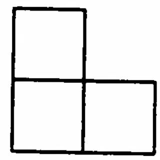
Lecture 16

The Trominoes

Definition: A (*right*) *tromino* is an object made of three unit squares as shown:



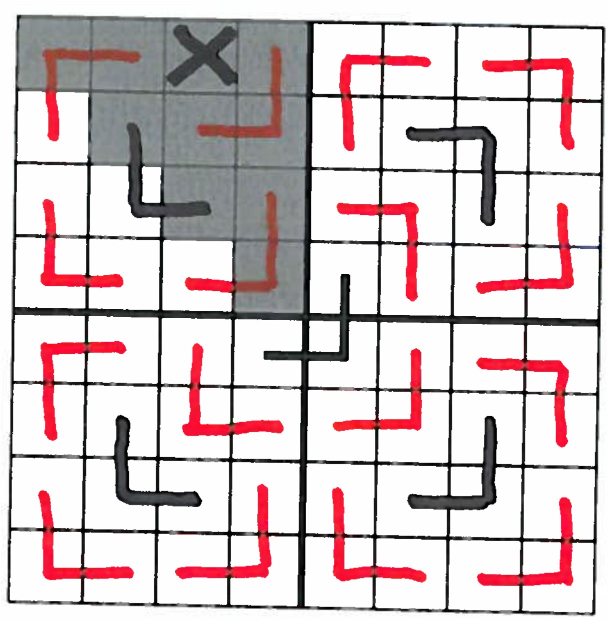
a tromino may appear as above, or it may be rotated through some multiple of 90°.

Definition: A board is called *deficient* if one unit squares is missing.

Definition: A board is said to be *tiled* if you can fit individual tiles together with no gaps or overlaps to fill the board.

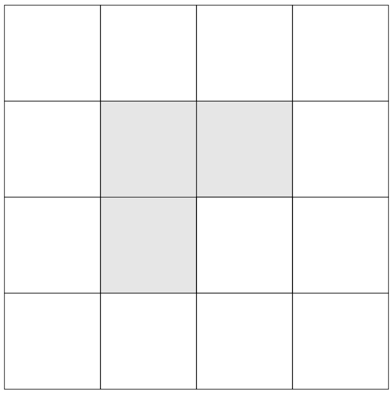
Note 1: Suppose we are given a deficient $n \times n$ board. We can rotate the board so the missing square is in the top left quadrant. Further we could reflect the board along its main diagonal. Using this symmetry we only need to look at cases where the missing square above or on the main diagonal (see the example for the 8×8 board below).

Note 2: Given a deficient 8×8 board, by symmetry we need only look at cases where the missing square is in the shaded region below. Notice that by dividing the board into four quadrants and placing one tromino in the center as shown we have four deficient 4×4 boards. Next, divide the board into sixteen 2×2 boards and place four trominoes to make each one deficient. Finally we can complete the tiling of the 8×8 board by tiling the sixteen deficient 2×2 boards.

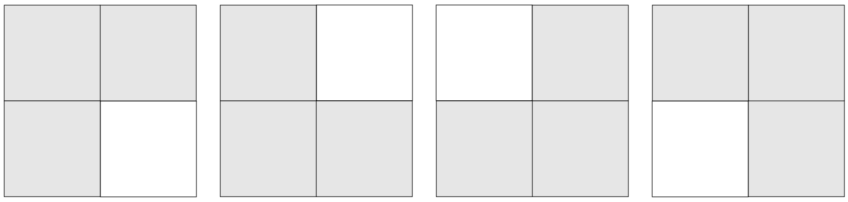


Example 1. Let n be a positive integer. Show that any $2^n \times 2^n$ chessboard with one square removed can be tiled using L–shaped pieces, where these pieces cover three squares at a time.

