Lecture 14

More Recurrence Relations

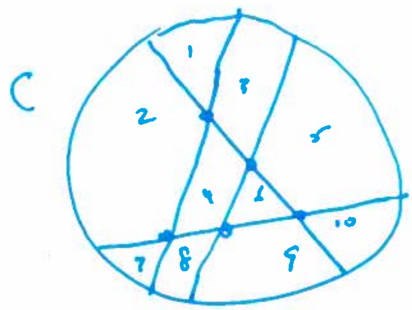
Recall that a subset C of the plane is said to be **convex** if and only if whenever A and B are in C, then the whole line segment $\overline{AB}$ joining A and B is contained in the set C.

convex

not convex

Theorem. Given a convex region C of the plane which is crossed by ℓ lines with p interior points of intersection, and let r be the number of disjoint regions created, then

(No 3 lines concurrent). $$r = \ell + p + 1. \qquad (*)$$



$\ell = 4$, $p = 5$

$r = 4 + 5 + 1 = 10$

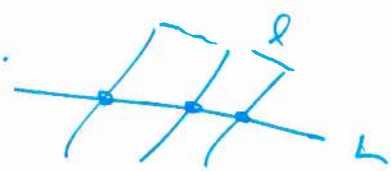
proof: By induction on ℓ.

Base Case: $\ell = 0$, $p = 0$ and $r = 1$ so $r = \ell + p + 1$

So $(*)$ holds for $\ell = 0$.

Induction Step: Assume $(*)$ is true for $\ell \geq 0$ number of lines.

Add another line L. Suppose L intersects the ℓ lines in s intersection points in the interior.

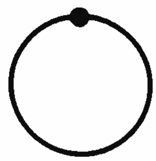


L is divided into $s+1$ interval.

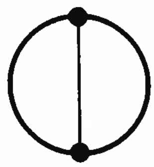
So ℓ has increased by 1, p has increased by s and r increases by $s+1$. So $(*)$ is true for $\ell+1$.

Example 1: There are $n \geq 1$ points on a circle, every two of which are joined by a chord. No three chords pass through a common point which lies inside the circle. Let a_n be the number of regions within the circle. Start by finding $a_1, a_2, a_3, a_4, a_5, a_6$ and then find a formula for a_n.

