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Chap 5 in
Scarlett's Notebook

§ Graph Theory

Lecture 17

Warm up problem: **The Odd Doors Problem**. (Section 1.3 of Ecco)

Lawrence Terrence III has a problem. His recently departed father has hidden a cache of jewels in one of two underground labyrinths. Lawrence knows the following facts about the labyrinths:

- The jewels are in a room with an odd number of doors.
- Only one of the labyrinths has a room with an odd number of doors.
- One labyrinth has two doors leading to the outside and the other has three.
- His father was an amateur mathematician.

Since it will cost a small fortune to explore each labyrinth, Lawrence Terrence III wants to know which labyrinth he should search.

Solution: Parity Theorem or Handshaking Lemma.

Definition 1: A *graph* is a collection of dots and lines, with every line terminated by a dot at each end. The dots are called *vertices* and the lines are called *edges*. ~~Note we can have an edge that starts and ends at the same vertex~~; ~~also, we could have multiple edges joining the same two vertices.~~

More formally, a graph or simple graph is an ordered pair $G = (V, E)$ where V is the vertex set (contains the vertices) and E is called the edge set, that is, E is a set containing 2-element subsets of V.
If $u, v \in V$ and $\{u, v\} \in E$, means that the vertices u and v are joined by an edge.

Definition 2: An edge that is joined to a vertex is said to be *incident* to the vertex, and the vertices that are joined to an edge are *incident* to the edge. More formally, the edge that connects the vertices u and v can be written as: $e = \{u, v\}$. Here we also say that u and v are *adjacent* to each other, and that they are the *endpoints* of the edge $\{u, v\}$. Two edges incident with the same vertex are also said to be *adjacent*.

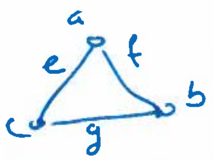
Examples:

$V = \{a\}$

$E = \emptyset$

$V = \{u, v\}$

$E = \{\{u, v\}\}$



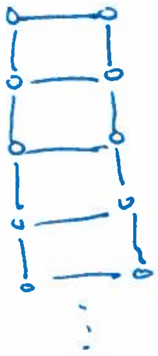
$V = \{a, b, c\}$

$E = \{\{a, b\}, \{a, c\}, \{b, c\}\}$

- e is incident to vertices a and c.
- f is incident to a and b.
- g is incident with b and c

Note: In a simple graph G = (V, E)
the vertex set need *not* *be* *finite*

G :



If the vertex set is finite, then we say it is a
finite graph.

Note: If in a "Graph" we have *multiple edges*,
that is, there exist two vertices say u and v such
that

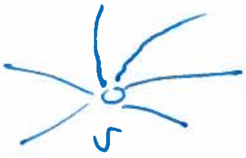
there are multiple edges
joining u and v, then the "graph" is called a *multigraph*.

If there is an edge joining a vertex to itself,
that is, a *loop*.

then the "graph" is called a *pseudograph*.

- Given a graph $G = (V, E)$, (finite simple graph) the degree of a vertex $v \in V$ is the number of edges that are incident with v.

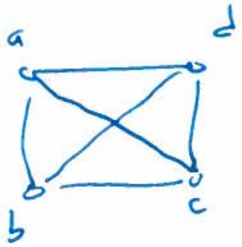
Ex:



In this case, $\deg v = 6$.

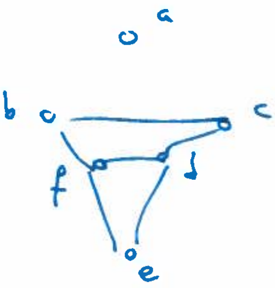
Ex:

G:



$\deg a = \deg b = \deg c = \deg d = 3$

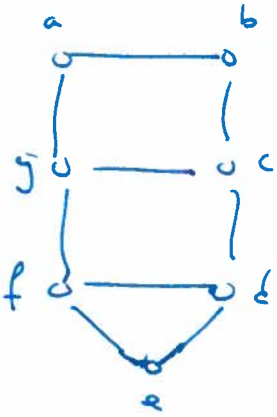
Ex:



$\deg a = 0$
$\deg b = 2$
$\deg c = 2$
$\deg d = 3$ ←
$\deg f = 3$ ←
$\deg e = 2$

Note: There are only two vertices with odd degree, i.e. the # of vertices with odd degree is even.

Ex:



$\underbrace{\deg c = \deg g = \deg d = \deg f}_{\text{even } \#} = 3$

Theorem: (Parity Theorem)

In a finite simple graph, the sum of the degrees of all the vertices is twice the number of edges, that is, if $G = (V, E)$, then

$$\sum_{v \in V} \deg(v) = 2|E|$$

$|E| = \#$ of edges.

proof:

Each edge contributes 2 to the sum of the degrees.

∎

Theorem (Handshaking Lemma)

In any finite simple graph $G = (V, E)$, the number of vertices with odd degree is even.