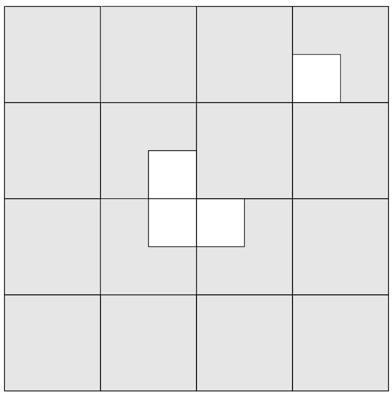
Solution: The geometry of the L–shaped pieces is as shown in the figure.



The figure below shows that any $2^1 \times 2^1$ chessboard with one square removed can be covered by a single L–shaped piece., so the result is true for $n = 1$.



Now assume that the result is true for some $n \geq 1$, if we divide a $2^{n+1} \times 2^{n+1}$ chessboard in half horizontally and vertically, we get four $2^n \times 2^n$ chessboards, exactly one of which has a square removed. From each of the other $2^n \times 2^n$ chessboards, remove the square that touches the center of the $2^{n+1} \times 2^{n+1}$ chessboard. By the induction hypothesis, each of the four $2^n \times 2^n$ chessboards with one square removed can be covered by L–shaped pieces, and now we cover the three center pieces with one more L–shaped piece.

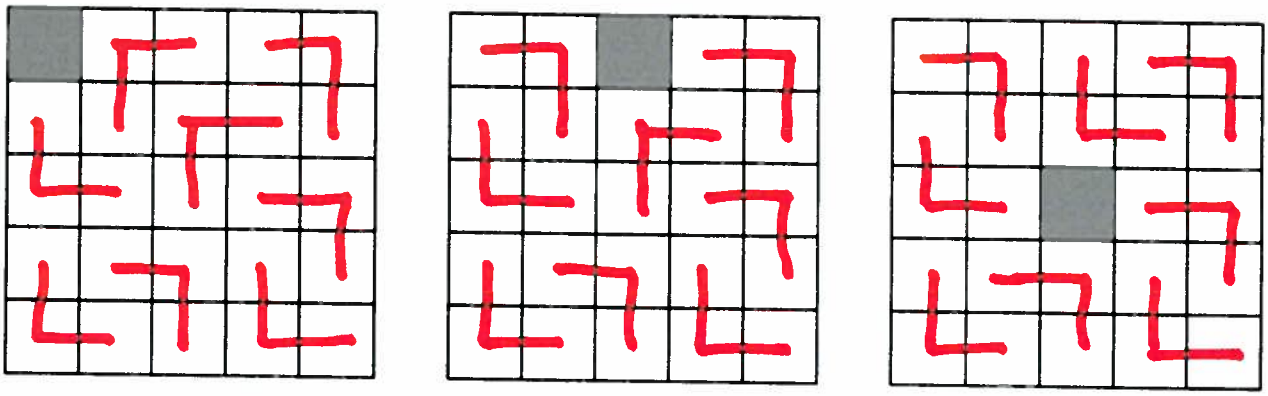


So the result is true for $n + 1$. By the principle of mathematical induction, it is true for all $n \geq 1$.

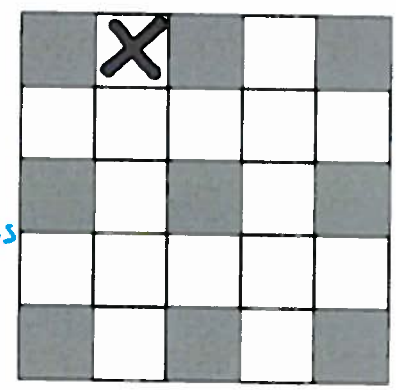
<u>The Deficient 5 × 5 Boards</u>

Example 2:

a) Tile the following deficient 5 × 5 boards with trominoes.



b) From the boards above, using symmetry, we know that a deficient 5 × 5 board made by removing one of the black squares below can be tiled with trominoes. Show that a deficient 5 × 5 board made by removing one of the white squares below cannot be tiled with trominoes.



Suppose one white square is missing.
The 9 black squares need exactly 9 trominoes to cover them.

