Review for Final Exam

Ex 1: if $n \neq 3$, then any $n \times n$ board can be tiled with trominoes if and only if $n \equiv 0 \pmod 3$

(Not a deficient board).

(i) Suppose that for all $n \neq 3$, we can tile a $n \times n$ board, then the area of board

$$\text{area} = 3(\#\text{ trominoes})$$

that is,

$$n^2 = \text{area} = 3k$$

where k is # of trominoes.

So $n^2 \equiv 0 \pmod 3$

if $n \equiv 1 \pmod 3$ then $n = 1 + 3\ell$

$$n^2 = (1+3\ell)^2 = 1 + 6\ell + 9\ell^2$$

so $n^2 \equiv 1 \pmod 3$

if $n \equiv 2 \pmod 3$ then $n = 2 + 3\ell$

$$n^2 = (2+3\ell)^2 = 4 + 12\ell + 9\ell^2$$

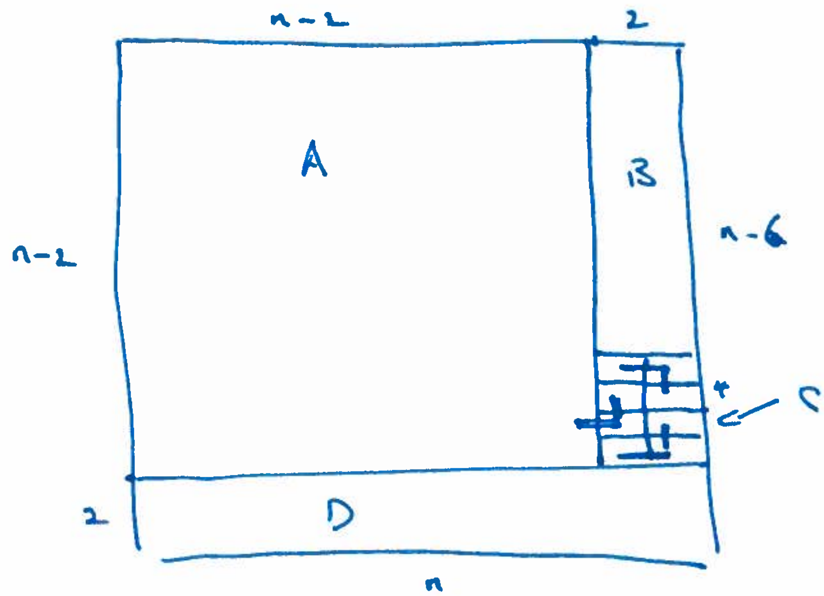
so $n^2 \equiv 1 \pmod 3$.

Therefore, if $n^2 \equiv 0 \pmod 3$, then $n \equiv 0 \pmod 3$.

Ex 1:

(iii) Assume that $n \neq 3$ and $n \equiv 0 \pmod 3$

then we section the board as follows:



- Tile Section C as shown

- Section A is a deficient $(n-2) \times (n-2)$ board

  and $n-2 \equiv 3-2 \equiv 1 \not\equiv 0 \pmod 3$

  and by assignment 4 Section A can be tiled by trominoes.

- Sections B and D

  $2 \equiv 0 \pmod 2$

  $n \equiv n-6 \equiv n-0 \equiv 0 \pmod 3$

  So by Proposition in Lecture 16, we can tile Sections B and D with trominoes.

Ex 2: From the Practice Problems on the web:

Show

$$\frac{1}{2^{2n}}\binom{2n}{n} > \frac{1}{2n} \qquad (*)$$

for all $n > 1$.

Use induction.

Base Case: $n = 1$, L.H.S. of $(*)$ is $\frac{1}{2^{2\cdot 1}}\binom{2}{1} = \frac{1}{4}\cdot 2 = \frac{1}{2}$

R.H.S. of $(*)$ is $\frac{1}{2\cdot 1} = \frac{1}{2}$

So we have equality in $(*)$ when $n=1$.

So take $n=2$ for the base case:

LHS of $(*)$ is $\frac{1}{2^{2\cdot 2}}\binom{2\cdot 2}{2} = \frac{1}{2^4}\binom{4}{2} = \frac{1}{16}\binom{4}{2} = \frac{1}{16}\cdot\frac{4\cdot 3}{2\cdot 1} = \frac{6}{16}$

$= \frac{3}{8}$

RHS of $(*)$ is $\frac{1}{2\cdot 2} = \frac{1}{4} = \frac{2}{8}$

So for $n=2$, we have

$$\frac{3}{8} > \frac{2}{8}$$

and $(*)$ holds for $n=2$.

