Lecture 13

Recurrence relations:

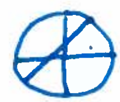
Warm up problem: **The Carrot Cake Problem.** (page 20 or page 4 in Ecco) How can we cut a cake into 16 equal pieces using four straight (vertical) cuts?

Definition: A *recurrence relation* is an equation that defines a sequence where each term of the sequence is defined as a function of the preceding terms.

Example 1: Find a recurrence relation for the maximum number of pieces of cake that can be obtained using n straight (vertical) cuts. Solve the recurrence relation.

Let a_n = max # of pieces using n cuts.

$a_0 = 1$ 

$\cancel{a_3 = 7}$ 

$a_1 = 2$ 

but if we move the 4 pieces and stack them vertically,

$a_2 = 4$ 

now cut it vertically get 8 pieces.

So $a_3 = 8$

By stacking a_{n-1} pieces and cutting all of them with 1 vertical cut, we obtain the maximum of $2a_{n-1}$ pieces.

$$\boxed{\begin{aligned} a_n &= 2a_{n-1}, \quad n \geq 1 \\ a_0 &= 1 \end{aligned}}$$

The recurrence relation is

$$a_n = 2a_{n-1} \qquad (*) \quad \leftarrow \text{difference eqn}$$

and the initial condition is

$$a_0 = 1 \qquad (**) \quad \leftarrow \text{initial condition}$$

The recurrence relation $(*)$ is called a 1st order difference equation, it is linear and has constant coefficients.

Together $(*)$ and $(**)$ form what is called a discrete initial value problem. (Discrete I.V.P.)

Ex: Solve the discrete I.V.P.

$$\begin{cases} a_n = 2a_{n-1}, & n \geq 1 \\ a_0 = 1 \end{cases}$$

Bottom up: start with $a_0 = 1$

$$a_1 = 2 \cdot a_0 = 2$$
$$a_2 = 2 \cdot a_1 = 2^2$$
$$a_3 = 2 \cdot a_2 = 2^3$$
$$\vdots$$
$$a_n = 2^n$$

Ex: Top down:

$$a_n = 2a_{n-1} = 2^2 a_{n-2} = 2^3 \cdot a_{n-3} = \cdots = 2^n a_0$$

So: $a_n = 2^n \cdot 1 = 2^n$

Example 2: What is the largest number of pieces you can get by cutting a pizza (2 dimensional) with n straight (vertical) cuts if the pieces are not to be moved between cuts?

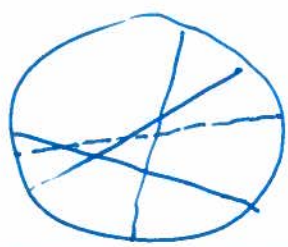
Let a_n = max # of pieces using n cuts.

$a_0 = 1$  $a_3 = 7$ 

$a_1 = 2$ 

$a_2 = 4$ 

If we have made n cuts with the max # of pieces then 3 of the cuts intersect at a point.



If we make an $n+1^{st}$ cut, then it is divided into $n+1$ pieces, so we get $n+1$ new pieces, and

$$\boxed{\begin{array}{l} a_{n+1} = a_n + n + 1, \quad n \geq 0 \\ a_0 = 1 \end{array}}$$

Discrete I.V. P.
1^{st} order diff eqn
w/ constant coeff,
and is nonhomogeneous

Solve by iteration: from the bottom up.

$a_0 = 1$, $a_1 = 1+1$, $a_2 = 1+1+2$

$a_3 = 1+1+2+3$, $a_4 = 1+1+2+3+4$

$\vdots$

$$a_n = 1 + 1 + 2 + \cdots + n = 1 + \frac{n(n+1)}{2} = \underline{\frac{n^2+n+2}{2}}, \quad n \geq 0.$$

Ex: The Fibonacci Sequence is

F_0	F_1	F_2	F_3	F_4	F_5	F_6	F_7	F_8	F_9	F_{10}	F_{11}	F_{12}	
0,	1,	1,	2,	3,	5,	8,	13,	21,	34,	55,	89,	144,	...