But, on a deficient 5x5 board, in terms of area there is room for only

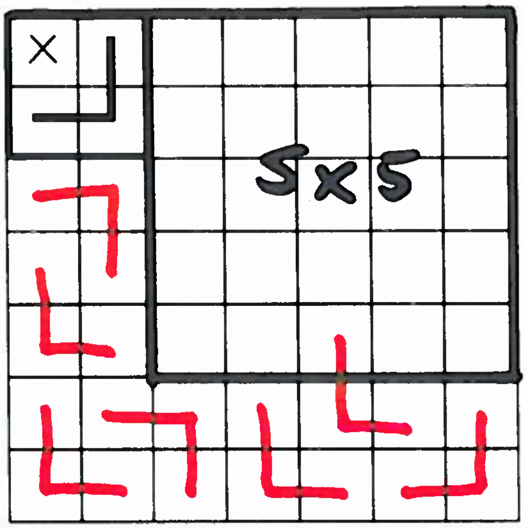
$$\frac{5^2-1}{3} = \frac{24}{3} = 8 \text{ trominoes.}$$

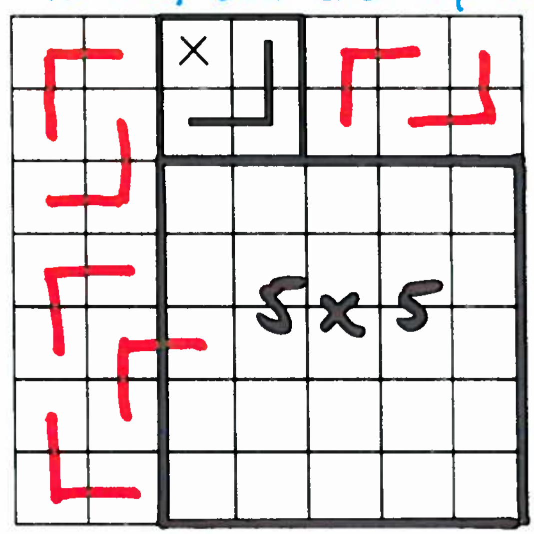
Therefore the above board with one white square missing can not be tiled.

The Deficient 7 × 7 Boards

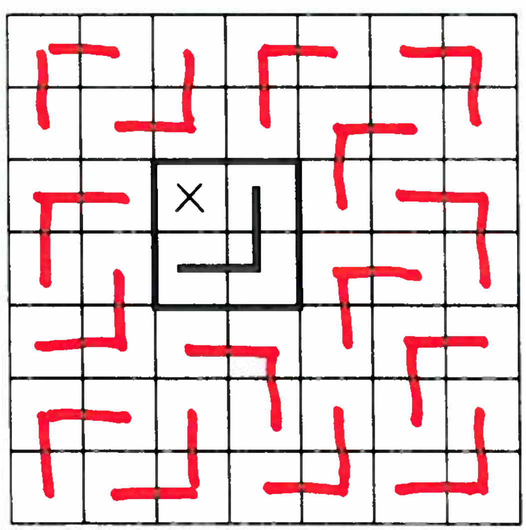
Example 3: By the symmetry described in note 1, the deficiency on a 7 × 7 board is in one of the three bold 2 × 2 sections below. Each deficient 2 × 2 section can be tiled with one tromino. After the deficient 2 × 2 sections are tiled they can be rotated to cover every missing square that needs to be considered. Tile the remaining squares to show any deficient 7 × 7 board can be tiled by trominoes.

- By symmetry, we need only consider cases where missing square is in of bold 2x2 squares





- The first two tilings can be completed using Example 2.
- By rotating each 2x2 bold square, we get all the cases.



Therefore, any deficient 7x7 board can be tiled by trominoes.