Ex 2: Inductive Step

Assume that (*) is true for some $n \geq 2$

show this implies it is true for $n+1$.

Replacing n by $n+1$ in the LHS of (*):

$$\frac{1}{2^{2(n+1)}}\binom{2(n+1)}{n+1} = \frac{1}{2^{2n+2}}\binom{2n+2}{n+1}$$

$$= \frac{1}{2^{2n+2}}\left(\frac{(2n+2)!}{(n+1)!\cdot(n+1)!}\right)$$

$$= \frac{1}{2^{2n+2}}\frac{(2n+2)(2n+1)(2n)!}{(n+1)\cdot n!\cdot(n+1)\cdot n!}$$

$$= \frac{1}{2^{2n+2}}\cdot\frac{2(2n+1)(2n)!}{(n+1)(n!)^2}$$

$$= \frac{1}{2^{2n+1}}\frac{(2n+1)}{(n+1)}\binom{2n}{n}$$

$$> \frac{1}{2}\frac{(2n+1)}{(n+1)}\cdot\frac{1}{2n}$$

$$= \frac{(2n+1)}{2(n+1)}\cdot\frac{1}{2n}$$

$$= \frac{1}{2(n+1)}\cdot\left(1+\frac{1}{2n}\right)$$

$$> \frac{1}{2(n+1)}$$

and (*) is true for $n+1$.

Therefore, by the principle of mathematical induction,

(*) is true for all $n \geq 2$.

Ex 3: Let $\{a_n\}_{n\geq 0}$ be the sequence of real numbers satisfying the discrete initial value problem

$$a_n = a_{n-1} + 2a_{n-2} + 3a_{n-3} + \cdots + n a_0 + 1, \qquad n \geq 1$$

$$a_0 = 1$$

(called a Full-History Recurrence Relation)

(a). Compute a_1, a_2, a_3, a_4, a_5.

(b). Conjecture the value of a_n by looking for a pattern.

(c) Use induction to prove your conjecture is correct.

(a) $a_0 = 1$

$a_1 = a_0 + 1 = 1 + 1 = 2$

$a_2 = a_1 + 2a_0 + 1 = 2 + 2 + 1 = 5$

$a_3 = a_2 + 2a_1 + 3a_0 + 1 = 5 + 4 + 3 + 1 = 13$

$a_4 = a_3 + 2a_2 + 3a_1 + 4a_0 + 1 = 13 + 10 + 6 + 4 + 1 = 34$

$a_5 = a_4 + 2a_3 + 3a_2 + 4a_1 + 5a_0 + 1 = 34 + 26 + 15 + 8 + 5 + 1 = 89$

(b) Recall: The Fibonacci sequence:

F_0	F_1	F_2	F_3	F_4	F_5	F_6	F_7	F_8	F_9	F_{10}	F_{11}	
0,	1,	1,	2,	3,	5,	8,	13,	21,	34,	55,	89,	...
	↑ a_0		↑ a_1		↑ a_2		↑ a_3		↑ a_4		↑ a_5	

and it looks like $a_n = F_{2n+1}$ for all $n \geq 0$.

Ex 3:

(c). What is the recurrence relation satisfied by F_{2n+1} ?

For $n \geq 1$

$$F_{2n+3} = F_{2n+2} + F_{2n+1} = (F_{2n+1} + F_{2n}) + F_{2n+1}$$

$$= 2F_{2n+1} + F_{2n}$$

$$= 2F_{2n+1} + (F_{2n+1} - F_{2n-1})$$

$$= 3F_{2n+1} - F_{2n-1}$$

Therefore, the sequence $\{F_{2n+1}\}_{n\geq 0}$ satisfies the discrete initial value problem as the odd-indexed Fibonacci ~~sequence~~ numbers

$$F_{2n+3} = 3F_{2n+1} - F_{2n-1}, \quad n \geq 1$$

$$F_1 = 1$$

$$F_3 = 2$$

- Now look at the original sequence $\{a_n\}_{n\geq 0}$.