the sequence $\{F_n\}_{n\geq 0}$ satisfies the 2nd order linear homogeneous constant coeff difference equation:

$$F_{n+2} = F_{n+1} + F_n, \quad n \geq 0 \quad \leftarrow \text{diff eqn}$$

$$\left.\begin{array}{l} F_0 = 0 \\ F_1 = 1 \end{array}\right. \quad \leftarrow \text{initial conditions.}$$

To find a solution in closed form, assume that

$a_n = \lambda^n$ satisfies:

$$\begin{cases} a_{n+2} = a_{n+1} + a_n \\ a_0 = 0 \\ a_1 = 1 \end{cases}$$

Plug $a_n = \lambda^n$ into the D.I.V.P., get

$$\lambda^{n+2} = \lambda^{n+1} + \lambda^n$$

and if $\lambda \neq 0$, divide by λ^n to get

$$\lambda^2 = \lambda + 1$$

or $\quad \lambda^2 - \lambda - 1 = 0$

and complete the square: $\quad \lambda^2 - \lambda + \frac{1}{4} - \frac{1}{4} - 1 = 0$

i.e. $\quad \lambda^2 - \lambda + \frac{1}{4} - \frac{5}{4} = 0$

Ex: The characteristic eqn for this diff. eqn

is
$$\underbrace{\lambda^2 - \lambda + \tfrac{1}{4}} - \tfrac{5}{4} = 0$$

$$(\lambda + \tfrac{1}{2})^2 = \tfrac{5}{4}$$

so
$$\lambda + \tfrac{1}{2} = \pm \frac{\sqrt{5}}{2}$$

The roots are

$$\alpha = \frac{1+\sqrt{5}}{2}, \qquad \beta = \frac{1-\sqrt{5}}{2}$$

and the general soln to the Discrete I.V.P.

$$a_{n+2} = a_{n+1} + a_n$$
$$a_0 = 0$$
$$a_1 = 1$$

is
$$\boxed{a_n = A\alpha^n + B\beta^n}$$ for $n \geq 0$.

where A and B are arbitrary constants; these are determined from the initial conditions.

$$a_0 = 0 = A + B \quad \text{so} \quad B = -A$$

and now the soln is

$$a_n = A(\alpha^n - \beta^n)$$

set $n=1$ and we get

$$a_1 = A(\alpha - \beta) = 1, \qquad \text{so} \quad A = \frac{1}{\alpha - \beta} = \frac{1}{\sqrt{5}}$$

Ex. So the solution is

$$a_n = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right], \quad n \geq 0$$

Since $\{F_n\}_{n\geq 0}$ and $\{a_n\}_{n\geq 0}$ satisfy the same 2nd order linear homogeneous const coeff discrete I.V.P., then

$$F_n = a_n = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right], \quad n \geq 0$$

Called <u>Binet's formula</u> (discovered by De Moivre).

Example 3: An old puzzle called The Tower of Hanoi consists of three pegs; A, B, and C. On peg A there are n disks of different diameters arranged by decreasing size from the bottom to the top. You wish to transfer all of the n disks from peg A to peg B. The rules for moving the disks are as follows:

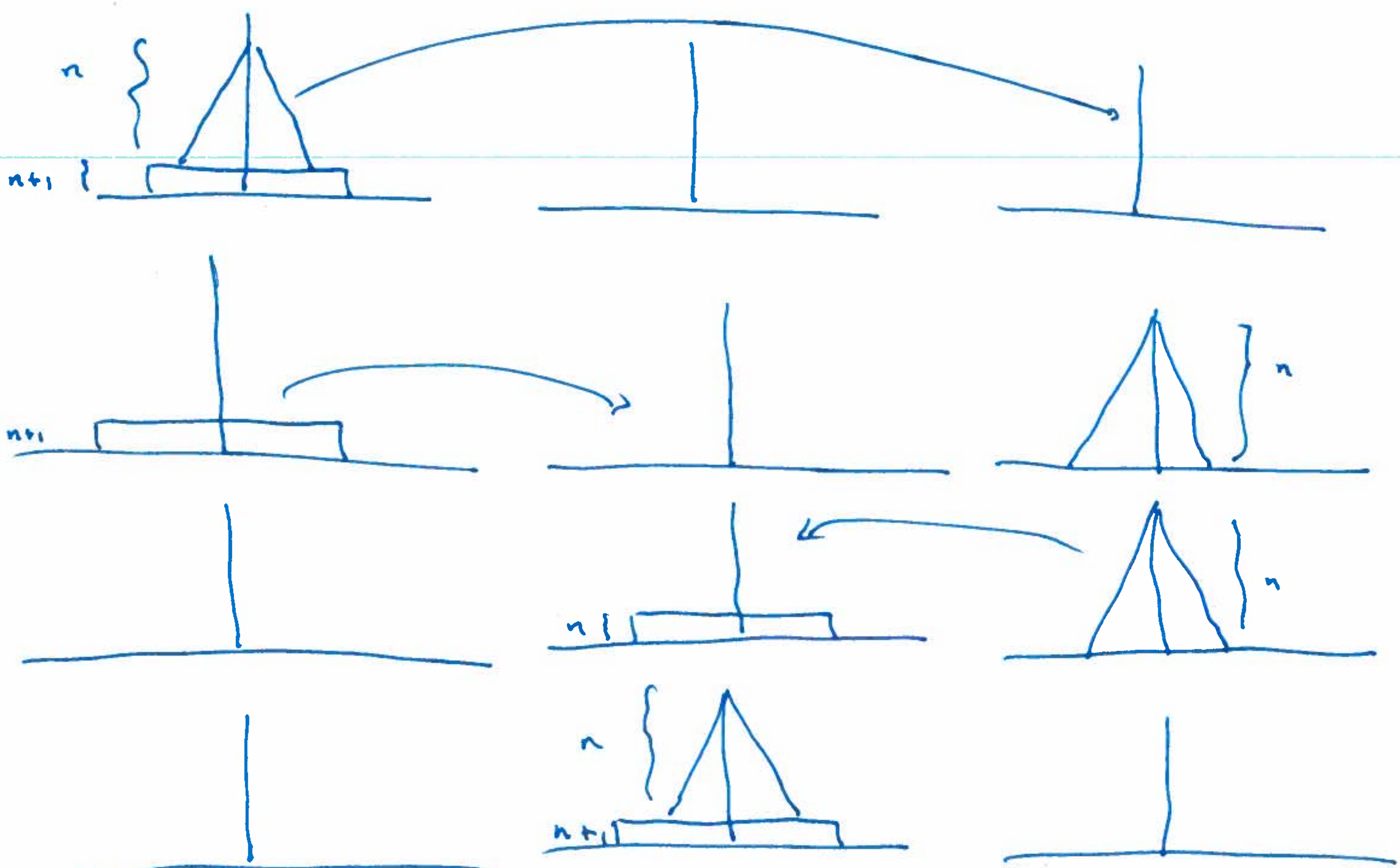
Only one disk may be moved at a time, and it may be moved from one peg to either of the other two pegs. No disk may be placed on top of one of smaller diameter.

Set up a recurrence relation to solve the puzzle. Then solve the recurrence relation.

Hint:

Let a_n = min # of moves needed to move all n disks from peg A to peg B.

Suppose we know how to move n disks



Ex 3: So a_n satisfies the recurrence relation

$$a_{n+1} = a_n + 1 + a_n$$

i.e.

$$\underline{a_{n+1} = 2a_n + 1,} \quad n \geq 1 \quad \leftarrow \text{diff. eqn.}$$

when $n = 1$

$$\underline{a_1 = 1}$$

The discrete I.V.P. satisfied by a_n is

$$\boxed{\begin{aligned} a_{n+1} &= 2a_n + 1, \quad n \geq 1 \\ a_1 &= 1 \end{aligned}}$$

Soln 1:

$$a_{n+1} + 1 = 2a_n + 2 = 2(a_n + 1)$$

let $b_n = a_n + 1$, then b_n satisfies

$$\boxed{\begin{aligned} b_{n+1} &= 2b_n \\ b_1 &= 2 \end{aligned}}$$

Bottom up:

$$b_1 = 2, \quad b_2 = 2 \cdot b_1 = 2^2$$

$$b_3 = 2b_2 = 2^2 \cdot 2 = 2^3$$

$$b_4 = 2 \cdot b_3 = 2 \cdot 2^3 = 2^4$$

$$\vdots$$

$$b_n = 2^n$$

So

$$\boxed{a_n = b_n - 1 = 2^n - 1 \quad n \geq 1}$$

Ex 3:

