

Lecture 15

The Principle of Mathematical Induction: to show a statement P_n is true for all positive integers greater or equal to n_0 there are two steps:

- **Base Case:** Show P_{n_0} is true
- **Inductive Step:** Show: $P_n \Rightarrow P_{n+1}$ (for all $n \geq n_0$)

Example 1: Prove:

$$1 + 2 + 3 + \dots + n = \frac{(n+1) \cdot n}{2} \quad (*)$$

for $n \geq 1$ by using mathematical induction.

Base Case: $P_1 =$ the statement $1+2+\dots+n = \frac{n(n+1)}{2}$ for $n=1$.

$$1 = \frac{(1+1) \cdot 1}{2} = \frac{2}{2} = 1 \quad \checkmark$$

LHS of (*)

RHS of (*)

So (*) is true for $n=1$.

Inductive Step:

Suppose that (*) is true for some $n \geq 1$. Show this implies (*) is true for $n+1$.

$$\underbrace{1+2+3+\dots+n}_{\frac{n(n+1)}{2}} + n+1 = \frac{n(n+1)}{2} + (n+1) = \frac{(n+1)}{2} [n+2]$$

by inductive hypothesis

which is (*) for $n+1$. So if (*) holds for some $n \geq 1$,

then (*) holds for $n+1$. \checkmark

By principle of mathematical induction (*) all $n \geq 1$.

Example 2: Prove:

$$\frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \dots + \frac{1}{n \cdot (n+1)} = \frac{n}{n+1}$$

for $n \geq 1$ by using mathematical induction. (exercise)

$$\frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \frac{1}{3 \cdot 4} + \dots + \frac{1}{n(n+1)}$$

$$= \left(1 - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) + \dots + \left(\frac{1}{n} - \frac{1}{n+1}\right)$$

The sum collapses or telescopes to

$$= 1 - \frac{1}{n+1} = \frac{n+1-1}{n+1} = \frac{n}{n+1}$$

Example 3: Let m be an integer and prove

$$m + (m + 1) + \dots + n = \frac{(n + m)(n - m + 1)}{2}$$

for $n \geq m$ by using mathematical induction. (exercise).

$$\begin{aligned} m + (m + 1) + \dots + n &= (1 + 2 + \dots + n) - (1 + 2 + \dots + (m - 1)) \\ &= \frac{n(n + 1)}{2} - \frac{(m - 1)m}{2} \\ &= \frac{1}{2} [n^2 + n - m^2 + m] \\ &= \frac{1}{2} [n^2 - m^2 + (n + m)] \\ &= \frac{1}{2} [(n - m)(n + m) + (n + m)] \\ &= \frac{1}{2} [(n + m)(n - m + 1)] \end{aligned}$$

Example 4: Find a closed form for the following expression.

$$\frac{1 + 2 + \dots + n}{n + (n + 1) + \dots + 2n}$$

$$\begin{aligned} \frac{1 + 2 + \dots + n}{n + (n + 1) + \dots + 2n} &= \frac{\frac{n(n + 1)}{2}}{\frac{2n(2n + 1)}{2} - \frac{(n - 1)n}{2}} = \frac{n(n + 1)}{2n(2n + 1) - n(n - 1)} \\ &= \frac{n(n + 1)}{3n^2 + 3n} = \frac{n(n + 1)}{3 \cdot n(n + 1)} = \frac{1}{3} \end{aligned}$$

Example 5: For $n \geq 1$ prove:

$$\underbrace{\sqrt{2 + \sqrt{2 + \dots + \sqrt{2}}}}_n = 2 \cdot \cos\left(\frac{\pi}{2^{n+1}}\right) \quad (*)$$

where the statement on the left has n 2's. Recall the half angle formula $\frac{1 + \cos 2\theta}{2} = \cos^2 \theta$.

Base Case: For $n=1$, $(*)$ says $\sqrt{2} = 2 \cos\left(\frac{\pi}{4}\right)$, i.e. $\cos\left(\frac{\pi}{4}\right) = \frac{1}{\sqrt{2}}$.
So $(*)$ is true for $n=1$.