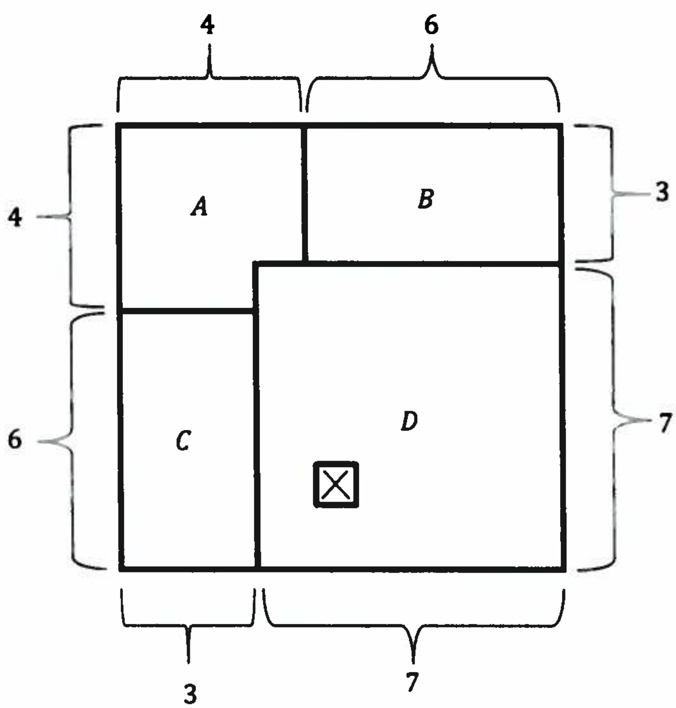
Proposition 1: Given that $n \equiv 0 \pmod 3$ and $m \equiv 0 \pmod 2$ an $n \times m$ board can be tiled with trominoes.

Proof. Any such board can be tiled by 2×3 blocks made by two trominoes.

<u>The Deficient 10 × 10 Boards</u>

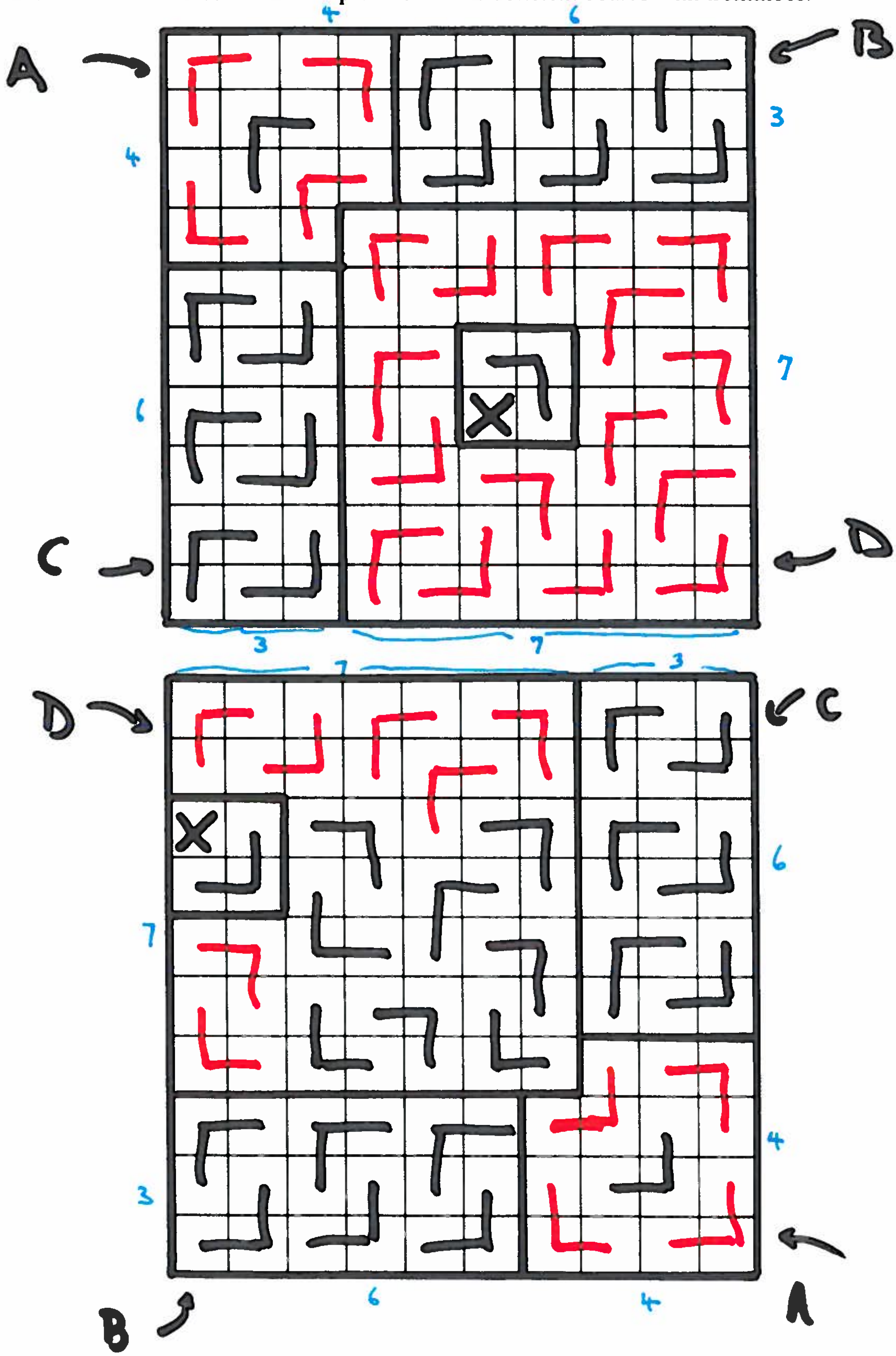
Example 4: Show that any deficient 10×10 board can be tiled by trominoes. To do this, start by splitting the board into the four sections shown below. By symmetry we can assume the deficiency is always in section D. Explain why each section can be tiled.