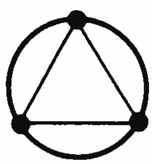
$a_1 = 1$



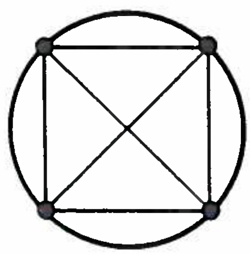
$a_1 = 2$



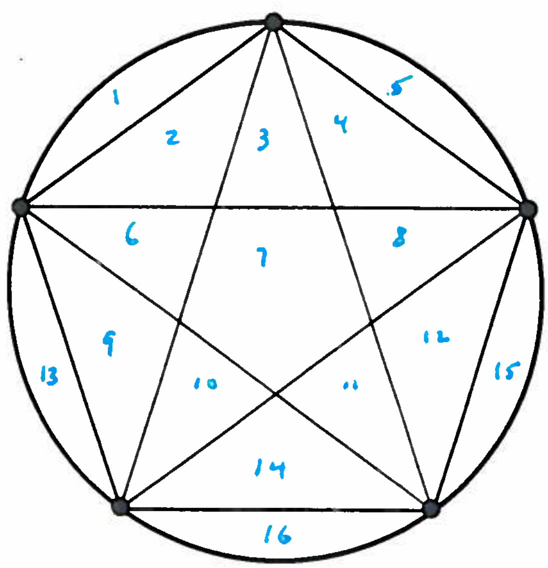
$a_3 = 4$



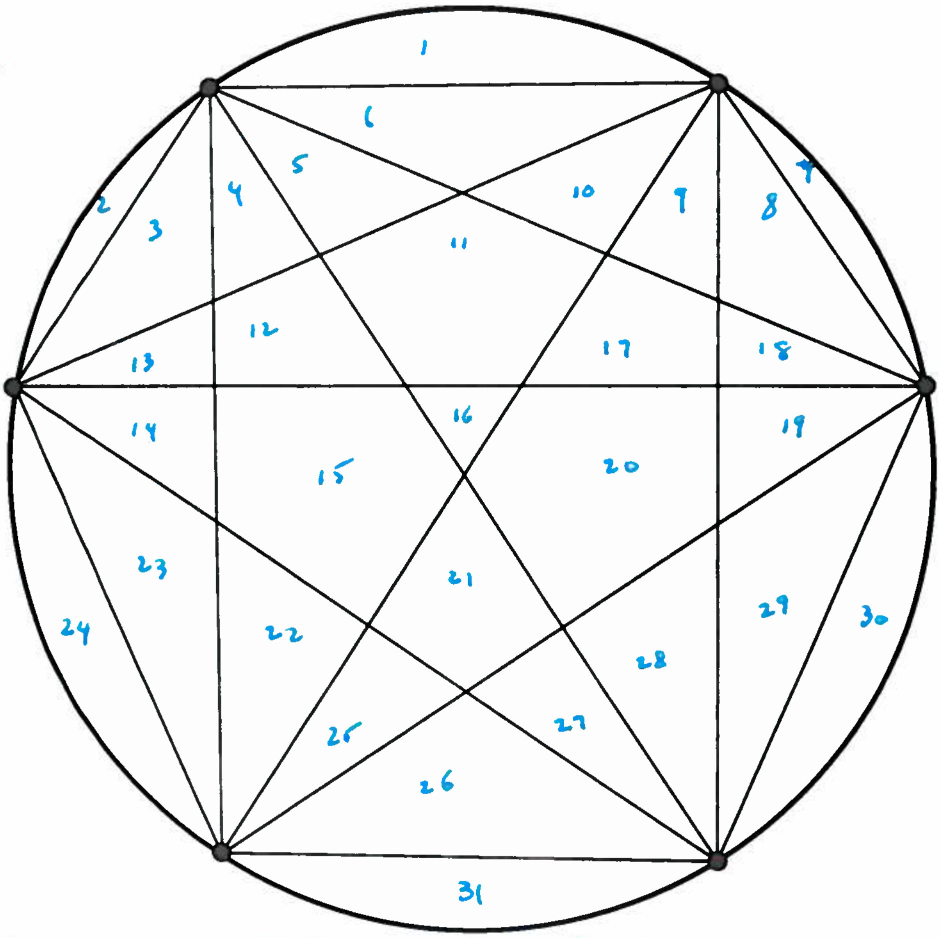
$a_4 = 8$



$a_5 = 16$



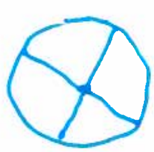
$a_6 = 31$



Use the previous theorem to find a_n

Suppose that we have n points on the circle, no three chords are concurrent at an interior point.

The number of chords is $\lambda = \binom{n}{2}$



Every intersection pt lies on exactly 2 chords

Given any 4 pts on the circle, then exactly 2 chords determined by them form pts that intersect in the interior.

The number of points is $p = \binom{n}{4}$.

So $\boxed{a_n = \binom{n}{2} + \binom{n}{4} + 1}$ for $n \geq 1$

Example 2: On the planet Jubilation, the weather each day is either good or bad. If it is good, then it remains good the next day 70% of the time. If it is bad, then it remains bad the next day 60% of the time. Let a_n be the probability that weather is good on day n. Find a recurrence relation and solve the recurrence. If you landed on Jubilation on a random day what are the chances of having good weather (in other words, what happens as n goes to ∞)?

Consider the weather on day n and day $n-1$.

Day $n-1$: Good Bad

$(.7)a_{n-1}$ $.3\,a_{n-1}$ $.4(1-a_{n-1})$ $.6(1-a_{n-1})$

Day n: Good Bad Good Bad

Two ways the weather good on day n:

$$a_n = (.7)\,a_{n-1} + (.4)(1-a_{n-1})$$

$$\boxed{\begin{aligned} a_n &= (.3)\,a_{n-1} + (.4) \qquad n \geq 1 \\ a_0 &= p \end{aligned}}$$

$$a_n = (.3)^n p + \frac{(.4)(1-(.3)^n)}{(.7)}$$

As $n \to \infty$

$$a_n \to \frac{.4}{.7} = \frac{4}{7} \approx .57$$

Example 3: Let a_n be the number of words of length $n \geq 1$ containing the digits $\{0, 1, 2, 3\}$ with an even number of 2's. Start by calculating a_1, a_2, and a_3 then find recurrence relation and solve it.