Inductive Step: Assume that $(*)$ is true for some $n \geq 1$,
show this implies $(*)$ is true for $n+1$.

$$\begin{aligned} \underbrace{\sqrt{2 + \sqrt{2 + \dots + \sqrt{2}}}}_{n+1 \text{ sqrts}} &= \sqrt{2 + \underbrace{\sqrt{2 + \dots + \sqrt{2}}}_n} \\ &= \sqrt{2 + 2 \cos\left(\frac{\pi}{2^{n+1}}\right)} && \text{by the inductive hypothesis} \\ &= \sqrt{\frac{4 + 4 \cos\left(\frac{\pi}{2^{n+1}}\right)}{2}} \\ &= \sqrt{\frac{4(1 + \cos\left(\frac{\pi}{2^{n+1}}\right))}{2}} \\ &= 2 \sqrt{\frac{1 + \cos\left(\frac{\pi}{2^{n+1}}\right)}{2}} = 2 \sqrt{\cos^2\left(\frac{\pi}{2^{n+2}}\right)} && \text{by double angle formula} \\ &= 2 \cos\left(\frac{\pi}{2^{n+2}}\right) && \text{since } \cos\left(\frac{\pi}{2^{n+2}}\right) > 0 \text{ for } n \geq 1. \end{aligned}$$

So $(*)$ is true for $n+1$ whenever it is true for n .

By P.M.I., $(*)$ is true for all $n \geq 1$.

Example 6: Define a sequence of shapes as follows:

- K_1 is an equilateral triangle
- for $n > 1$, K_n is formed by replacing each line segment



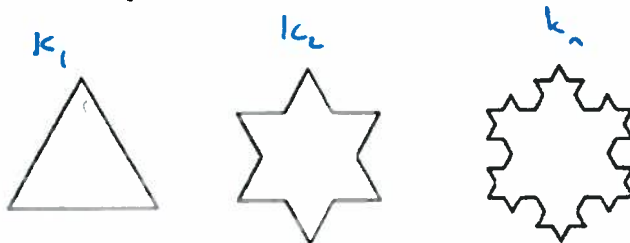
of K_{n-1} with the shape



according to the following three steps

- 1) The line segment was divided into three segments of equal length.
- 2) An equilateral triangle was drawn pointing outward that has its middle segment from step 1 as its base.
- 3) The line segment that is the base of the triangle from step 2 was removed.

The limit of this sequence of shapes is known as the Koch snowflake:



- a) Write down a recurrence relation for a_n the number of line segments in K_n .

$$a_1 = 3$$

$$a_n = 4 \cdot a_{n-1} \quad \text{for } n \geq 2.$$

- b) Show by induction that the number of line segments in K_n is: (exercise)

$$3 \cdot 4^{n-1}$$

By induction: (bottom up)

$$a_1 = 3$$

$$a_2 = 4 \cdot a_1 = 4 \cdot 3$$

$$a_3 = 4 \cdot a_2 = 4^2 \cdot 3$$

$$a_4 = 4 \cdot a_3 = 4^3 \cdot 3$$

$$\vdots$$

$$a_n = 4^{n-1} \cdot 3 \quad \text{for } n \geq 1.$$

Example 7: Conjecture a formula for the sum of the first n Fibonacci numbers with even indices and prove your formula works by using mathematical induction. That is, find and prove a formula for $F_2 + F_4 + \dots + F_{2n}$.

Note:

$$F_1 = 1,$$

$$F_2 = 1,$$

$$F_n = F_{n-1} + F_{n-2}.$$

$$F_0 = 0$$

$$F_1 = 1$$

$$F_2 = 1$$

$$F_3 = 2$$

$$F_4 = 3$$

$$F_5 = 5$$

$$F_6 = 8$$

$$F_7 = 13$$

$$F_8 = 21$$

$$F_9 = 34$$

$$F_2 = 1 = F_3 - 1$$

$$F_2 + F_4 = 1 + 3 = 4 = 5 - 1 = F_5 - 1$$

$$F_2 + F_4 + F_6 = 4 + 8 = 12 = 13 - 1 = F_7 - 1$$

$$F_2 + F_4 + F_6 + F_8 = 33 = 34 - 1 = F_9 - 1$$

So it looks like

$$F_2 + F_4 + \dots + F_{2n} = F_{2n+1} - 1 \quad (*)$$

B.C. $F_2 = 1 = 2 - 1 = F_3 - 1$ ✓ So (*) is true for $n=1$.