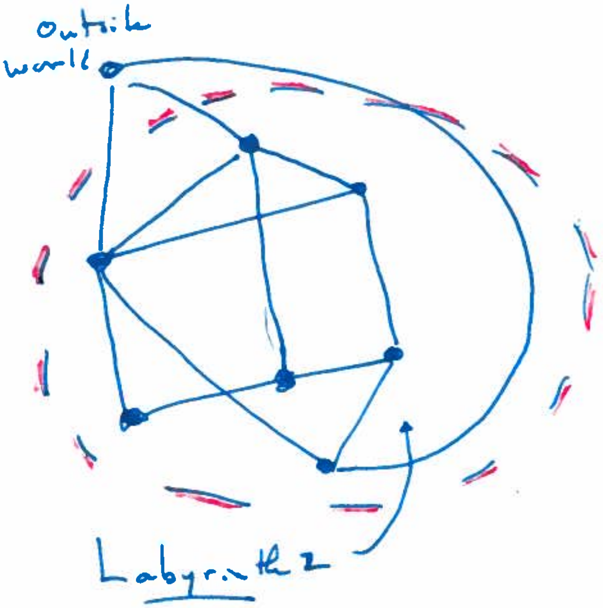
proof By the parity theorem:

$$\sum_{v \in V} \deg(v) = \underset{\overset{\uparrow}{\text{this is even}}}{\sum_{\substack{v \in V \\ \deg(v)\text{ even}}} \deg(v)} + \underset{\overset{\uparrow}{\text{so, this must also be even.}}}{\sum_{\substack{v \in V \\ \deg(v)\text{ odd}}} \deg(v)} = \underset{\overset{\uparrow}{\text{this is even}}}{2|E|}$$

∎

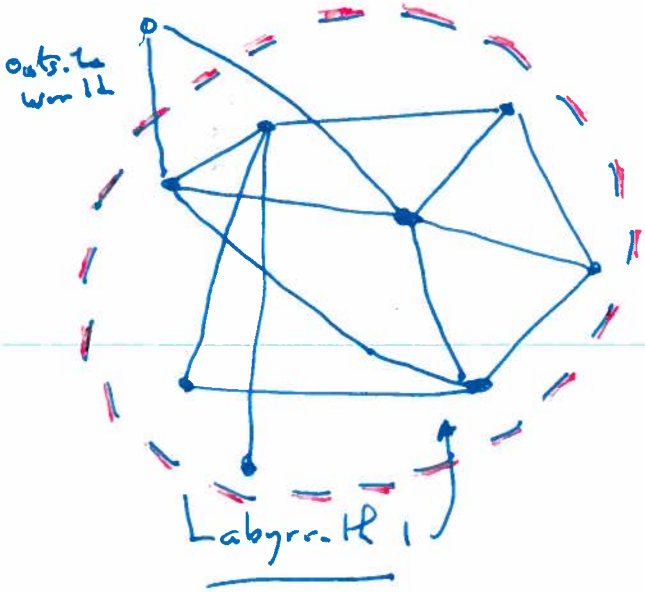
Ex 1: Solve the odd doors problem:

Represent the Labyrinths using a graph.
A vertex corresponds to a room and an edge corresponds to a door between the two rooms.



The father was an amateur mathematician.

He guaranteed his son that one of the labyrinths has an odd number of [redacted] doors

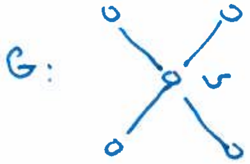


Answer: Janet's one is: Labyrinth 2.

Recall: Graph ≡ finite simple graph $G = (V, E)$

V is the vertex set, E is the edge set.

Definition 3: In a graph, the *degree* of a vertex v is the number of edges incident to v, and is denoted $deg(v)$. Each edge contributes 1 to the degree of each of the two vertices incident with it.



$\deg(v) = 4.$

Theorem 1 (The Parity Theorem): The sum of the degrees of all vertices of a graph is equal to twice its number of edges. $\sum_{v\in V}\deg(v) = 2|E|$

Proof. Each edge contributes 2 to the total degree.

The Parity Theorem is also called the Handshaking Lemma and is stated as follows:

Theorem 2 (The Handshaking Lemma): Every graph has an even number of vertices of odd degree.

~~**Example 1: Solve the Odd Doors Problem:**~~

Proof:

$$\sum_{v\in V}\deg(v) = \underset{\overset{\uparrow}{\text{this is even}}}{\sum_{\substack{v\in V\\ \text{even}\\ \text{degree}}}\deg(v)} + \sum_{\substack{v\in V\\ \text{odd}\\ \text{degree}}}\deg(v) = \underset{\overset{\uparrow}{\text{this is even}}}{2|E|}$$

Therefore, $\sum_{\substack{v\in V\\ \text{odd}\\ \text{degree}}}\deg(v)$ is even.

Therefore, the # of vertices with odd degree is even.

∎

Example 2: The University of Two Hills has 25 professors each with a telephone. If any professor collaborates over the phone with more than 5 others confusion spreads through the university and nothing gets done. To maximize collaboration without risking confusion they have decided to connect each phone to exactly 5 other phones. Can this be done?

Represent this by a graph:

Vertices $\equiv$ phones

Edges $\equiv$ phones that are connected

As a finite simple graph $G = (V, E)$

$V = 25$, each with odd degree $\deg(v) = 5$ for each $v \in V$.

So the # of vertices of odd degree is odd.

This can't happen in a graph, so it can't be done.

Example 3: In the Kingdom of Glee roads do not intersect nor do they lead to dead ends.

a) If there are 100 cities, and four roads lead out of each city, how many roads are there altogether in the kingdom?

b) If 3 roads lead out of each city can the kingdom have exactly 100 roads?

Represent this by a graph.

Cities $\equiv$ Vertices $\quad G = (V, E)$

Roads $\equiv$ Edges.

(a). Use Parity theorem

$$2|E| = \sum_{v \in V} \deg(v) = 100 \cdot 4$$

So $|E| = 200$.

