+\frac{1}{(n+1)(n+2)}$$

$$\overset{*}{=} \frac{n}{n+1}+\frac{1}{(n+1)(n+2)}$$

$$= \frac{1}{n+2}\left(\frac{n(n+2)+1}{n+1}\right)$$

$$= \frac{1}{n+2}\cdot\frac{(n^2+2n+1)}{n+1}$$

$$= \frac{1}{n+2}\cdot\frac{(n+1)^2}{n+1}$$

$$= \frac{n+1}{n+2}$$

**Example 3:** Let $m$ be an integer and prove

$$m + (m+1) + \cdots + n = \frac{(n+m)(n-m+1)}{2}$$

for $n \geq m$ by using mathematical induction.

$\underline{BC}$ $\quad n=m \implies m = \frac{2m}{2} = \frac{(m+m)(m-m+1)}{2}$

$\underline{IS}$ $\quad$ show: $\quad m + (m+1) + \cdots + n \overset{*}{=} \frac{(n+m)(n-m+1)}{2}$

$$\implies m + (m+1) + \cdots + n+1 = \frac{(n+m+1)(n-m+1+1)}{2}.$$

$$\big(m + (m+1) + \cdots + n\big) + n+1 \overset{*}{=} \frac{(n+m)(n-m+1)}{2} + n + 1$$

$$= \frac{(n+m)(n-m+1) + 2n+2}{2}$$

$$= \frac{(n+m)(n-m+1) + (n+m)\cdot 1 + 1\cdot(n-m+1) + 1}{2}$$

$$\therefore = \frac{(n+m+1)(n-m+1+1)}{2}$$

"Think

$\big((n+m)+1\big)\big((n-m+1)+1\big)$

FOIL

**Example 4:** Find a closed form for the following expression.

$$\frac{1 + 2 + \cdots + n}{n + (n+1) + \cdots + 2n}$$

$$\underset{\text{(Ex 3)}}{=} \frac{\frac{(n+1)n}{2}}{\frac{(2n+n)(n+1)}{2}} = \frac{1}{3}\,\frac{\frac{(n+1)n}{2}}{\frac{(n+1)n}{2}} = \boxed{\frac{1}{3}}$$

**Example 5:** Prove:

$$\sqrt{2+\sqrt{2+\cdots+\sqrt{2}}} = 2\cdot\cos\left(\frac{\pi}{2^{n+1}}\right)$$

where the statement on the left has $n$ 2's. Recall the half angle formula $\frac{1+\cos 2\theta}{2} = \cos^2\theta$.

BC $\quad n=1 \implies \sqrt{2} = 2\cdot\cos\left(\frac{\pi}{4}\right) \iff \cos\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}$

IS $\quad$ Show: $\underbrace{\sqrt{2+\sqrt{2+\cdots+\sqrt{2}}}}_{n} \overset{*}{=} 2\cdot\cos\left(\frac{\pi}{2^{n+1}}\right)$

$$\implies \underbrace{\sqrt{2+\sqrt{2+\cdots+\sqrt{2}}}}_{n+1} = 2\cdot\cos\left(\frac{\pi}{2^{n+2}}\right) \quad \text{for } n\geq 1$$

$$\underbrace{\sqrt{2+\underbrace{\sqrt{2+\cdots+\sqrt{2}}}_{n}}}_{n+1} \overset{*}{=} \sqrt{2+2\cdot\cos\frac{\pi}{2^{n+1}}}$$

$$= \sqrt{\frac{4+4\cdot\cos\frac{\pi}{2^{n+1}}}{2}}$$

$$= \sqrt{4\left(\frac{1+\cos\left(2\cdot\frac{\pi}{2^{n+2}}\right)}{2}\right)}$$

$$= 2\sqrt{\cos^2\left(\frac{\pi}{2^{n+2}}\right)}$$

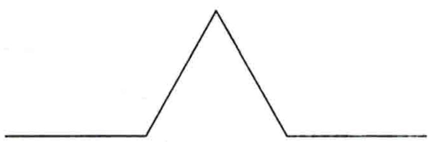
$$= 2\cos\left(\frac{\pi}{2^{n+2}}\right) \quad \text{since } \cos\left(\frac{\pi}{2^{n+2}}\right) > 0 \text{ for } n\geq 1$$