I.S. Suppose (*) is true for some $n \geq 1$, then

$$F_2 + F_4 + \dots + F_{2n} + F_{2n+2}$$

$$= F_{2n+1} - 1 + F_{2n+2} = \underbrace{F_{2n+1} + F_{2n+2}}_{F_{2n+3}} - 1$$

$$= F_{2n+3} - 1$$

wh if (*) true for n , then it is true for $n+1$.

By induction, (*) is true for all integers $n \geq 1$.

Example 8: Consider the recurrence relation defined by:

$$\begin{cases} a_0 = 1, \\ a_n = 3a_{n-1} - 2^{n-1} \text{ for } n \geq 1. \end{cases}$$

Conjecture a solution to this recurrence relation and prove it by mathematical induction.

Make a substitution: Let $b_n = a_n - 2^n$ for $n \geq 0$

$$b_n + 2^n = 3 \cdot (b_{n-1} + 2^{n-1}) - 2^{n-1}$$

$$b_n + 2^n = 3 \cdot b_{n-1} + 3 \cdot 2^{n-1} - 2^{n-1} = 3b_{n-1} + (3-1)2^{n-1}$$

$$b_n + 2^n = 3b_{n-1} + 2^n \quad \therefore \quad b_n = 3 \cdot b_{n-1} \quad \text{for } n \geq 1.$$

Initial condition: $b_0 = a_0 - 2^0 = 1 - 1 = 0$

So $\{b_n\}_{n \geq 0}$ satisfies the discrete initial value problem

$$\begin{cases} b_n = 3 \cdot b_{n-1}, & n \geq 1 \\ b_0 = 0 \end{cases}$$

Solve by iteration: (bottom up)

$$b_1 = 3 \cdot b_0 = 3 \cdot 0 = 0$$

$$b_2 = 3 \cdot b_1 = 3 \cdot 0 = 0$$

$$b_3 = 3 \cdot b_2 = 3 \cdot 0 = 0$$

⋮

prove by induction that $b_n = 0$ for any $n \geq 0$.

Therefore,

$$a_n = b_n + 2^n = 2^n$$

for all $n \geq 0$.

Ex 8:

Go back to the original discrete I.V.P.

$$a_n = 3a_{n-1} - 2^{n-1} \quad \text{for } n \geq 1$$

$$a_0 = 1$$

Solve this by iteration (from the bottom up).

$$a_1 = 3 \cdot a_0 - 2^0 = 3 \cdot 1 - 1 = 2 = 2^1$$

$$a_2 = 3 \cdot a_1 - 2^1 = 3 \cdot 2^1 - 2^1 = (3-1)2^1 = 2^2$$

$$a_3 = 3 \cdot a_2 - 2^2 = 3 \cdot 2^2 - 2^2 = (3-1)2^2 = 2^3$$

⋮

Looks like $a_n = 2^n$ for all $n \geq 0$. (*)Prove by induction that $a_n = 2^n$ for all $n \geq 0$.Base case: $n=0$, $a_0 = 1 = 2^0$ ✓Induction step: Suppose that (*) holds for some $n \geq 0$, look at a_{n+1} . From the recurrence relation

$$\begin{aligned} a_{n+1} &= 3 \cdot a_n - 2^n \\ &= 3 \cdot (2^n) - 2^n \quad (\text{by induction hypothesis}) \\ &= (3-1) \cdot 2^n = 2 \cdot 2^n = 2^{n+1} \end{aligned}$$

So have shown that if (*) is true for n , this implies it is true for $n+1$. ✓By the principle of mathematical induction (*) is true for all $n \geq 0$, i.e. $a_n = 2^n$ for all $n \geq 0$.

