Lecture 13

<u>Recurrence relations:</u>

Warm up problem: **The Carrot Cake Problem.** (page 20 or page 4 in Ecco) How can we cut a cake into 16 equal pieces using four straight (vertical) cuts?

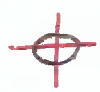
Definition: A *recurrence relation* is an equation that defines a sequence where each term of the sequence is defined as a function of the preceding terms.

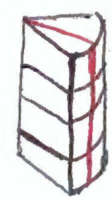
Example 1: Find a recurrence relation for the maximum number of pieces of cake that can be obtained using n straight (vertical) cuts. Solve the recurrence relation.

Let $a_n =$ max # of pieces using n cuts.

Now $a_0 = 1$ 

$a_1 = 2$ 

$a_2 = 4$ 

$a_3 = 8$ 

Next, by stacking a_{n-1} pieces and cutting all of them with one vertical cut we obtain the max of $2a_{n-1}$ pieces.

$\therefore$ $\boxed{\begin{array}{l} a_0 = 1 \\ a_n = 2a_{n-1} \end{array}}$ is the r.r. (recurrence relation)

solve

$a_0 = 1 = 2^0$

$a_1 = 2 \cdot 1 = 2^1$

$a_2 = 2 \cdot 2^1 = 2^2$

$\vdots$

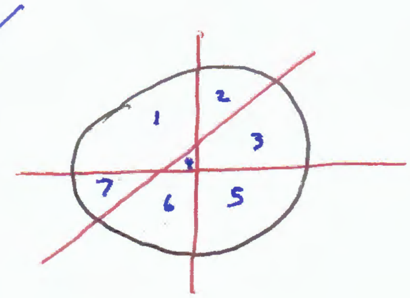
$\boxed{a_n = 2^n}$

Example 2: What is the largest number of pieces you can get by cutting a pizza (2 dimensional) with n straight (vertical) cuts if the pieces are not to be moved between cuts?

Let $a_n =$ max # of pieces using n cuts.

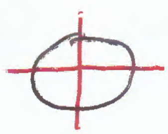
$a_0 = 1$

$a_3 = 7$



$a_1 = 2$

$a_2 = 4$



Note: for every $n-1$ new intersection points we have n new pieces.

$\therefore$ $\boxed{\begin{array}{l} a_0 = 1 \\ a_n = a_{n-1} + n \end{array}}$ is the r.r.

Solve

$a_0 = 1$

$a_1 = 1 + 1$

$a_2 = 1 + 1 + 2$

$a_3 = 1 + 1 + 2 + 3$

$\vdots$

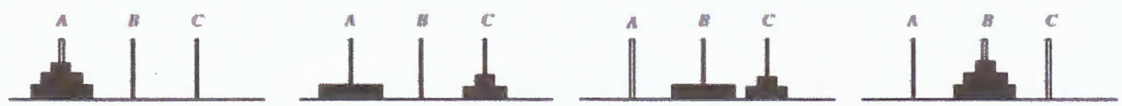
$a_n = 1 + (1 + 2 + \cdots + n) = 1 + \dfrac{n(n+1)}{2}$

Example 3: An old puzzle called The Tower of Hanoi consists of three pegs; A, B, and C. On peg A there are n disks of different diameters arranged by decreasing size from the bottom to the top. You wish to transfer all of the n disks from peg A to peg B. The rules for moving the disks are as follows:

Only one disk may be moved at a time, and it may be moved from one peg to either of the other two pegs. No disk may be placed on top of one of smaller diameter.

Set up a recurrence relation to solve the puzzle. Then solve the recurrence relation.

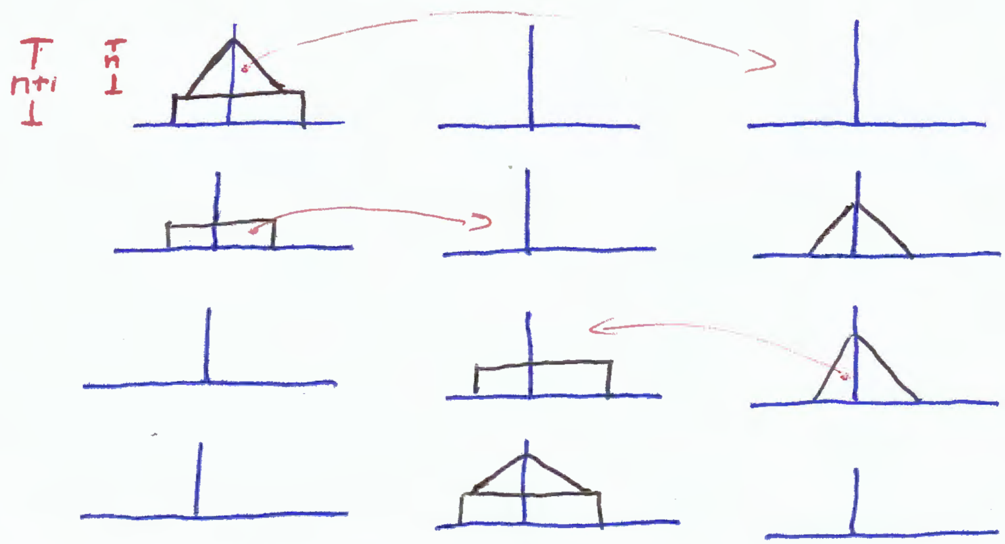
Hint:



Let a_n = "the number of moves needed to move a tower of n disks.

Suppose we know how to move n disks and we need to move $n+1$ disks.

given: # of moves?



1. a_n

2. 1

3. a_n

$\therefore$ $\boxed{\begin{array}{l} a_1 = 1 \\ a_{n+1} = 2a_n + 1 \end{array}}$ is the r.r.

Solve

$$\begin{aligned} a_1 &= 1 = 2^0 \\ a_2 &= 2\cdot 1 + 1 = 2^1 + 2^0 \\ a_3 &= 2(2+1) + 1 = 2^2 + 2^1 + 2^0 \\ &\vdots \\ a_n &= 2^{n-1} + 2^{n-2} + \cdots + 2^0 \end{aligned}$$

$$\Rightarrow a_n = \frac{2^n - 1}{2 - 1}$$

$$\Rightarrow \boxed{a_n = 2^n - 1}$$

Example 4: A certain basketball team can only sink foul shots and lay-ups, worth 1 and 2 points, respectively. Let a_n denote the number of ways the team can score n points. (Scoring 1 then 2 is considered to be different than scoring 2 then 1). Write down a recurrence relation for a_n with initial conditions for a_0 and a_1; and explain why it holds for all $n \geq 2$.

$a_0 = 1$

$a_1 = 1$ 1

$a_2 = 2$ 11 2

$a_3 = 3$ 111 12 21

$a_4 = 5$ 1111 112 121 211 22

<u>CASE 1</u> (The first point is worth 1)

$\therefore$ there are a_{n-1} ways to score n points,

<u>CASE 2</u> The first point is worth 2

$\therefore$ there are a_{n-2} ways to score n points.

$\therefore$ $\boxed{\begin{array}{l} a_0 = 1 \\ a_1 = 1 \\ a_n = a_{n-1} + a_{n-2} \end{array}}$ is the r.r.