Soln 2:

$$a_{n+1} = 2a_n + 1$$
$$a_1 = 1$$

start iterations (bottom up)

$$a_1 = 1 = 2^0$$
$$a_2 = 2\cdot 1 + 1 = 2^0 + 2^1$$
$$a_3 = 2(2^0 + 2^1) + 1 = 2^0 + 2^1 + 2^2$$
$$a_4 = 2(2^0 + 2^1 + 2^2) + 1 = 2^0 + 2^1 + 2^2 + 2^3$$
$$\vdots$$
$$a_n = 2^0 + 2^1 + 2^2 + 2^3 + \cdots + 2^{n-1}$$

this is a partial sum for the geometric series

and

$$a_n = 2^0 + 2^1 + \cdots + 2^{n-1} = \frac{2^n - 1}{2-1} = 2^n - 1, \quad n \geq 1.$$

Ex: Find a recurrence relation for the number of n-bit binary numbers that have no consecutive zeros

Let a_n = # of n-bit strings with no consecutive 0's.

If x is such a string, then it either starts with a 1 or a 0.

$$\underline{1}\ \underbrace{\_\ \_\ \cdots\ \_\ \_}_{n-1} \qquad\qquad \underline{0}\ \underline{1}\ \underbrace{\_\ \_\ \cdots\ \_\ \_}_{n-2}$$

So

$$a_n = a_{n-1} + a_{n-2} \quad \text{for } n \geq 2.$$

$a_2 = 3$ $\quad$ $\underline{1}\,\underline{1}$, $\underline{1}\,\underline{0}$, $\underline{0}\,\underline{1}$

$a_3 = 5$ $\quad$ $\underline{1}\,\underline{1}\,\underline{1}$, $\underline{1}\,\underline{1}\,\underline{0}$, $\underline{1}\,\underline{0}\,\underline{1}$

$\underline{0}\,\underline{1}\,\underline{1}$, $\underline{0}\,\underline{1}\,\underline{0}$

So a_n satisfies the Discrete I.V.P

$$\boxed{\begin{aligned} a_n &= a_{n-1} + a_{n-2}, \quad n \geq 2 \\ a_2 &= 3 \\ a_3 &= 5 \end{aligned}}$$

Ex:

$$a_n = a_{n-1} + a_{n-2}$$

$$a_4 = a_3 + a_2 = 5 + 3 = 8 = F_6$$

$$a_5 = a_4 + a_3 = 8 + 5 = 13 = F_7$$

$$\vdots$$

So looks like $a_n = F_{n+2}$, $n \geq 2$.

and

$$a_n = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^{n+2} - \left(\frac{1-\sqrt{5}}{2}\right)^{n+2}\right]$$

for $n \geq 2$.

Example 4: A certain basketball team can only sink foul shots and lay-ups, worth 1 and 2 points, respectively. Let a_n denote the number of ways the team can score n points. (Scoring 1 then 2 is considered to be different than scoring 2 then 1). Write down a recurrence relation for a_n with initial conditions for a_0 and a_1; and explain why it holds for all $n \geq 2$.

$a_0 = 1$

$a_1 = 1$ 1

$a_2 = 2$ 11, 2

$a_3 = 3$ 111, 12, 21

$a_4 = 5$ 1111, 112, 121, 211, 22

So it looks like $a_n = F_{n+1}$ (starts out later in Fibonacci sequence).

Case 1: (The 1st point is worth 1)

So there are a_{n-1} ways to score n points.

Case 2: (The 1st point is worth 2)

So there are a_{n-2} ways to score n points.

Therefore,

$$\begin{cases} a_n = a_{n-1} + a_{n-2}, & n \geq 2 \\ a_0 = 1 \\ a_1 = 1 \end{cases}$$

The solution to this discrete I.V.P. is

$$a_n = F_{n+1} = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^{n+1} - \left(\frac{1-\sqrt{5}}{2}\right)^{n+1}\right]$$

for $n \geq 0$.