**Example 6:** Define a sequence of shapes as follows:

- $K_1$ is an equilateral triangle
- for $n > 1$, $K_n$ is formed by replacing each line segment

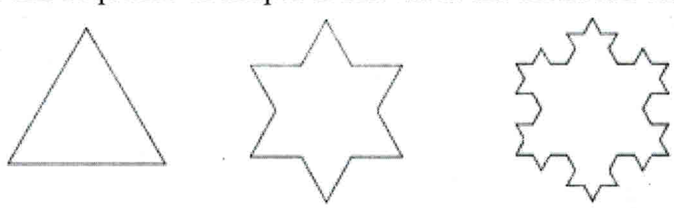


of $K_{n-1}$ with the shape



according to the following three steps
1) The line segment was divided into three segments of equal length.
2) An equilateral triangle was drawn pointing outward that has its middle segment from step 1 as its base.
3) The line segment that is the base of the triangle from step 2 was removed.

The limit of this sequence of shapes is known as the Koch snowflake:



a) Write down a recurrence relation for $a_n$ the number of line segments in $K_n$.

$$a_1 = 3$$
$$a_n = 4 \cdot a_{n-1}$$

b) Show by induction that the number of line segments in $K_n$ is:

$$3 \cdot 4^{n-1}$$

BC when $n=1 \Rightarrow a_1 = 3 \cdot 4^{1-1} = 3$

IS Show: $a_n \overset{*}{=} 3 \cdot 4^{n-1} \Rightarrow a_{n+1} = 3 \cdot 4^n$ for $n \geq 1$

$$a_{n+1} \overset{a)}{=} 4\, a_n \overset{*}{=} 4\left(3 \cdot 4^{n-1}\right) = 3 \cdot 4^n$$

**Example 7:** Conjecture a formula for the sum of the first $n$ Fibonacci numbers with even indices and prove your formula works by using mathematical induction. That is, find and prove a formula for $F_2 + F_4 + \cdots + F_{2n}$.

Note:

$$F_1 = 1,$$
$$F_2 = 1,$$

$$F_n = F_{n-1} + F_{n-2}.$$

$F_1 = 1$

$F_2 = 1$ $\quad +1$

$F_3 = 2$

$F_4 = 3$ $\quad +1$

$F_5 = 5$ $\quad +1$

$F_6 = 8$

$F_7 = 13$

$F_2 = 1$

$F_4 + F_2 = 4$

$F_6 + F_4 + F_2 = 12$

Guess : $F_2 + F_4 + \cdots + F_{2n} = F_{2n+1} - 1$

BC $\quad F_2 = F_3 - 1$ ✓

IS $\quad$ Show : $F_2 + F_4 + \cdots + F_{2n} \overset{*}{=} F_{2n+1} - 1$

$\Longrightarrow F_2 + F_4 + \cdots + F_{2n+2} = F_{2n+3} - 1$ , for $n \geq 1$.

$$F_2 + F_4 + \cdots + F_{2n} + F_{2n+2} \overset{*}{=} F_{2n+1} - 1 + F_{2n+2}$$

$$= F_{2n+3} - 1$$ ✓

**Example 8:** Consider the recurrence relation defined by:

$$a_n = 3\,a_{n-1} - 2^{n-1}, \qquad \text{for } n \geq 1,$$
$$a_0 = 1.$$

Conjecture a solution to this recurrence relation and prove it my mathematical induction.