Example 9: Use mathematical induction to show that $n^3 - n$ is divisible by 3 for every positive integer n .

$$n^3 - n = n(n^2 - 1) = n(n-1)(n+1) = (n-1)n(n+1)$$

given any 3 consecutive integers, one of them is congruent to 0 mod 3, is one of them is divisible by 3, so $n^3 - n$ is divisible by 3.

Proof by Induction:

Base case: $n=1, 1^3 - 1 \equiv 0 \pmod{3}$ ✓

Induction step: Assume that for some $n \geq 1$, $n^3 - n$ is divisible by 3,

then

$$(n+1)^3 - (n+1) = (n+1)[(n+1)^2 - 1] = (n+1)[n^2 + 2n] = n^3 + 2n^2 + n^2 + 2n$$

$$\therefore (n+1)^3 - (n+1) = n^3 + 3n^2 + 2n = \underbrace{n^3 - n}_{\text{divisible by 3}} + 3n^2 + 3n$$

$$\therefore (n+1)^3 - (n+1) \equiv 0 + 0 + 0 \equiv 0 \pmod{3}$$

$\therefore (n+1)^3 - (n+1)$ is divisible by 3 if $n^3 - n$ is divisible by 3.

By induction, $n^3 - n$ is divisible by 3 for all $n \geq 1$.

Example 10:

All girls have the same hair colour. We claim that all girls in any group of n girls have the same hair colour, for each $n = 1, 2, \dots$

Step 1: When $n = 1$, there is only one girl in the group, so all girls within the group clearly have the same hair colour. *Base case true.*

Step 2: Assume that the case $n = k$ holds. Given a group of $k + 1$ girls, remove one of them from the group. By assumption, each of the remaining k girls have the same hair colour. Now swap one of these girls with the girl we removed. Since every girl in this new group of k girls also have the same hair colour, we know now that all $k + 1$ girls have the same hair colour!

By induction, the claim holds.

Is there an error in the above "proof"? If so, where is the flaw?

P_1 holds but $P_1 \not\Rightarrow P_2$

Strong Induction; to show a statement P_n is true for all positive integers greater or equal to n_0 there are two steps:

- **Base Case:** Show P_{n_0} is true (you may need to do this for other values also)
- **Inductive Step:** Show: $(P_{n_0} \& P_{n_0+1} \& P_{n_0+2} \& \dots \& P_n) \Rightarrow P_{n+1}$ (for all $n \geq n_0$)

Example 11: P_n : "Postage of n cents can be formed using 4-cent and 5-cent stamps." Show: P_n is true for $n \geq 12$.

Base Case: $n_0 = 12 = 3 \cdot 4$ ✓

Inductive Step: Suppose postage of n cents can be formed using 4¢ and 5¢ stamps, for some $n \geq 12$.

then $n = a \cdot 4 + b \cdot 5$

where a and b are nonnegative integers.

To form $n+1$ cents of postage:

Case 1: Suppose $a > 0$ we can replace a 4¢ stamp with a 5¢ stamp, i.e.

$$n+1 = 4(a-1) + 5(b+1)$$

Case 2: Suppose $a=0$, then there are no 4¢ and all there must be at least 3 5¢ stamps.

So replace 3 5¢ stamps with 4 4¢ stamps, i.e.

$$n+1 = 4(a+1) + 5(b-3) \quad \checkmark$$

By induction, P_n is true for all $n \geq 12$.

Ex 11: Alternative Soln

Base Cases:

$$\begin{array}{l} n=12, \quad 3 \cdot 4 + 0 \cdot 5 = 12 \\ n=13, \quad 2 \cdot 4 + 1 \cdot 5 = 13 \\ n=14, \quad 1 \cdot 4 + 2 \cdot 5 = 14 \\ n=15, \quad 0 \cdot 4 + 3 \cdot 5 = 15 \end{array}$$

So $P_{12}, P_{13}, P_{14}, P_{15}$ are all true.