(b). If $\deg(v) = 3$ for each $v \in V$, can $|E| = 100$

use the parity theorem

$$2 \cdot 100 = \sum_{v \in V} \deg(v) = 3 \cdot |V|, \quad \text{i.e. } 200 = 3|V|.$$

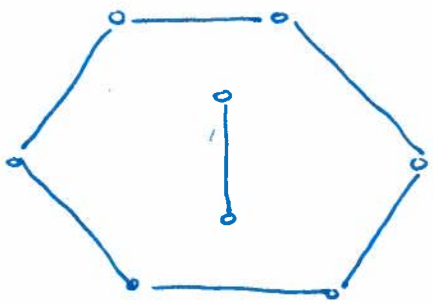
$\not\equiv$ This is a contradiction since $200 \not\equiv 0 \pmod 3$

Answer: No!

- Given a finite simple graph $G = (V, E)$ a subgraph of G is a graph $H = (V', E')$ where $V' \subseteq V$ and $E' \subseteq E$.

Ex:

G:

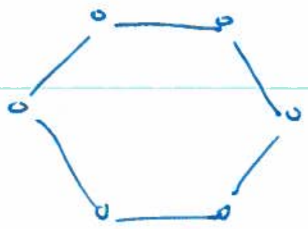


The following are subgraphs of G:

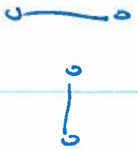
H_1:

H_2:

H_3:



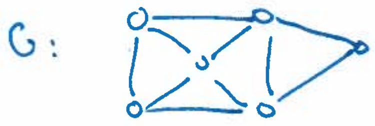
H_4:

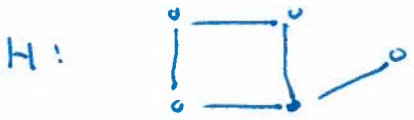


H_5:

H_6:

Definition 4: A *subgraph* of a graph is a subset of its vertices and edges, provided that all vertices incident with edges in the subgraph are included. In other words, a subgraph is a subset of the vertices and edges that itself forms a graph.

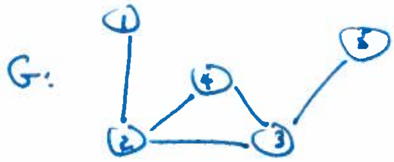




H is a subgraph of G.

Certain types of subgraphs have specific names:

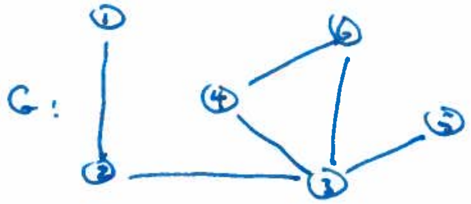
Definition 5: A *walk* is a subgraph that consists of a sequence of vertices and edges $v_0, e_1, v_1, e_2, v_2, \ldots, e_n, v_n$ such that for $1 \le i \le n$ the edge e_i joins vertices v_{i-1} and v_i .



Walk:

1 → 2 → 3 → 4 → 2 → 3 → 5

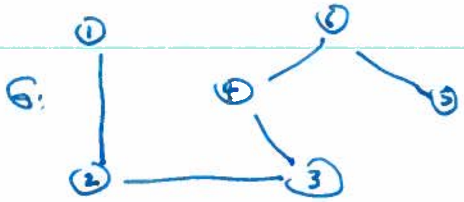
Definition 6: A *trail* is a walk in which no edges are repeated.



Trail:

1 → 2 → 3 → 4 → 6 → 3 → 5

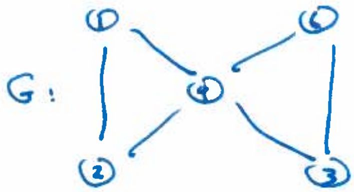
Definition 7: A *path* is a trail in which no vertices are repeated except perhaps for the first and last vertex.



Path

1 → 2 → 3 → 4 → 6 → 5

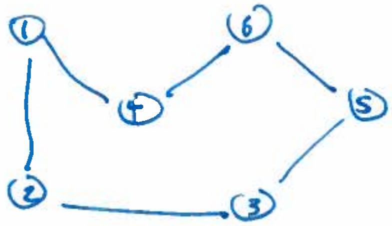
Definition 8: A *circuit* is a trail that's first and last vertices are the same.



Circuit:

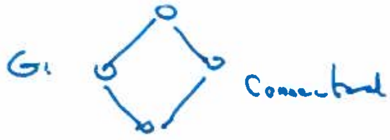
1 → 2 → 4 → 3 → 6 → 4 → 1

Definition 9: A *cycle* is a path that's first and last vertices are the same.



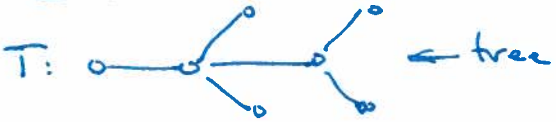
Cycle:

1 → 2 → 3 → 5 → 6 → 4 → 1



Definition 10: Two vertices of a graph that are joined by a path are said to belong to the same *component* of the graph. If the whole graph is one component, then it is said to be *connected*.

Definition 11: A *tree* is defined as a graph T such that for any two vertices u and v in T, there is exactly one path which joins u and v.



Definition 12: A collection of disjoint trees is called a *forest*.

F is a forest. has three components, each of which is a tree.

Theorem 3: (Trees). A tree has the following three properties.