- A can be tiled by Example 1.
- D can be tiled by Example 3.
- B and C can be tiled by Proposition 1.



Therefore, any deficient 10×10 board can be tiled by trominoes.

Example 5: Remove any single square from each of the 10×10 boards below to make two deficient 10×10 boards. Use example 4 to tile the deficient boards with trominoes.



Proposition 2: If $n \equiv 1 \pmod 3$ a deficient $n \times n$ board can be tiled with trominoes.

Proof. Let P_n be the statement that "if $n \equiv 1 \pmod 3$ then a deficient $n \times n$ board can be tiled by trominoes."

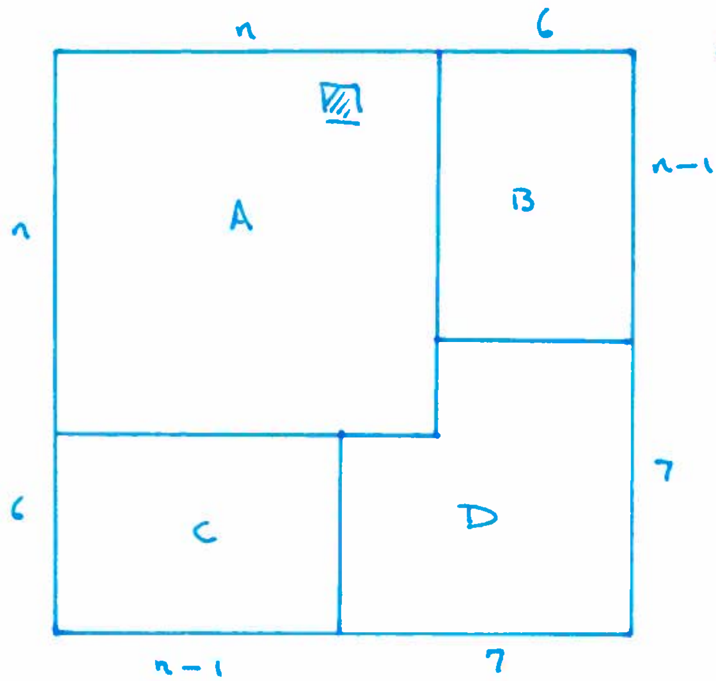
We will prove by induction that P_n is true for all $n \geq 1$.