Induction Step:

Show $P_n \Rightarrow P_{n+4}$ for all $n \geq 12$.

$$n+4 = \underbrace{4x + 5y}_n + 4 = 4(x+1) + 5y \quad \text{where } x, y \geq 0.$$

So if P_n is true for some $n \geq 12$, this implies P_{n+4} is true.

We have shown:

$$P_{12} \Rightarrow P_{16} \Rightarrow P_{20} \Rightarrow P_{24} \dots$$

$$P_{13} \Rightarrow P_{17} \Rightarrow P_{21} \Rightarrow P_{25} \dots$$

$$P_{14} \Rightarrow P_{18} \Rightarrow P_{22} \Rightarrow P_{26} \dots$$

$$P_{15} \Rightarrow P_{19} \Rightarrow P_{23} \Rightarrow P_{27} \dots$$

By induction, P_n is true for all $n \geq 12$.

Ex: Let a be a positive real number

such that $a + \frac{1}{a}$ is an integer.

Use the principle of strong induction to show that

$$a^n + \frac{1}{a^n} \quad (*)$$

is also an integer for all positive integers $n \geq 1$.

Base case: We want to show that $(*)$ is also true for $n=2$, so look at

$$\left(a + \frac{1}{a}\right)^2 = a^2 + 2a \cdot \frac{1}{a} + \frac{1}{a^2} = a^2 + 2 + \frac{1}{a^2}$$

$$\text{So } a^2 + \frac{1}{a^2} = \left(a + \frac{1}{a}\right)^2 - 2$$

and since $a + \frac{1}{a}$ is an integer, then $a^2 + \frac{1}{a^2}$ is an integer.

Inductive step: Assume that $(*)$ is true for all integers k such that $1 \leq k \leq n$, then we want to show that this implies $(*)$ is true for $n+1$.

So

$$\begin{aligned} \left(a^n + \frac{1}{a^n}\right) \left(a + \frac{1}{a}\right) &= a^{n+1} + a^n \cdot \frac{1}{a} + a \cdot \frac{1}{a^n} + \frac{1}{a^{n+1}} \\ &= a^{n+1} + \frac{1}{a^{n+1}} + a^{n-1} + \frac{1}{a^{n-1}} \end{aligned}$$

So

$$a^{n+1} + \frac{1}{a^{n+1}} = \left(a^n + \frac{1}{a^n}\right) \left(a + \frac{1}{a}\right) - \left(a^{n-1} + \frac{1}{a^{n-1}}\right)$$

By the induction hypothesis, the expression on the r.h.s. is an integer, so $a^{n+1} + \frac{1}{a^{n+1}}$ is also an integer. And $(*)$ is true for $n+1$. By strong induction, $(*)$ is true for all $n \geq 1$.

Example 12: Show that every positive integer can be written as a sum of distinct powers of two.

We showed this earlier when we were talking about the well-ordering property for the positive integers, and the division algorithm.

Look at notes on the web.

Let P_n be the statement that n can be written as a sum of distinct powers of 2. Show P_n is true for all $n \geq 1$.

Base case: $n=1$, $1 = 1 \cdot 2^0$ ✓

Inductive Step: Assume that P_k is true for all k 's with $1 \leq k < n$, we have to show this implies P_n is true for n .

Case 1: If n is even, then $1 \leq \frac{n}{2} < n$, so by the inductive hypothesis, we can write

$$\frac{n}{2} = a_0 + a_1 \cdot 2^1 + a_2 \cdot 2^2 + \dots + a_l \cdot 2^l$$

$$\text{So } n = a_0 \cdot 2^1 + a_1 \cdot 2^2 + a_2 \cdot 2^3 + \dots + a_l \cdot 2^{l+1}$$

Case 2: If n is odd, then $1 \leq \frac{n-1}{2} < n$ by the inductive hypothesis, we can write

$$\frac{n-1}{2} = a_0 + a_1 \cdot 2^1 + a_2 \cdot 2^2 + \dots + a_l \cdot 2^l$$

$$\text{So } n = 1 + a_0 \cdot 2^1 + a_1 \cdot 2^2 + a_2 \cdot 2^3 + \dots + a_l \cdot 2^{l+1}$$

So P_n is true.