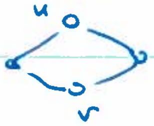
1. It is connected.
2. It has no cycles.
3. It satisfies the Tree Formula $V = E + 1$, where V and E are the numbers of vertices and edges respectively.

Proof.

Given a tree $T = (V, E)$

(1) It is connected, since is a tree there is exactly one path connecting any two vertices.

(2). If T had a cycle:



then there are two paths joining u and v. which is a contradiction. So T has no cycles.

(3). Consider vertices as beads joined by strings. Pick up 1 vertex, then it is connected so everything comes up. No cycles, so every string has exactly one bead at end of it, so $|V| = |E| + 1$. ∎

Theorem 4: (Two out of three is a tree). A graph is a tree if it has any two of the following properties:

1. It is connected.
2. It has no cycles.
3. It satisfies the Tree Formula $V = E + 1$, where V and E are the numbers of vertices and edges respectively.

Proof Exercise. Show

$$\left.\begin{array}{l} 1+2 \Rightarrow \text{tree} \\ 1+3 \Rightarrow \text{tree} \\ 2+3 \Rightarrow \text{Tree} \end{array}\right\}$$

See: Notes on Trees on my webpage.

Read: Ecco: Spies + Acquaintances (2.1).

Example 4: (Ecco 2.1) The police have captured seven criminals Al, Bob, Carl, Dan, Ed, Frank, and Gary. When questioned by the police, Al admitted to having known all of the other six criminals. Bob admitted to having known five, Carl admitted to having known four, Dan to having known three, Ed to having known two, Frank to having known two, and Gary to having known one. Represent this by a graph: $V = \{A, B, C, D, E, F, G\}$; edge iff X knows Y

Degree Sequence:

A	B	C	D	E	F	G
6	5	4	3	2	2	1

a) Is it possible that all seven criminals are telling the truth? Explain. If they all tell the truth,

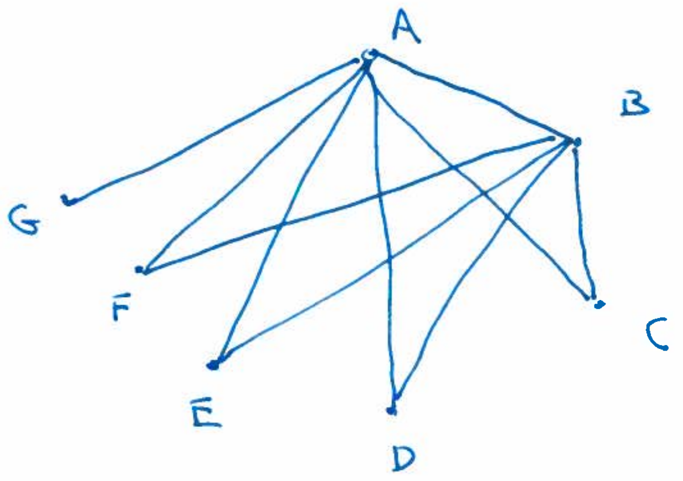
From the handshaking lemma: The # of vertices with odd degree must be even.

But only B, D, G have odd degree, and this can't happen.

So one of them is lying.

b) Suppose we know that there is only at most one liar and that no criminal would say that they know more people than they actually do. Further we know Frank is not lying. Which of the other criminals could possibly be lying?

- A tells the truth $\deg(A) \geq 6$ and $\deg(A) \leq 6$.
- F tells the truth, by hypothesis.
- B tells the truth. $\deg(B) \geq 5$ and if B is lying, then he would know 6 others. This implies that G knows A and B, so G is also lying. This contradicts the fact that there is exactly one liar.
- If E and G both tell the truth, since $\deg(G) = 1$, then B is connected to A, C, D, E, F.



Degree Sequence

A	B	C	D	E	F	G
6	5	4	3	2	2	1

- If E and G both tell the truth, then C or D is lying
  - if C lies, then C is connected to 5 or more vertices this is a contradiction, since F tells the truth.
  - if D lies, D is connected to 4 or more vertices this is a contradiction since F tells the truth and E and G both tell the truth

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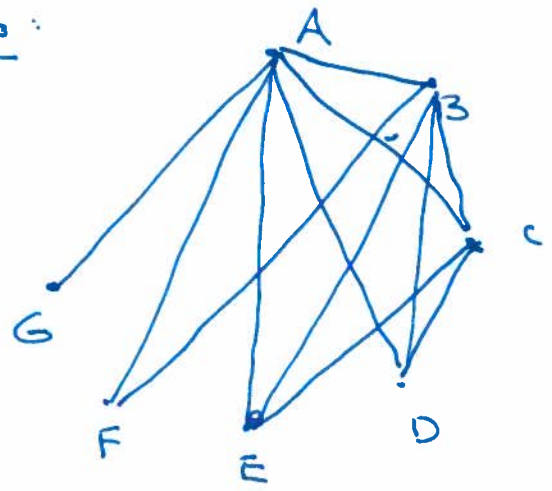
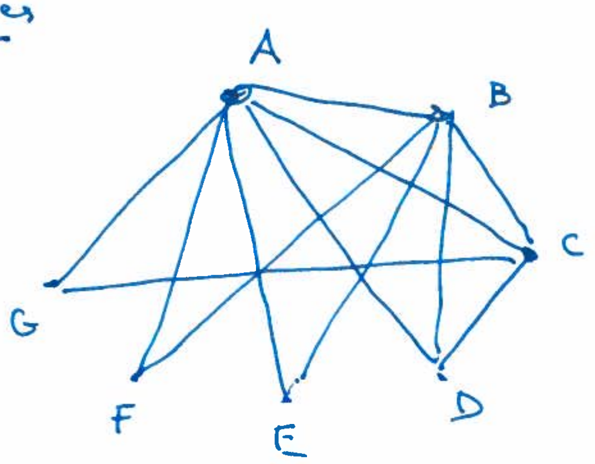
- Therefore one of E or G is lying.