Base Cases: P_1, P_4, P_7, P_{10} are true by examples 1, 3, 4

Induction Step: we will show $P_n \Rightarrow P_{n+6}$ for $n \geq 7$.

Given an $(n+6) \times (n+6)$ board with $n \equiv 1 \pmod 3$ and $n \geq 7$.

Divide the board as follows:



• By symmetry, we can assume the missing tile is in A.

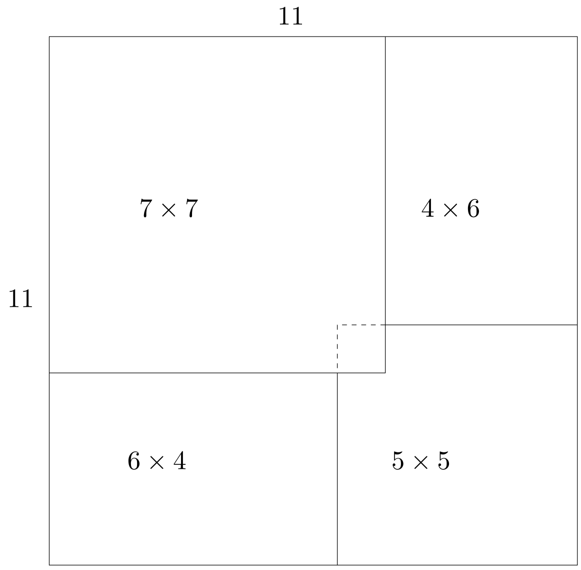
- A can be tiled since we are assuming P_n is true.
- B and C can be tiled since $n-1 \equiv 0 \pmod 3$ and $6 \equiv 0 \pmod 2$
- D can be tiled by Example 3.

So the $(n+6) \times (n+6)$ deficient board can be tiled by trominoes.

Therefore, by the principle of strong induction, if $n \equiv 1 \pmod 3$ a deficient $n \times n$ board can be tiled by trominoes.

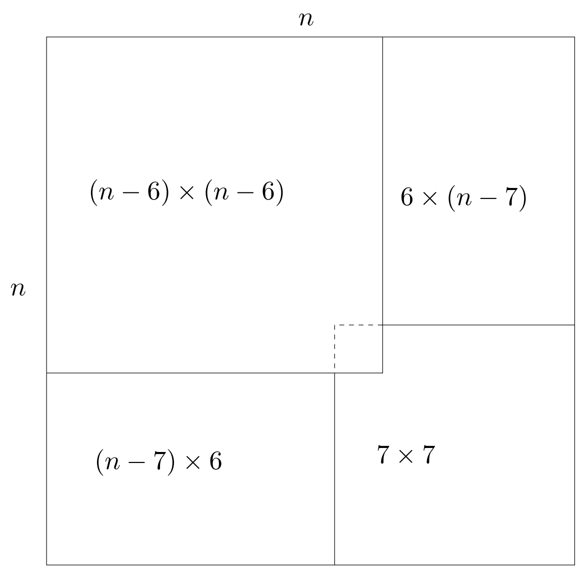
Proposition 3. If n is odd, $n > 5$, and $n \not\equiv 0 \pmod 3$, then a deficient $n \times n$ board can be tiled with trominoes.

Proof. The case $n = 7$ is given by Example 3. The solution for $n = 11$ is given in the figure below.



We first rotate the board so that the missing square is located in the 7×7 subboard. By Example 3, this deficient 7×7 subboard can be tiled. The 6×4 and 4×6 subboards can be tiled by Proposition 1. The 5×5 subboard with a corner square missing can be tiled by Example 2. The above results give the base cases for a proof by strong induction.

For the inductive step, suppose that n is odd, $n > 11$, and $n \not\equiv 0 \pmod 3$, and assume that all $k \times k$ deficient boards can be tiled, where k is odd, $5 < k < n$, and $k \not\equiv 0 \pmod 3$. The figure below shows a tiling of the deficient $n \times n$ board.