By strong induction, P_n is true for all $n \geq 1$.

Ex 12: Alternate proof. For $n \geq 1$,

P_n = integer can be written as a distinct sum of powers of 2

Base Case: $n=1$, $1 = 2^0$ so P_1 is true.

Inductive Step: Show that $P_n \Rightarrow (P_n \text{ and } P_{n+1})$ are true for $n \geq 1$.

Assume that P_n is true, then

$$P_n: 2n = 2 \cdot (\underbrace{2^i + \dots + 2^j}_{\text{distinct powers}}) = \underbrace{2^{i+1} + \dots + 2^{j+1}}_{\text{distinct powers}}$$

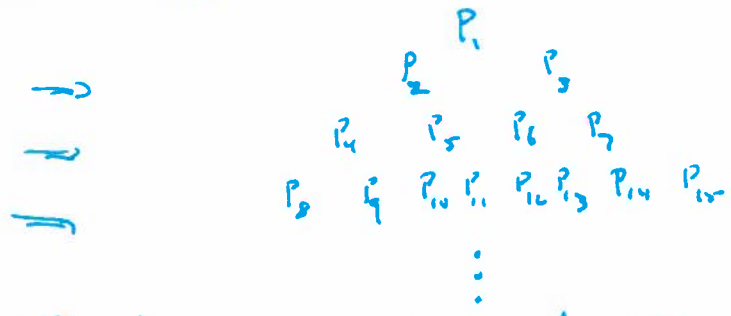
So P_n true.

$$P_{n+1}: 2n+1 = 2(\underbrace{2^i + \dots + 2^j}_{\text{distinct powers}}) + 1 = \underbrace{2^0 + 2^{i+1} + \dots + 2^{j+1}}_{\text{distinct powers}}$$

so P_{n+1} is true.

Therefore, if P_n is true for some $n \geq 1$, then

P_n and P_{n+1} are both true.



Therefore P_n is true for all $n \geq 1$.

Example 13: Define the recurrence relation:

$$\begin{aligned} a_0 &= 1 \\ a_1 &= 1 \\ a_n &= 2a_{n-1} + a_{n-2} \end{aligned}$$

Show that a_n is odd for all $n \geq 0$.

By induction:

Base case: $a_0 = 1$ and $a_1 = 1$ are both odd. ✓

Inductive step: Assume that a_k is odd for all k satisfying $1 \leq k \leq n$, show this implies that a_{n+1} is odd.

Recurrence relation:

$$a_{n+1} = 2a_n + a_{n-1}$$

by the inductive hypothesis a_{n-1} and a_n are odd,

$$\text{so } a_{n+1} = 2 \cdot \text{odd} + \text{odd} = \text{even} + \text{odd} = \text{odd}.$$

Therefore a_{n+1} is also odd.

So a_0, a_1 odd, and if a_k is odd for $1 \leq k \leq n$, this implies a_{n+1} is odd. Therefore, by strong induction, a_n is odd for all $n \geq 0$.

Solve the discrete initial value problem

$$a_n = 2a_{n-1} + a_{n-2}, \quad n \geq 2$$

$$a_0 = 1$$

$$a_1 = 1$$

the difference equation is 2nd order, linear, constant coefficients and is homogeneous, so we can assume a solution of the form $a_n = \lambda^n$.

Ex 13: Plug this into the difference eqn:

$$\lambda^n = 2 \cdot \lambda^{n-1} + \lambda^{n-2}$$

assuming that $\lambda \neq 0$, divide by λ^{n-2} and get the characteristic eqn

$$\lambda^2 = 2\lambda + 1$$

which is a quadratic eqn.

$$\left. \begin{aligned} \lambda^2 - 2\lambda + 1 &= 2 \\ (\lambda - 1)^2 &= 2 \end{aligned} \right\} \Rightarrow \lambda = 1 \pm \sqrt{2}$$

The roots are $\lambda_1 = 1 + \sqrt{2}$, $\lambda_2 = 1 - \sqrt{2}$.