- If E lies:



This is the solution if E is lying.

- If G lies



Degree sequence

A	B	C	D	E	F	G
6	5	4	3	2	2	1

This is the solution if G is lying.

- Read Trevor's solution.

Example 5: Kids and Chocolate bars. What are the possible values of $n > 9$ such that n children can equally share 9 identical chocolate bars, with the restriction that no bar be cut into more than two pieces.

Step 1. Think about different possibilities for n.

Step 2. Represent the problem as a graph.

Step 3. Explain why this graph has no cycles.

Step 4. Explain why this graph is a tree or a forest.

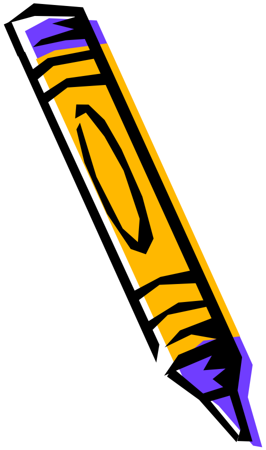
Step 5. Find all the possible choices for n.

Solution on Following Pages

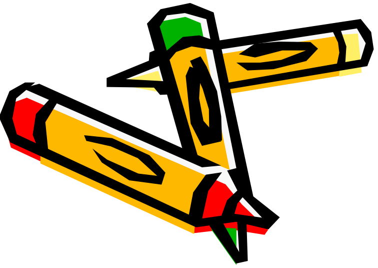
What is the Problem?

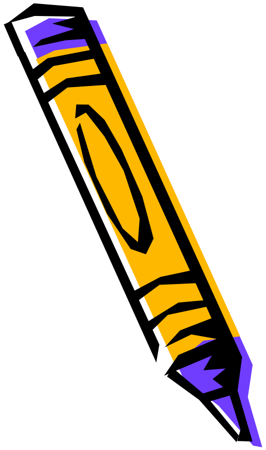
- What are the values of $n > 9$ such that n children can share 9 identical chocolate bars equally, with the restriction that no bar be cut into more than two pieces ?

What are the solutions?



- The first solution is easy to find.
- n = 18: Each of 18 children gets half a chocolate bar



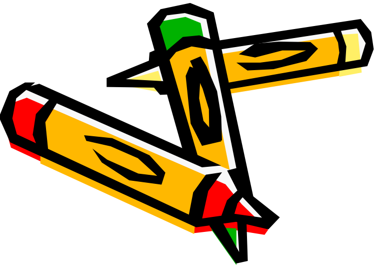


n = 18: Each of 18 children gets 1/2 of a chocolate bar.

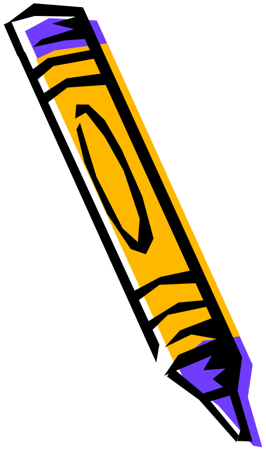
(1/2, 1/2), (1/2, 1/2), (1/2, 1/2)

(1/2, 1/2), (1/2, 1/2), (1/2, 1/2)

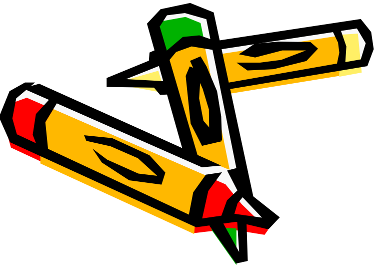
(1/2, 1/2), (1/2, 1/2), (1/2, 1/2)



Are there other solutions?



- A second solution is not quite as easy to find.
- n = 10: Each of 10 children gets 9/10 of a chocolate bar.



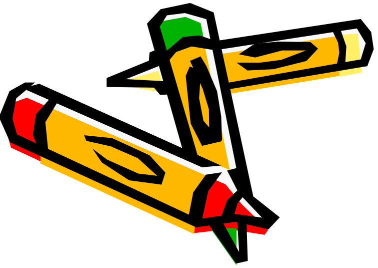
n = 10:

Each of 9 children gets
9/10 of a chocolate bar,
10th child gets $9 \cdot 1/10$.

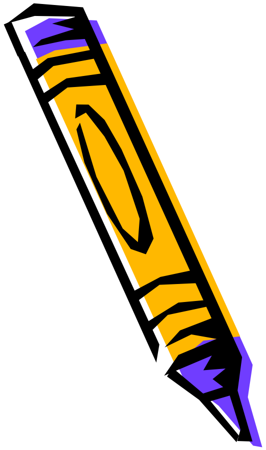
(9/10, 1/10), (9/10, 1/10), (9/10, 1/10)

(9/10, 1/10), (9/10, 1/10), (9/10, 1/10)

(9/10, 1/10), (9/10, 1/10), (9/10, 1/10)



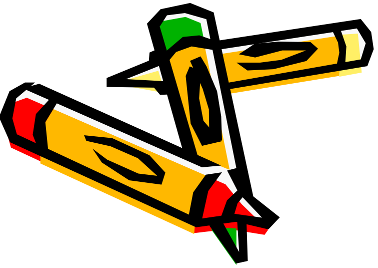
n = 10: Another Solution. Not all bars have to be divided the same way.