Length	Words	a_n
1	0 1 3	$a_1 = 3$ ✓
2	00 10 30 01 11 31 03 13 33 22	$a_2 = 10$ ✓
3	000 100 300 001 101 301 003 103 303 010 110 310 011 111 311 013 113 313 030 130 330 031 131 331 033 133 333 022 122 322 220 221 223 202 212 232	$a_3 = 36$ ✓

Call a code word legitimate if it contains an even # of 2's.

Let x be a legit code word of length n

$$x = \underline{\begin{matrix}0\\1\\3\end{matrix}}\ \underbrace{\_\ \_\ \cdots\ \_}_{n-1 \text{ even } \# 2's} \qquad 3a_{n-1}$$

$$x = \underline{2}\ \underbrace{\_\ \_\ \_\ \_\ \cdots\ \_}_{n-1 \text{ odd } \# 2's} \qquad 4^{n-1} - a_{n-1}$$

Ex 3:

Recurrence relation is

$$a_n = \underbrace{3a_{n-1}}_{\text{even}} + \underbrace{4^{n-1} - a_{n-1}}_{\text{odd}}$$

So

$$\left.\begin{aligned} a_n &= 2a_{n-1} + 4^{n-1} \\ a_1 &= 3 \end{aligned}\right\} \quad \text{Discrete I.V.P.}$$

For $n \geq 2$ let $\boxed{b_n = a_n - \frac{4^n}{2}}$

$$a_n - \frac{4^n}{2} = 2a_{n-1} + 4^{n-1} - \frac{4^n}{2} = 2a_{n-1} + 4^{n-1}(1-2)$$

$$a_n - \frac{4^n}{2} = 2a_{n-1} - 4^{n-1} = 2\left(a_{n-1} - \frac{4^{n-1}}{2}\right)$$

So

$$b_n = 2 \cdot b_{n-1}$$

$$b_1 = a_1 - 2 = 3 - 2 = 1$$

and $\{b_n\}$ satisfies the D.I.V.P.

$$\left.\begin{aligned} b_n &= 2 \cdot b_{n-1}, \quad n \geq 2 \\ b_1 &= 1 \end{aligned}\right\}$$

Iterate to get $b_n = 2^{n-1}$ for $n \geq 2$.

and

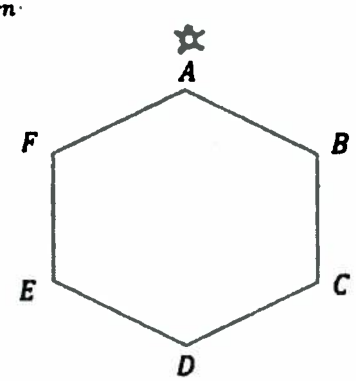
$$a_n = b_n + \frac{4^n}{2} = 2^{n-1} + \frac{4^n}{2} = \frac{1}{2}(2^n + 4^n)$$

$$\boxed{a_n = \frac{1}{2}(2^n + 4^n)} \quad n \geq 1.$$

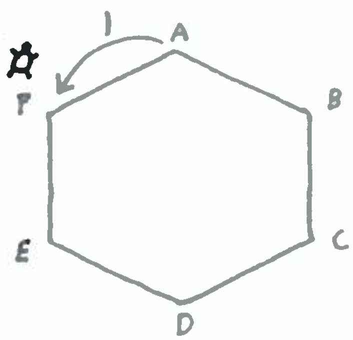
Example 4: A bug starts at vertex A of the regular hexagon below and each minute travels to an adjacent vertex. There is a spider web on vertex D if the bug moves there it is stuck. Let

a_n be the number of different ways the bug can travel from vertex A to vertex D after n minutes.
b_n be the number of different ways the bug can travel from vertex B to vertex D after n minutes.
f_n be the number of different ways the bug can travel from vertex F to vertex D after n minutes.

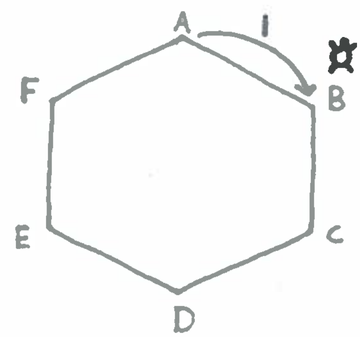
Set up a recurrence relation for a_n.



