A Little Combinatorics

- If n is a positive integer, and we have n distinct objects, then the # of ways to arrange these objects is:

$$\underset{n \text{ choices}}{\_\_} \quad \underset{n-1 \text{ choices}}{\_\_} \quad \underset{n-2 \text{ choices}}{\_\_} \quad \_\_ \quad \cdots \quad \underset{2 \text{ choices}}{\_\_} \quad \underset{1 \text{ choices}}{\_\_}$$

So there are

$$n \cdot (n-1)(n-2) \cdots 2 \cdot 1 = n!$$

arrangements of n distinct objects.

Ex: $\{a, b, c\}$

The arrangements

a b c	b a c	c a b
a c b	b c a	c b a

here $n = 3$, and $3! = 3 \cdot 2 \cdot 1 = \underline{6}$

By convention: $0! = 1$.

- Let n, k ~~positive~~ nonnegative integers, with $0 \leq k \leq n$, and let

$P(n, k)$ be the # of ways of arranging n distinct objects k at a time.

Ex: $P(3,2) = 6$

$\{a, b, c\}$

ab, ac,
ba, bc,
ca, cb

In general, $P(n,k)$ can be counted as follows

$$\underset{\substack{\uparrow \\ n \text{ choices}}}{\_\_} \quad \underset{\substack{\uparrow \\ n-1}}{\_\_} \quad \underset{\substack{\uparrow \\ n-2}}{\_\_} \quad \_\_ \quad \cdots \quad \underset{\substack{\uparrow \\ n-(k-2)}}{\_\_} \quad \underset{\substack{\uparrow \\ n-(k-1)}}{\_\_}$$

So $P(n,k) = n(n-1)\cdots(n-k+1) = \dfrac{n(n-1)\cdots(n-k+1)}{(n-k)!}(n-k)!$

$$\boxed{P(n,k) = \frac{n!}{(n-k)!}}$$

For the example, if $n=3$, and $k=2$

$$\frac{n!}{(n-k)!} = \frac{3!}{(3-2)!} = \frac{3!}{1!} = 3\cdot 2\cdot 1 = 6.$$

- Let $C(n,k)$ = # of selections of n distinct objects taken k at a time.
  = # of combinations of n distinct objects taken k at a time.

Ex: $C(3,2) = 3$

ab, ac, bc

In general,

$$P(n,k) = C(n,k)\cdot k!$$

So $$\boxed{C(n,k) = \frac{P(n,k)}{k!} = \frac{n!}{k!\,(n-k)!}}$$

- $C(n,k)$ is called the <u>binomial coefficient</u>

and is denoted by $\binom{n}{k}$ ← read "n <u>choose</u> k"

So
$$\binom{n}{k} = \frac{n!}{k!\,(n-k)!}$$

for $0 \le k \le n$

- Usually, $$\binom{n}{k} = \begin{cases} \dfrac{n!}{k!\,(n-k)!}, & 0 \le k \le n \\ 0, & \text{otherwise} \end{cases}$$

<u>Note</u>: $\binom{n}{k} = \binom{n}{n-k}$

[L a combinatorial proof: # of ways of choosing k objects from n distinct obj is exactly the same as the # of ways of <u>not</u> choosing n−k obj from n distinct obj.]]

<u>Ex</u>: $$\binom{7}{3} = \frac{7\cdot 6\cdot 5\cdot \cancel{4\cdot 3\cdot 2\cdot 1}}{3\cdot 2\cdot 1\cdot \cancel{4\cdot 3\cdot 2\cdot 1}} = \frac{7\cdot 6\cdot 5}{3\cdot 2\cdot 1} = 35$$

$$\binom{7}{4} = \frac{7\cdot 6\cdot 5\cdot 4}{4\cdot 3\cdot 2\cdot 1} = \frac{7\cdot 6\cdot 5}{3\cdot 2\cdot 1} = 35.$$

Theorem: Pascal's Identity

If n, k are positive integers, then

$$\binom{n-1}{k} + \binom{n-1}{k-1} = \binom{n}{k}$$

proof: (a combinatorial proof).

Suppose $\{a_1, a_2, \ldots, a_n\}$ are distinct objects

# of ways of choosing k objects from the set of n objects

$= \binom{n}{k}$.

If a_1 is chosen then have $n-1$ obj. and we want to select $k-1$ from them, can do this in $\binom{n-1}{k-1}$ ways.

If a_1 is not chosen, then we have $n-1$ objects to choose k obj from, and can do this in $\binom{n-1}{k}$ ways.