$a_0 = 1$

$a_1 = 3 \cdot 1 - 2^0 = 2$

$a_2 = 3^2 - 3 - 2 = 9 - 3 - 2 = 4$

$a_3 = 3^3 - 3^2 - 3 \cdot 2 - 2^2 = 27 - 9 - 6 - 4 = 8$

$\vdots$

$$a_n = 3^n - \sum_{i=0}^{n-1} 2^i \cdot 3^{n-1-i} \stackrel{?}{=} 2^n$$

Guess: $a_n = 2^n$

BC $\quad a_0 = 1 = 2^0$ ✓

IS $\quad$ Show: $a_n \stackrel{*}{=} 2^n \Rightarrow a_{n+1} = 2^{n+1}$, for $n \geq 0$.

$$a_{n+1} = 3\,a_n - 2^n$$

$$\stackrel{*}{=} 3 \cdot 2^n - 2^n$$

$$= 2 \cdot 2^n$$

$$= 2^{n+1}$$ ✓

**Example 9:** Use mathematical induction to show that $n^3 - n$ is divisible by 3 for every positive integer $n$.

BC $\quad n=1 \qquad 1^3 - 1 \equiv 0 \mod 3$ ✓

IS $\quad$ Show: $\quad n^3 - n \overset{*}{\equiv} 0 \mod 3$

$\Rightarrow \quad (n+1)^3 - (n+1) \equiv 0 \mod 3$, for $n \geq 1$.

$$(n+1)^3 - (n+1) = n^3 + 3n^2 + 3n + \cancel{1} - n - \cancel{1}$$
$$= (n^3 - n) + 3n^2 + 3n$$
$$\overset{*}{\equiv} 0 + 0 + 0$$
$$\equiv 0 \mod 3$$ ✓

**Example 10:**

All girls have the same hair colour. We claim that all girls in any group of $n$ girls have the same hair colour, for each $n = 1, 2, \ldots$

*Step 1:* When $n = 1$, there is only one girl in the group, so all girls within the group clearly have the same hair colour.

*Step 2:* Assume that the case $n = k$ holds. Given a group of $k + 1$ girls, remove one of them from the group. By assumption, each of the remaining $k$ girls have the same hair colour. Now swap one of these girls with the girl we removed. Since every girl in this new group of $k$ girls also have the same hair colour, we know now that all $k + 1$ girls have the same hair colour!

By induction, the claim holds.

Is there an error in the above "proof"? If so, where is the flaw?

Yes, $P_1$ does not imply $P_2$.

**Strong Induction**; to show a statement $P_n$ is true for all positive integers greater or equal to $n_0$ there are two steps:

- *Base Case*: Show $P_{n_0}$ is true (you may need to do this for other values also)

- *Inductive Step:* Show: $(P_{n_0} \,\&\, P_{n_0+1} \,\&\, P_{n_0+2} \,\&\, \cdots \,\&\, P_n) \Rightarrow P_{n+1}$ (for all $n \geq n_0$)

**Example 11:** $P_n$: "Postage of $n$ cents can be formed using 4-cent and 5-cent stamps." Show: $P_n$ is true for $n \geq 12$.

BC

$$\left.\begin{aligned} 3(4) + 0(5) &= 12 \\ 2(4) + 1(5) &= 13 \\ 1(4) + 2(5) &= 14 \\ 0(4) + 3(5) &= 15 \end{aligned}\right\} \Rightarrow P_{12}, P_{13}, P_{14}, P_{15} \text{ are true}$$

IS Show: $P_n \Rightarrow P_{n+4}$ for $n \geq 12$

$$n + 4 \overset{P_n}{=} 4x + 5y + 4$$

(since $P_n$ is true $n = 4x + 5y$ for some non-negative integers $x, y$.)

$$= 4(x+1) + 5y$$

$\therefore$ $P_{n+4}$ is true.

Note we have shown:

$$P_{12} \Rightarrow P_{16} \Rightarrow P_{20} \Rightarrow \cdots$$

$$P_{13} \Rightarrow P_{17} \Rightarrow P_{21} \Rightarrow \cdots$$

$$P_{14} \Rightarrow P_{18} \Rightarrow P_{22} \Rightarrow \cdots$$

$$P_{15} \Rightarrow P_{19} \Rightarrow P_{23} \Rightarrow \cdots$$

**Example 12:** Show that every positive integer can be written as a sum of distinct powers of two.

Let $P_n$ be the statement for $n \geq 1$.