The bugs first move is to F or B:



OR



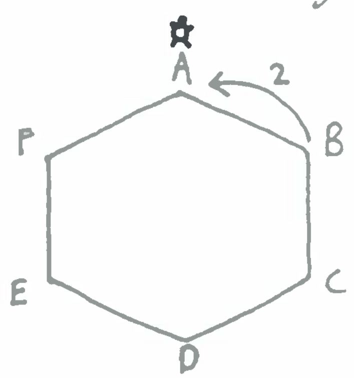
$$\therefore \quad a_n = f_{n-1} + b_{n-1}$$

By the vertical symmetry of the hexagon we have:

$$f_n = b_n \quad \text{for all } n > 0.$$

$$\therefore \quad \boxed{a_n = 2\, b_{n-1}} \qquad n > 0 \qquad (1)$$

If the bugs first move is to B then the second move is to A or C. If the bugs second move is to C and $n>3$ then the bug must return to B or else it gets stuck at D too soon.



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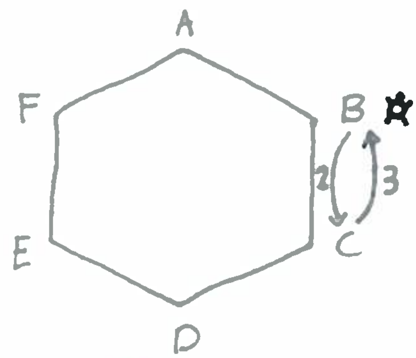


$$\boxed{b_{n-1} = a_{n-2} + b_{n-3}} \quad n>3 \qquad (2)$$

Now, sub equation (2) into equation (1)

$$a_n = 2(a_{n-2} + b_{n-3}) \quad \text{for } n>3$$

$$= 2a_{n-2} + 2b_{n-3}$$

$$= 2a_{n-2} + a_{n-2} \quad \text{by equation (1)}$$

$$= 3a_{n-2}$$

$$\therefore \boxed{\begin{array}{l} a_2 = 0 \\ a_3 = 2 \\ a_n = 3a_{n-2} \quad \text{for } n>3 \end{array}}$$

is the r.r.

Ex4: Recurrence relation is

$$\left.\begin{aligned} a_n &= 3a_{n-2} \quad \text{for } n > 3 \\ a_2 &= 0 \\ a_3 &= 2 \end{aligned}\right\}$$

Assume $a_n = \lambda^n$ plug into the r.r.

$\lambda^n = 3\lambda^{n-2}$ $\quad 1 = 3 \cdot \lambda^{-2}$ $\quad$ (assume $\lambda \neq 0$)

$\lambda^2 = 3$ and $\lambda_1 = \sqrt{3}$, $\lambda_2 = -\sqrt{3}$

General soln:

$$a_n = A\,3^{n/2} + B\,(-3^{1/2})^n = A\,3^{n/2} + B(-1)^n\,3^{n/2}$$

Apply I.C. to find A and B.

$0 = a_2 = A \cdot 3 + B \cdot 3$ $\quad$ so $\quad B = -A$

and

$$a_n = A \cdot 3^{n/2}\,[1 - (-1)^n]$$

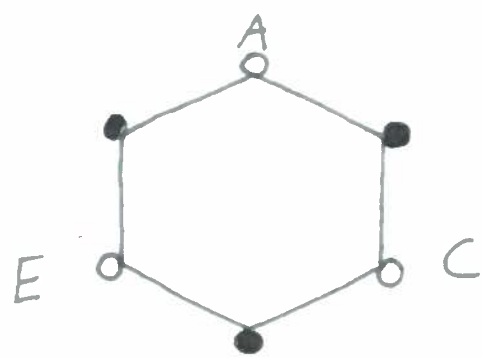
$2 = a_3 = A \cdot 3^{3/2}\,[2]$ $\quad$ so $\quad A = 3^{-3/2}$

and soln is

$$a_n = 3^{\frac{n-3}{2}}\,[1 - (-1)^n] \quad \text{for } n \geq 3.$$

$a_n = 0$ for n even.

Note: if the bug makes an even # of moves it will never be on vertex D. To see why we can colour the vertices:



$\therefore$ after an even # of moves the bug is on vertex A, C, or E.

So $a_n = 0$ for n even

and $a_n = 2 \cdot 3^{\frac{n-3}{2}}$ for n odd.