$$a_n = a_{n-1} + 2a_{n-2} + 3a_{n-3} + \cdots + na_0 + 1 \qquad (*)$$

and replace n by $n+1$, to get

$$a_{n+1} = a_n + 2a_{n-1} + 3a_{n-2} + \cdots + (n+1)a_0 + 1 \qquad (**)$$

Now subtract $(*)$ from $(**)$ to get

$$a_{n+1} - a_n = a_n + a_{n-1} + a_{n-2} + \cdots + a_1 + a_0 \qquad (***)$$

So $\quad a_{n+1} - 2a_n = \sum_{k=0}^{n-1} a_k = a_n - a_{n-1} \quad$ from $(***)$

Ex 3: Therefore,

$$a_{n+1} = 3a_n - a_{n-1} \quad \text{for } n \geq 1$$

$$a_0 = 1$$
$$a_1 = 2$$

and the sequence $\{a_n\}_{n\geq 0}$ satisfies the same discrete initial value problem that $\{F_{2n+1}\}_{n\geq 0}$ does.

- To show that $a_n = F_{2n+1}$ for all $n \geq 0$, define

$$b_n = a_n - F_{2n+1}$$

for $n \geq 0$.

Then $b_0 = a_0 - F_1 = 0$
$b_1 = a_1 - F_3 = 0$

and

$$b_{n+1} = a_{n+1} - F_{2n+3} = (3a_n - a_{n-1}) - (3F_{2n+1} - F_{2n-1})$$

$$= 3(a_n - F_{2n+1}) - (a_{n-1} - F_{2n-1})$$

$$= 3b_n - b_{n-1}$$

So $\{b_n\}_{n\geq 1}$ satisfies

$$b_{n+1} = 3b_n - b_{n-1}, \quad n \geq 1$$

$$b_0 = 0$$
$$b_1 = 0$$

and now an easy induction proof shows that $b_n = 0$ for all $n \geq 0$, that is, $a_n = F_{2n+1}$ for all $n \geq 0$.

Spies and Acquaintances

There are two solutions to the Spies and Acquaintances problem that Dr Ecco overlooked.

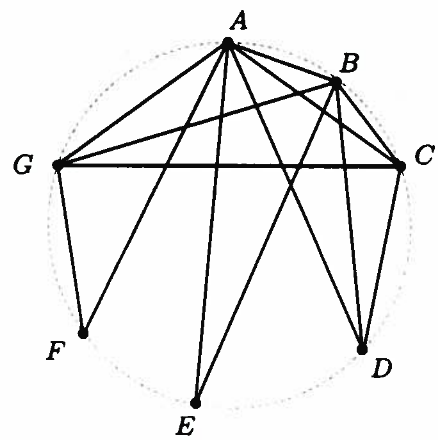
- A is telling the truth, otherwise A would claim to know fewer than he actually does, that is, the number A actually knew would be greater than A claimed.

- B is telling the truth, otherwise B knows 6 others, and then G knows A and B, which means G is also lying, contradicting the fact that there is only one liar.

So far, A, B and F are known to be telling the truth.

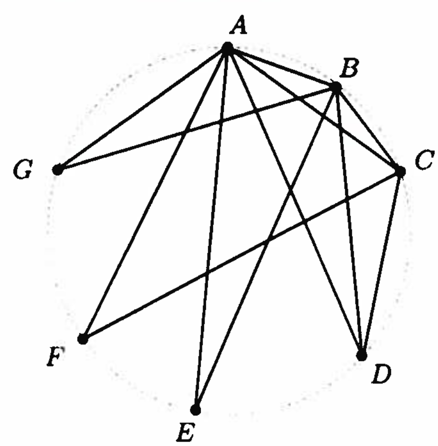
How do we proceed from here? Eventually, we come to the question: Who does F know? F knows A and one other person. In the previous solutions, F knew A and B.

Can F know G? If so, then G is the liar, so all the others are telling the truth, and B does **not** know F. So the graph looks like:



This solution is consistent with the facts since C knows two others besides A and B, and C cannot know F (F is telling the truth). Thus, C knows two of G, D, and E. However, if G is the liar, C cannot know E (E is telling the truth), and so C must know D and G. Thus, this solution is consistent with the facts.

Another solution, where F knows A and C is shown below:



Final Exam

Tuesday, Apr 23, 9:00 – 12:00

in CCIS 1-430

Covers:

1. Cryptography.
2. Induction
3. Strong Induction
4. Triminoes
5. Recurrence Relations
6. Graph Theory
7. Kids + Chocolate
8. Nim.

The exam will have between 7 and 10 problems.

(depending on how long the problems are).

Bring your One Card.