BC $1 = 2^0 \Rightarrow P_1$ is true

IS Show: $P_n \Rightarrow (P_{2n} \ \& \ P_{2n+1})$ for $n \geq 1$

$$2n = 2\underbrace{(2^i + \cdots + 2^j)}_{\text{distinct powers}} = \underbrace{2^{i+1} + \cdots + 2^{j+1}}_{\text{distinct powers}}$$

$$\Rightarrow 2n+1 = \overbrace{2^{i+1} + \cdots + 2^{j+1}}^{\text{distinct powers}} + 1 = \overbrace{2^{i+1} + \cdots + 2^{j+1} + 2^0}^{\text{distinct powers}}$$

$\therefore$ $P_{2n}$ & $P_{2n+1}$ are true.

Note we have shown:

$$
\begin{array}{ccccccccc}
 & & & & P_1 & & & & \\
\Rightarrow & & P_2 & & & & P_3 & & \\
\Rightarrow & P_4 & & P_5 & & P_6 & & P_7 & \\
\Rightarrow & P_8 \ P_9 & & P_{10} \ P_{11} & & P_{12} \ P_{13} & & P_{14} \ P_{15} & \\
 & & & & \vdots & & & &
\end{array}
$$

**Example 13:** Define the recurrence relation:

$$a_0 = 1$$
$$a_1 = 1$$
$$a_n = 2a_{n-1} + a_{n-2}$$

Show that $a_n$ is odd for all $n \geq 0$.

Let $P_n$ be the statement for $n \geq 0$.

<u>BC</u> $a_0 = a_1 = 1$ is odd $\therefore P_0, P_1$ are true

<u>IS</u> Show: $P_{n-2} \Longrightarrow P_n$ for $n \geq 2$

$$a_n \overset{\text{Def}}{=} 2a_{n-1} + a_{n-2}$$

$$\overset{P_{n-2}}{=} 2a_{n-1} + 2m+1 \quad \text{(for some } m\text{)}$$

$$= 2(a_{n-1} + m) + 1 \quad \text{is odd}$$

$\therefore P_n$ is true

Note we have shown:

$$P_0 \Rightarrow P_2 \Rightarrow P_4 \Rightarrow \cdots$$

$$P_1 \Rightarrow P_3 \Rightarrow P_5 \Rightarrow \cdots$$

**Example 14:** Prove that $5^{n+1} + 2 \cdot 3^n + 1$ is divisible by 8 for all $n \geq 1$.

Let $P_n$ be the statement for $n \geq 1$.

<u>BC</u>

$$5^2 + 2 \cdot 3 + 1 \equiv 1 + 6 + 1 \equiv 8 \equiv 0 \pmod 8 \Rightarrow P_1 \text{ is true}$$

$$5^3 + 2 \cdot 3^2 + 1 \equiv 5 + 2 + 1 \equiv 8 \equiv 0 \pmod 8 \Rightarrow P_2 \text{ is true}$$

<u>IS</u> Show: $P_{n-2} \Rightarrow P_n$ for $n \geq 3$.

$$\begin{aligned} 5^{n+1} + 2 \cdot 3^n + 1 &\equiv (5^2) 5^{n-1} + 2 \cdot (3^2) 3^{n-2} + 1 \\ &\equiv (1) 5^{n-1} + 2 \cdot (1) 3^{n-2} + 1 \\ &\equiv 5^{n-1} + 2 \cdot 3^{n-2} + 1 \\ &\overset{P_{n-2}}{\equiv} 0 \pmod 8 \end{aligned}$$

$\therefore P_n$ is true.

Note we have shown:

$$P_1 \Rightarrow P_3 \Rightarrow P_5 \Rightarrow \cdots$$

$$P_2 \Rightarrow P_4 \Rightarrow P_6 \Rightarrow \cdots$$