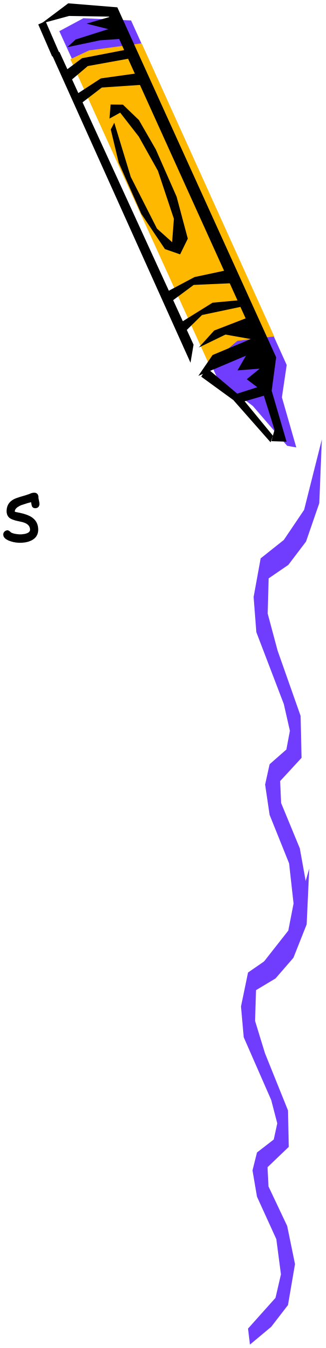
(9/10, 1/10), (8/10, 2/10), (7/10, 3/10)

(6/10, 4/10), (5/10, 5/10), (4/10, 6/10)

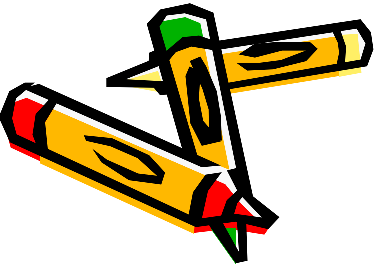
(3/10, 7/10), (2/10, 8/10), (1/10, 9/10)

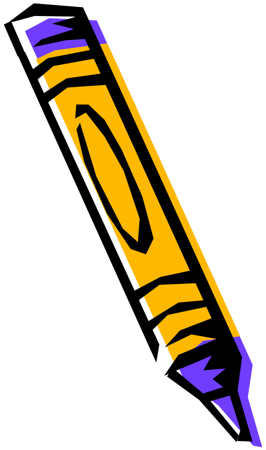


Are there any other solutions?



- A third solution is also not quite as easy to find.
- n = 12: Each of 12 children gets 3/4 of a chocolate bar.



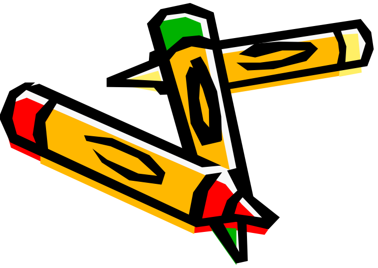


n = 12: Each of 9 children gets 3/4 of a chocolate bar,
remaining 3 children each get 3•1/4 of a chocolate bar.

(3/4, 1/4), (3/4, 1/4), (3/4, 1/4)

(3/4, 1/4), (3/4, 1/4), (3/4, 1/4)

(3/4, 1/4), (3/4, 1/4), (3/4, 1/4)

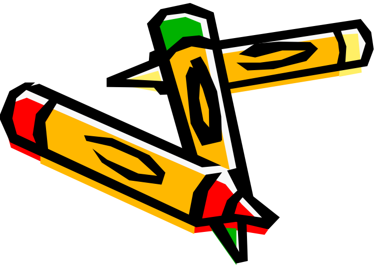


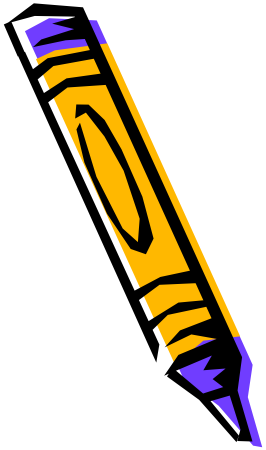
$n = 12$: Another Solution. Again, not all bars have to be divided in the same way.

(3/4, 1/4), (1/2, 1/2), (1/4, 3/4)

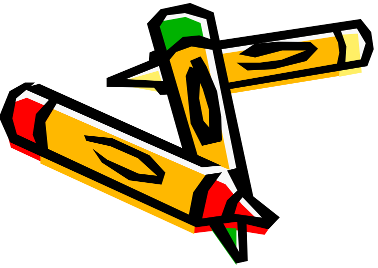
(3/4, 1/4), (1/2, 1/2), (1/4, 3/4)

(3/4, 1/4), (1/2, 1/2), (1/4, 3/4)

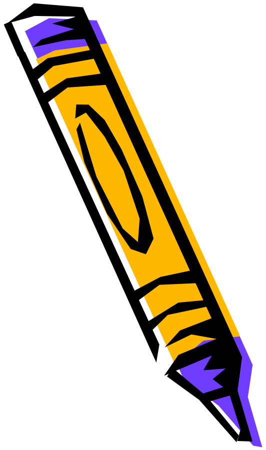




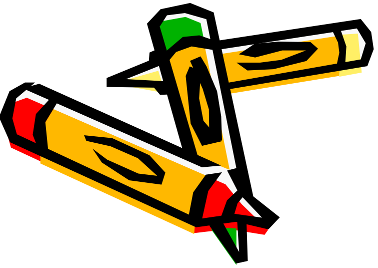
These are the ONLY solutions!!!