So

$$\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}$$

∎

· Can also prove this using the expression

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}$$

Theorem: (Binomial Theorem)

If n is a positive integer, and a, b are real numbers, then

$$(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^k b^{n-k}$$

proof (a combinatorial proof).

Look at the power $(a+b)^n$; expanding this, we get a sum of terms of the following form

$$a^k b^{n-k}$$

$$(a+b)^n = (a+b)(a+b)(a+b)\cdots(a+b)$$

The coeff of $a^k b^{n-k}$ is the # of ways of choosing k factors $(a+b)$ and taking the a, and we are left with $n-k$ factors $(a+b)$ where we take the b's, this is just $\binom{n}{k}$, so

$$(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^k b^{n-k}$$

∎

- Can also prove this by induction.

Ex:

$$(a+b)^0 = 1 \quad \text{if } a+b \neq 0.$$

$$(a+b)^1 = a + b$$

$$(a+b)^2 = a^2 + 2ab + b^2$$

$$(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$$

$$(a+b)^4 = a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4$$

$$\vdots$$

Pascal's Triangle:

$$\begin{array}{ccccccccccccccc}
 & & & & & & & 1 & & & & & & & \\
 & & & & & & 1 & & 1 & & & & & & \\
 & & & & & 1 & & 2 & & 1 & & & & & \\
 & & & & 1 & & 3 & & 3 & & 1 & & & & \\
 & & & 1 & & 4 & & 6 & & 4 & & 1 & & & \\
 & & 1 & & 5 & & 10 & & 10 & & 5 & & 1 & & \\
 & 1 & & 6 & & 15 & & 20 & & 15 & & 6 & & 1 & \\
1 & & 7 & & 21 & & 35 & & 35 & & 21 & & 7 & & 1 \\
 & & & & & & & \vdots & \binom{7}{4} & & & & & &
\end{array}$$

The integers in Pascal's triangle are just the coefficients in the binomial theorem. In fact, the entry in the n^{th} row and k^{th} column is $\binom{n}{k}$, $0 \le k \le n$

e.g. $\binom{7}{4} = \binom{7}{3} = \frac{7\cdot 6\cdot 5}{3\cdot 2\cdot 1} = 7\cdot 5 = 35$

Note: The sum of numbers in the nth row of Pascal's triangle looks like 2^n; ie

$$\boxed{\sum_{k=0}^{n} \binom{n}{k} = 2^n}$$

From the binomial theorem

$$2^n = (1+1)^n = \sum_{k=0}^{n} \binom{n}{k} 1^k 1^{n-k} = \sum_{k=0}^{n} \binom{n}{k}$$

Also

$$0 = (1-1)^n = \sum_{k=0}^{n} \binom{n}{k} 1^k (-1)^{n-k} = (-1)^n \sum_{k=0}^{n} \binom{n}{k} (-1)^k$$

So

$$\boxed{\sum_{k=0}^{n} \binom{n}{k} (-1)^k = 0} \qquad (*)$$

$$\sum_{\substack{k=0 \\ k \text{ even}}}^{n} \binom{n}{k} = ? \qquad \text{or} \qquad \sum_{\substack{k=0 \\ k \text{ odd}}}^{n} \binom{n}{k} = ?$$

Note that

$$\sum_{k=0}^{n} \binom{n}{k} = \sum_{\substack{k=0 \\ k \text{ even}}}^{n} \binom{n}{k} + \sum_{\substack{k=0 \\ k \text{ odd}}}^{n} \binom{n}{k}$$

but from $(*)$

$$\sum_{\substack{k=0 \\ k \text{ even}}}^{n} \binom{n}{k} + \sum_{\substack{k=0 \\ k \text{ odd}}}^{n} \binom{n}{k} (-1) = 0$$

So

$$\sum_{k=0}^{n} \binom{n}{k} = \sum_{k=0}^{n} \binom{n}{k}$$

So

$$2^n = 2 \sum_{\substack{k=0 \\ k \text{ even}}}^{n} \binom{n}{k}$$

$$\sum_{\substack{k=0 \\ k \text{ even}}}^{n} \binom{n}{k} = \frac{2^n}{2} = 2^{n-1}$$

and

$$\sum_{\substack{k=0 \\ k \text{ odd}}}^{n} \binom{n}{k} = 2^{n-1}$$

§ Recursion & Induction

Lecture 12

Open and Closed Forms:

Definition 1: A *closed form* expression is an expression which has a fixed number of operations.

$$a_n = \frac{n(n+1)(2n+1)}{6} \quad \text{for } n \geq 1.$$

this is a closed form.