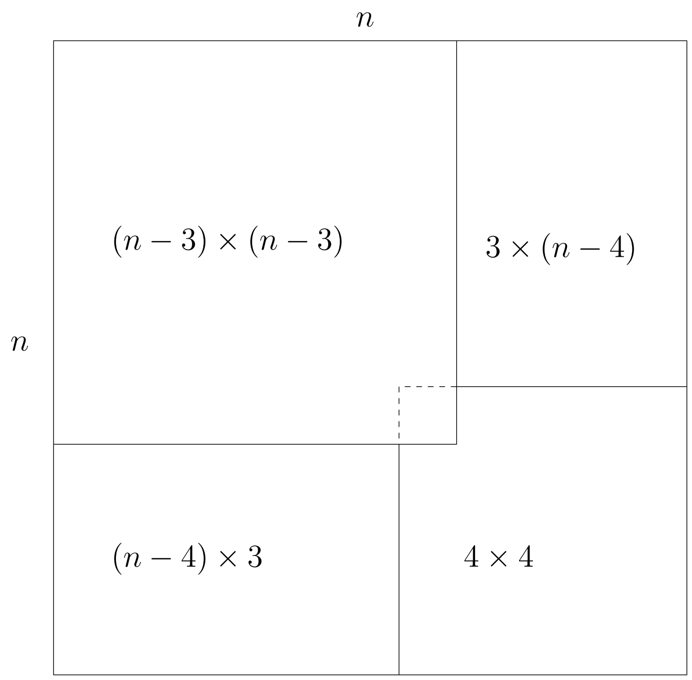


We first rotate the board so that the missing square is located in the $(n-6) \times (n-6)$ subboard. We have $n-6$ is odd, $n-6 > 5$, and $n-6 \not\equiv 0 \pmod 3$, so by the inductive hypothesis, this deficient $(n-6) \times (n-6)$ subboard can be tiled. Also, since n is odd, then $n-7$ is even, and by Proposition 1, both the $(n-7) \times 6$ and the $6 \times (n-7)$ subboards can be tiled. Finally, the deficient 7×7 subboard can be tiled. This completes the inductive step.

Therefore, by the principle of strong induction, if n is odd, $n > 5$, and $n \not\equiv 0 \pmod 3$, then a deficient $n \times n$ board can be tiled with trominoes. $\square$

Proposition 4. If n is even, $n > 1$, and $n \not\equiv 0 \pmod 3$, then a deficient $n \times n$ board can be tiled with trominoes.

Proof. The cases $n = 2$, 4, and 8 are given by Example 1. The figure below shows a tiling of the deficient $n \times n$ board where n is even, $n > 8$, and $n \not\equiv 0 \pmod 3$.



We first rotate the board so that the missing square is located in the $(n-3) \times (n-3)$ subboard. Since $n-3$ is odd, $n-3 > 5$, and $n-3 \not\equiv 0 \pmod 3$, we can use Proposition 3 to conclude that the deficient $(n-3) \times (n-3)$ subboard can be tiled. Since n is even, then by Proposition 1, the $(n-4) \times 3$ and the $3 \times (n-4)$ subboards can be tiled. By Example 1, the deficient 4×4 subboard can be tiled. Therefore, if n is even, $n > 1$, and $n \not\equiv 0 \pmod 3$, then a deficient $n \times n$ board can be tiled with trominoes. $\square$