Why are these the only solutions?



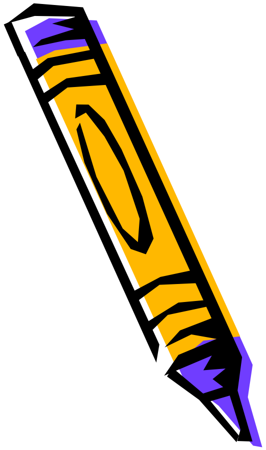
- We know there are no solutions for n > 18.
- So how do we show 10, 12, and 18 are the only solutions?

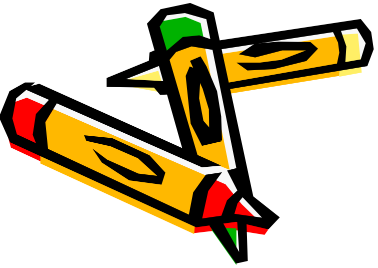


We represent the solutions using a GRAPH:

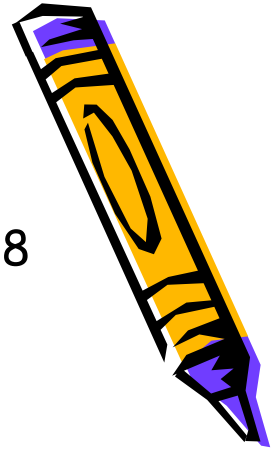
- The vertices of the graph are the n children.

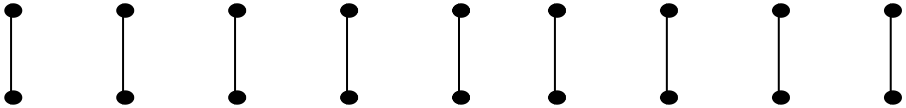
- There is an edge between two vertices if and only if the corresponding two children share a chocolate bar.



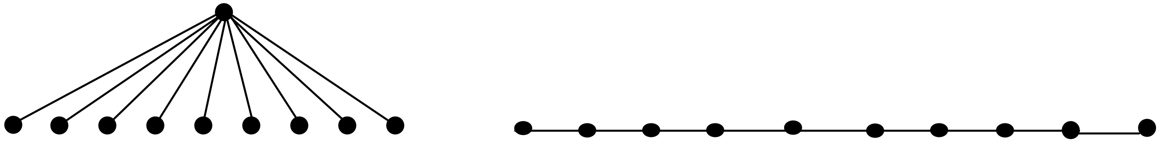


The solutions found so far:

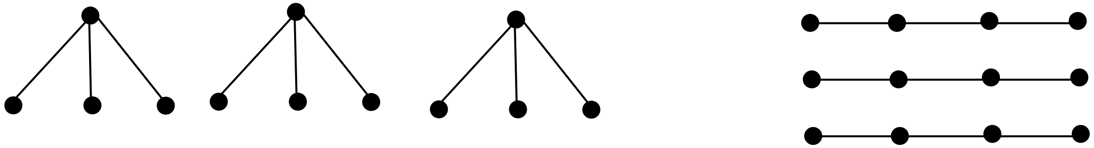




Graph n = 18

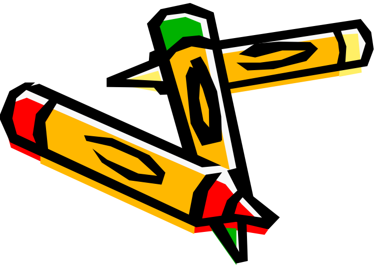


Graph n = 10



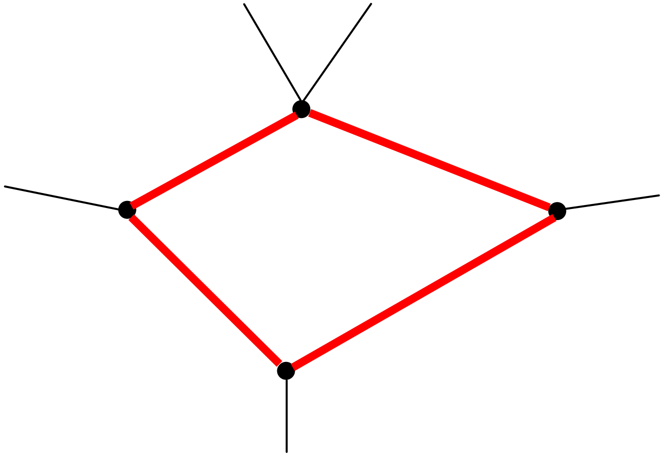
Graph n = 12

are all **forests**, that is, disjoint collections of **trees**.



Now we can show why n = 10, 12, and 18 are the only
solutions.

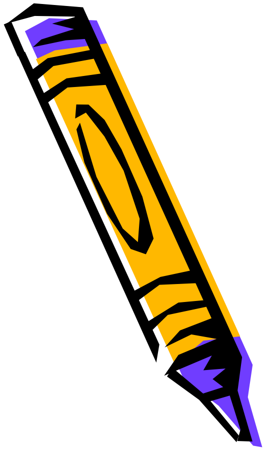
Suppose we have a solution with n > 9 vertices and 9 edges,
and suppose the graph contains a cycle.