Definition 2: An *open form* expression is an expression in which the number of operations grows depending on a variable in the expression.

$$S_2(n) = 1^2 + 2^2 + 3^2 + \cdots + n^2, \quad \text{for } n \geq 1.$$

Then this is an open form

Definition 3: The sum (in open form) of the first n natural numbers each to the power i is given by:

$$S_i(n) = 1^i + 2^i + \cdots + n^i, \quad \text{for } n \geq 1.$$

Example 1: Find a closed form for:

$$S_1(n) = 1 + 2 + 3 + \cdots + n$$

To find the closed form, use the expansion:

$$(x+1)^2 = x^2 + 2x + 1$$

by plugging in $x = 1, x = 2, x = 3, \ldots, x = n$.

$$S_1(n) = 1 + 2 + 3 + \cdots + n \quad \text{for } n \geq 1,$$

- In summation notation:

$$S_1(n) = \sum_{k=1}^{n} k = \sum_{k=1}^{n} (n-k+1) = \sum_{k=1}^{n}(n+1) - \sum_{k=1}^{n} k$$

$$= (n+1)\sum_{k=1}^{n} 1 - S_1(n) = n(n+1) - S_1(n)$$

So $2S_1(n) = n(n+1)$ and $\underline{S_1(n) = \frac{n(n+1)}{2}}$.

Due to Gauss (1796)

- Another method, which generalizes to $S_k(n) = 1^k + 2^k + \cdots + n^k$ is as follows: For each k, $1 \leq k \leq n$,

$$(k+1)^2 = k^2 + 2k + 1 \quad \text{for } k = 1, 3, \ldots, n$$

Write these down:

$$2^2 = 1^2 + 2\cdot 1 + 1$$
$$3^2 = 2^2 + 2\cdot 2 + 1$$
$$4^2 = 3^2 + 2\cdot 3 + 1$$
$$\vdots$$
$$(n+1)^2 = n^2 + 2\cdot n + 1$$

---

add: $\underline{2^2 + 3^2 + \cdots + n^2} + (n+1)^2 = \underline{1^2 + 2^2 + \cdots + n^2} + 2(1 + 2 + \cdots + n) + n$

$$(n+1)^2 - 1 = 2(1 + 2 + 3 + \cdots + n) + n$$

So $2(1 + 2 + \cdots + n) = (n+1)^2 - (n+1) = (n+1)[n+1-1] = n(n+1)$

and $1 + 2 + 3 + \cdots + n = \underline{\dfrac{n(n+1)}{2}}$

Example 2: Find a closed form for:

$$S_2(n) \quad \cancel{S_2} = 1^2 + 2^2 + 3^2 + \cdots + n^2$$

To find the closed form, use S_1 and the expansion:

$$\boxed{(x+1)^3 = x^3 + 3x^2 + 3x + 1}$$

by plugging in $x = 1, x = 2, x = 3, \ldots, x = n$.

$$2^3 = 1^3 + 3\cdot 1^2 + 3\cdot 1 + 1$$
$$3^3 = 2^3 + 3\cdot 2^2 + 3\cdot 2 + 1$$
$$4^3 = 3^3 + 3\cdot 3^2 + 3\cdot 3 + 1$$
$$\vdots$$
$$(n+1)^3 = n^3 + 3n^2 + 3n + 1$$

add:

$$\underline{2^3 + 3^3 + \cdots + n^3} + (n+1)^3 = \underline{1 + 2^3 + \cdots + n^3} + 3(1^2 + 2^2 + \cdots + n^2) + 3\cdot(1 + 2 + \cdots + n) + n$$

$$(n+1)^3 - 1 = 3\, S_2(n) + 3\, S_1(n) + n$$

$$(n+1)^3 - (n+1) = 3\cdot S_2(n) + 3\cdot \frac{n(n+1)}{2}$$

$$3\, S_2(n) = (n+1)^3 - (n+1) - \frac{3}{2} n(n+1) = (n+1)\left[(n+1)^2 - 1 - \frac{3}{2}n\right]$$

$$3\, S_2(n) = (n+1)\left[n^2 + 2n + 1 - 1 - \frac{3}{2}n\right] = (n+1)\left(n^2 + \frac{1}{2}n\right)$$

$$3\, S_2(n) = \frac{(n+1)}{2}\left[2n^2 + n\right] = \frac{n(n+1)}{2}\left[2n+1\right]$$

$$\boxed{S_2(n) = \frac{n(n+1)(2n+1)}{6}}$$

Example 3: Find a closed form expression for

$$a_n = 1\cdot n + 2\cdot(n-1) + 3\cdot(n-2) + \cdots + (n-1)\cdot 2 + n\cdot 1$$

valid for $n \geq 1$.

