

§ Recursion & Induction

Lecture 12

Open and Closed Forms:

Definition 1: A *closed form* expression is an expression which has a fixed number of operations.

Definition 2: An *open form* expression is an expression in which the number of operations grows depending on a variable in the expression.

Definition 3: The sum (in open form) of the first n natural numbers each to the power i is given by:

$$S_i = 1^i + 2^i + \dots + n^i$$

Example 1: Find a closed form for:

$$S_1 = 1 + 2 + 3 + \dots + n$$

To find the closed form, use the expansion:

$$(x+1)^2 = x^2 + 2x + 1$$

by plugging in $x = 1, x = 2, x = 3, \dots, x = n$.

Note: (Gauss in grade 1)

$$S_1 = 1 + 2 + \dots + n$$

$$S_1 = n + (n-1) + \dots + 1$$

$$2 S_1 = \underbrace{(n+1) + (n+1) + \dots + (n+1)}_n = n(n+1)$$

$$\Rightarrow S_1 = \frac{(n+1)n}{2}$$

$$x=1: (1+1)^2 = 1^2 + 2 \cdot 1 + 1$$

$$x=2: (1+2)^2 = 2^2 + 2 \cdot 2 + 1$$

\vdots

$$x=n: (1+n)^2 = n^2 + 2 \cdot n + 1$$

$$(n+1)^2 + \dots + 2^2 + 1^2 - 1^2 = S_2 + 2 S_1 + n$$

$$\Rightarrow (n+1)^2 + \cancel{S_2} - 1 = \cancel{S_2} + 2 S_1 + n$$

$$\Rightarrow 2 \cdot S_1 = (n+1)^2 - 1 - n = n^2 + n = (n+1)n$$

$$\Rightarrow \boxed{S_1 = \frac{(n+1)n}{2}}$$

Example 2: Find a closed form for:

$$S_2 = 1^2 + 2^2 + 3^2 + \dots + n^2$$

To find the closed form, use S_1 and the expansion:

$$(x+1)^3 = x^3 + 3x^2 + 3x + 1$$

by plugging in $x = 1, x = 2, x = 3, \dots, x = n$.

$$\begin{array}{rcl} x=1: & 2^3 & = 1^3 + 3 \cdot 1^2 + 3 \cdot 1 + 1 \\ x=2: & 3^3 & = 2^3 + 3 \cdot 2^2 + 3 \cdot 2 + 1 \\ & \vdots & \\ & \vdots & \\ x=n: & (n+1)^3 & = n^3 + 3n^2 + 3n + 1 \end{array}$$

$$\underbrace{(n+1)^3 + n^3 + \dots + 2^3 + 1^3}_{S_3} = S_3 + 3S_2 + 3S_1 + n$$

$$\Rightarrow (n+1)^3 + \cancel{S_3} - 1 = \cancel{S_3} + 3S_2 + 3S_1 + n$$

$$\begin{aligned} \Rightarrow 3S_2 &= (n+1)^3 - 3S_1 - (n+1) \\ &= (n+1) \left((n+1)^2 - \frac{3n}{2} - 1 \right) \\ &= (n+1) \left(n^2 + 2n + 1 - \frac{3n}{2} - 1 \right) \\ &= \frac{(n+1)}{2} (2n^2 + 4n - 3n) \\ &= \frac{(n+1)(2n+1)n}{2} \end{aligned}$$

since:
 $S_1 = \frac{n(n+1)}{2}$

$$\Rightarrow \boxed{S_2 = \frac{(2n+1)(n+1)n}{6}}$$

Example 3: Find a closed form expression for

$$a_n = 1 \cdot n + 2 \cdot (n-1) + 3 \cdot (n-2) + \dots + (n-1) \cdot 2 + n \cdot 1$$

valid for $n \geq 1$.

$$\begin{aligned} a_n &= \sum_{i=1}^n i \cdot (n+1-i) \\ &= \sum_{i=1}^n i(n+1) - i^2 \\ &= (n+1) \sum_{i=1}^n i - \sum_{i=1}^n i^2 \\ &= \frac{(n+1)^2 n}{2} - \frac{(2n+1)(n+1)n}{2} \end{aligned}$$

Example 4: For $x \neq 1$ find a closed form for:

$$\begin{aligned} S &= x^0 + x^1 + x^2 + \dots + x^n \\ \Rightarrow -xS &= x^1 + x^2 + \dots + x^n + x^{n+1} \\ \hline S - xS &= x^0 - x^{n+1} \\ \Rightarrow S(1-x) &= 1 - x^{n+1} \\ \Rightarrow S &= \frac{1 - x^{n+1}}{1-x} \\ \Rightarrow S &= \frac{x^{n+1} - 1}{x - 1} \end{aligned}$$

(Note: if $0 < x < 1$ then:
 $S = \frac{1}{1-x}$ as $n \rightarrow \infty$)

Definition 4: The number of ways to choose k objects from a group of n objects is given by:

$$\binom{n}{k} = \frac{n!}{(n-k)! \cdot k!}$$

Definition 5: Pascal's Formula:

$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$$

Example 5: Find a closed form for:

$$\begin{aligned} & \sum_{i=3}^{n-1} \binom{i}{3} \\ = & \sum_{i=3}^{n-1} \binom{i+1}{4} - \binom{i}{4} \quad \text{since } \binom{n-1}{k-1} = \binom{n}{k} - \binom{n-1}{k} \\ = & \binom{4}{4} + \binom{5}{4} + \dots + \binom{n-1}{4} + \binom{n}{4} \\ & - \binom{3}{4} - \binom{4}{4} - \binom{5}{4} - \dots - \binom{n-1}{4} \\ = & \binom{n}{4} \end{aligned}$$

Example 6: Show that

$$a_n = \frac{1}{n+1} \cdot \binom{2n}{n}$$

is an integer for $n = 1, 2, 3 \dots$

$$\begin{aligned} a_n &= \frac{(2n)!}{(n+1)n!n!} = \frac{(2n)!}{(n+1)!n!} = \frac{(2n)!((n+1)-n)}{(n+1)!n!} \\ &= \frac{(2n)!}{n!n!} - \frac{(2n)!}{(n+1)!(n-1)!} = \binom{2n}{n} - \binom{2n}{n+1} \in \mathbb{Z} \end{aligned}$$