4 children share at least 4 bars

For example, as in the figure, each of the children (vertices) in the cycle gets at least a full chocolate bar, which is not possible, since n > 9.

The graph is a Forest!

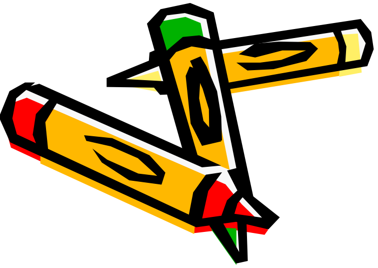


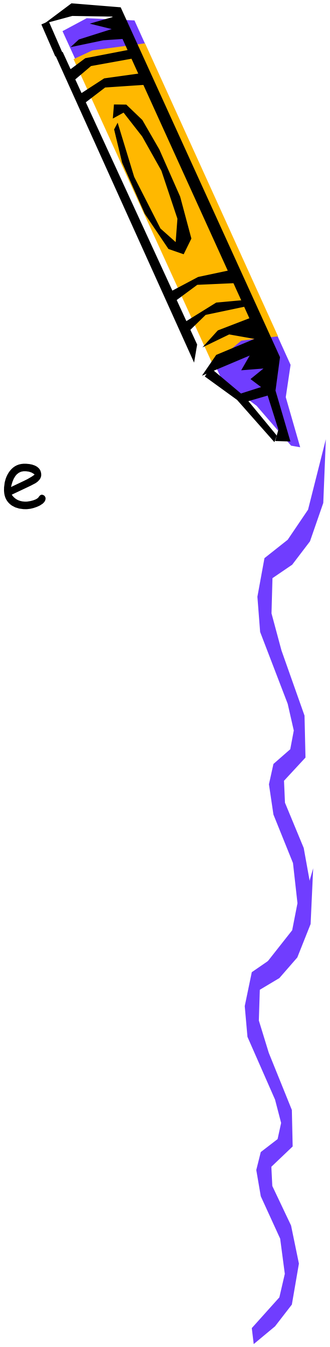
Now note that each tree in the forest contains the same number of edges:

If one tree has e edges and another has f edges, since each child gets same amount, then

$$e/(e+1) = f/(f+1)$$

and therefore $e = f$.

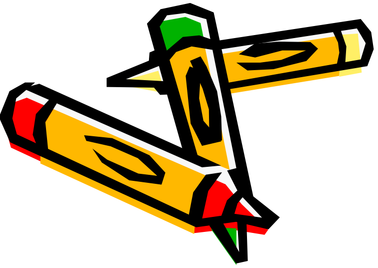


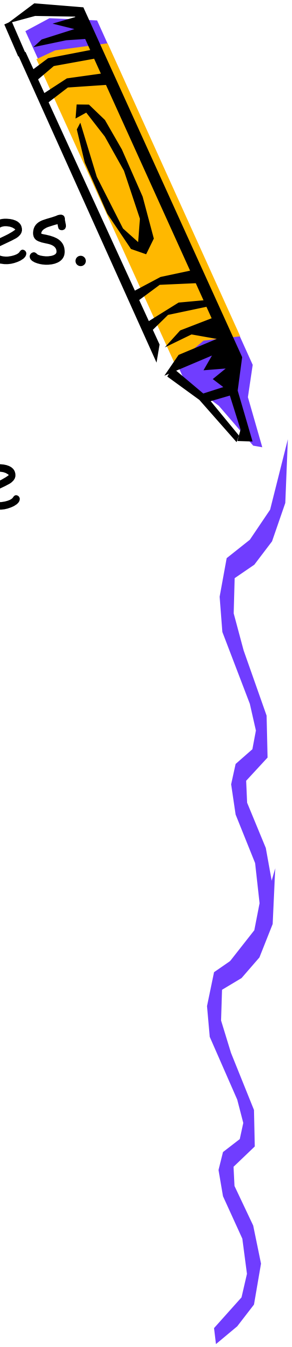


If there are k trees in the forest, and each tree has e edges, then the total number of edges is

$$k \cdot e = 9,$$

that is, the number of trees is a divisor of 9.

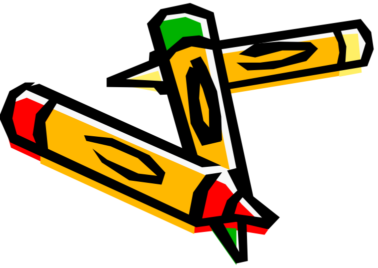




If $k = 1$, there is one tree in the forest with 9 edges and $n = 10$ vertices.

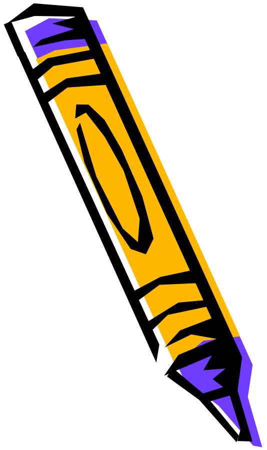
If $k = 3$, there are three trees in the forest, each with $9/3 = 3$ edges and $3 + 1 = 4$ vertices, so $n = 3 \cdot 4 = 12$

If $k = 9$, there are nine trees in the forest, each with $9/9 = 1$ edge and



$1 + 1 = 2$ vertices,

so $n = 9 \cdot 2 = 18$.



THAT'S ALL FOLKS !!!