The general soln to the difference eqn is

$$a_n = A \cdot (1 + \sqrt{2})^n + B (1 - \sqrt{2})^n \quad \text{for } n \geq 0$$

The constants A and B can be determined from the initial conditions.

$$a_0 = 1 = A + B$$

$$a_1 = 1 = A(1 + \sqrt{2}) + B(1 - \sqrt{2}) = (A + B) + \sqrt{2}(A - B)$$

$$\Rightarrow 1 = 1 + \sqrt{2}(A - B), \text{ and } \sqrt{2}(A - B) = 0$$

$$\Rightarrow A - B = 0 \text{ and } A = B = \frac{1}{2}$$

The solution is

$$a_n = \frac{1}{2} \left[(1 + \sqrt{2})^n + (1 - \sqrt{2})^n \right]$$

for $n \geq 0$.

Ex 13) From the binomial theorem

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$$(1+\sqrt{2})^n = \sum_{k=0}^n \binom{n}{k} 2^{k/2}$$

$$(1-\sqrt{2})^n = \sum_{k=0}^n \binom{n}{k} 2^{k/2} (-1)^k$$

So

$$a_n = \frac{1}{2} \left[\sum_{k=0}^n \binom{n}{k} 2^{k/2} \{1 + (-1)^k\} \right]$$

$$1 + (-1)^k = 0 \quad \text{if } k \text{ is odd}$$

$$1 + (-1)^k = 2 \quad \text{if } k \text{ is even}$$

So

$$a_n = \frac{1}{2} \sum_{\substack{k=0 \\ k \text{ even}}}^n \binom{n}{k} 2^{k/2} \cdot 2 = \sum_{\substack{k=0 \\ k \text{ even}}}^n \binom{n}{k} 2^{k/2}$$

$$a_n = \sum_{l=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n}{2l} 2^l \quad \begin{array}{l} 2l = k \\ l = k/2 \end{array}$$

$$\text{So } a_n = \sum_{l=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n}{2l} 2^l$$

Each term in the sum is even except for the 1st term: $\binom{n}{0} 2^0 = 1$

So the sum is odd and a_n is odd for all $n \geq 0$.

Example 14: Prove that $5^{n+1} + 2 \cdot 3^n + 1$ is divisible by 8 for all $n \geq 1$.

Modulo 8:

$$\left. \begin{array}{l} 5 \equiv 5 \\ 5^2 \equiv 25 \equiv 1 \\ 5^3 \equiv 5 \\ 5^4 \equiv 1 \end{array} \right\} \begin{array}{l} 5^n \equiv 1 \text{ if } n \text{ even} \\ 5^n \equiv 5 \text{ if } n \text{ odd} \end{array}$$

$$\left. \begin{array}{l} 3 \equiv 3 \\ 3^2 \equiv 9 \equiv 1 \\ 3^3 \equiv 3 \\ 3^4 \equiv 1 \end{array} \right\} \begin{array}{l} 3^n \equiv 1 \text{ if } n \text{ even} \\ 3^n \equiv 3 \text{ if } n \text{ odd} \end{array}$$

if n is even: $n+1$ is odd

$$\text{So } 5^{n+1} + 2 \cdot 3^n + 1 \equiv 5 + 2 + 1 \equiv 8 \equiv 0 \pmod{8}$$

if n is odd: $n+1$ even

So

$$5^{n+1} + 2 \cdot 3^n + 1 \equiv 1 + 2 \cdot 3 + 1 \equiv 1 + 6 + 1 \equiv 8 \equiv 0 \pmod{8}$$

In both cases $5^{n+1} + 2 \cdot 3^n + 1 \equiv 0 \pmod{8}$

i.e. $5^{n+1} + 2 \cdot 3^n + 1$ is divisible by 8 for all $n \geq 1$.