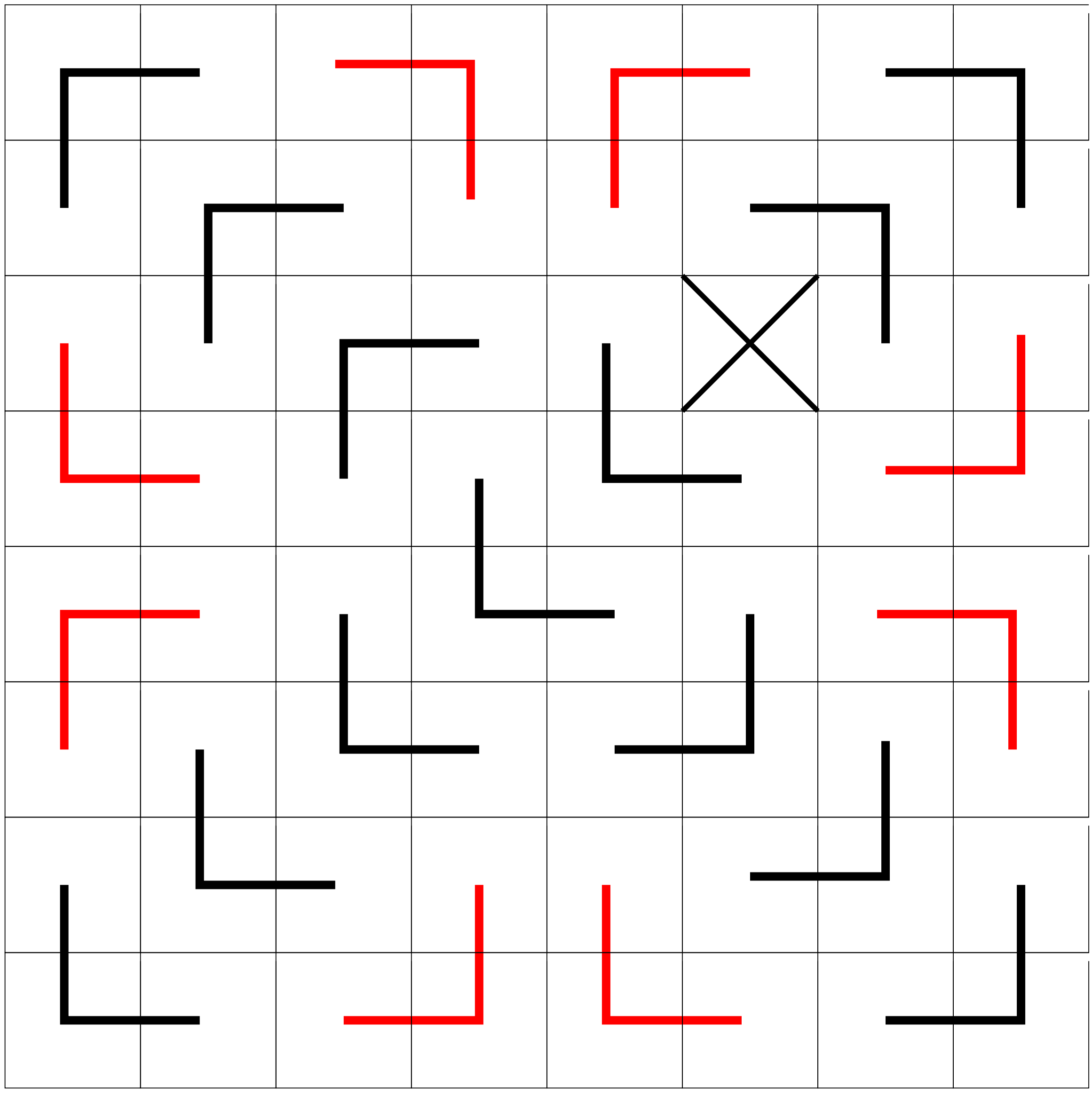
Combining Proposition 3 and Proposition 4, we have the following:

Theorem. If n is a positive integer with $n \neq 5$, then a deficient $n \times n$ board can be tiled with trominoes if and only if $n \not\equiv 0 \pmod 3$.

The proofs of Propositions 3 and 4 were taken from:

I. Chu and R. Johnsonbaugh, *Tiling Deficient Boards with Trominoes*, Mathematics Magazine, Vol 59, No 1, 1986, pp. 34-40.

Activity: Suppose that we remove any single square from an 8×8 board, can we fill in the rest of it with right trominoes?



What if one square is removed instead from a 10×10 board?