In summation notation:

$$a_n = \sum_{k=1}^{n} k(n-k+1) = \sum_{k=1}^{n}(n+1)k - \sum_{k=1}^{n} k^2$$

$$a_n = (n+1)\sum_{k=1}^{n} k - \sum_{k=1}^{n} k^2 = (n+1)\cdot\frac{(n+1)\cdot n}{2} - \frac{n(n+1)(2n+1)}{6}$$

$$a_n = \frac{n(n+1)^2}{2} - \frac{(2n+1)n(n+1)}{6}$$

$$a_n = \frac{n(n+1)}{6}\left[3(n+1) - (2n+1)\right] = \frac{n(n+1)}{6}[n+2]$$

$$\boxed{a_n = \frac{n(n+1)(n+2)}{6} = \binom{n+2}{3}}$$

Example 4: For $x \neq 1$ find a closed form for:

$$x^0 + x^1 + x^2 + \cdots + x^n$$

By convention when looking at $\sum_{k=0}^{n} x^k$

we assume $x^0 = 1$.

$$S_n(x) = 1 + x + x^2 + \cdots + x^n \quad \text{for } n \geq 0 \quad (*)$$

(an open form)

Multiply $(*)$ by x, and subtract,

$$x\,S_n(x) = x + x^2 + x^3 + \cdots + x^n + x^{n+1} \quad (**)$$

Subtract $(**)$ from $(*)$, we get

$$S_n(x) - x S_n(x) = 1 - x^{n+1}$$

That is,

$$(1-x) S_n(x) = 1 - x^{n+1}$$

if $x \neq 1$, then $S_n(x) = \dfrac{1-x^{n+1}}{1-x}$

if $x = 1$, $S_n(1) = n+1$

$$\boxed{S_n(x) = \begin{cases} \dfrac{1-x^{n+1}}{1-x} & \text{if } x \neq 1 \\ n+1 & \text{if } x = 1. \end{cases}}$$

Note: if $|x| < 1$, then $\lim_{n\to\infty} |x|^n = 0$

Look at $\lim_{n\to\infty} |x|^n = \gamma$

{ $|x|^n$ is a strictly decreasing seq. bounded below, so it converges. (order completeness of $\mathbb{R}$)

$$\gamma = \lim_{n\to\infty} |x|^{n+1} = |x| \cdot \lim_{n\to\infty} |x|^n = |x| \cdot \gamma$$

So $\gamma(1-|x|) = 0$

But $|x| < 1$ so $1-|x| > 0$ and this implies $\gamma = 0$.

So $\lim_{n\to\infty} |x|^n = 0$ if $|x| < 1$.

Go back to $S_n(x) = \dfrac{1-x^{n+1}}{1-x}$, if $|x| < 1$

$(-|x|^n \leq x^n \leq |x|^n)$ so $\lim_{n\to\infty} x^{n+1} = 0$

So $\lim_{n\to\infty} S_n(x) = \dfrac{1}{1-x}$ if $|x| < 1$.

i.e. $\lim_{n\to\infty} (1+x+x^2+\cdots+x^n) = \dfrac{1}{1-x}$ if $|x| < 1$

So $1+x+x^2+\cdots+x^n+\cdots = \dfrac{1}{1-x}$ if $|x| < 1$

So the geometric series converges to $\frac{1}{1-x}$ if $|x| < 1$.

Note: if $S_n(x) = \frac{1-x^n}{1-x} = \frac{x^n-1}{x-1}$ for $n \geq 0$, if $x \neq 1$,

then look at

$$T_n(x) = 1 + 2x + 3x^2 + 4x^3 + \cdots + nx^{n-1}$$

$$\left(S_n(x) = 1 + x + x^2 + x^3 + x^4 + \cdots + x^n \right)$$

so $T_n(x) = S_n'(x)$ provided $x \neq 1$.

Exercise:
Instead of differentiating, find a closed form for $T_n(x)$.

Hint:

$$T_n(x) = \begin{cases} 1 + x + x^2 + \cdots + x^{n-1} \\ \quad + x + x^2 + \cdots + x^{n-1} \\ \quad\quad + x^2 + \cdots + x^{n-1} \\ \quad\quad\quad \vdots \\ \quad\quad\quad + x^{n-1} \end{cases}$$

Theorem 4: The number of ways to choose k objects from a group of n distinct objects is given by

$$\binom{n}{k} = \frac{n!}{k! \cdot (n-k)!}$$

for $0 \leq k \leq n$.

proof.

If $P(n,k)$ = # of arrangements of n distinct objs taken k at a time,

then

$$P(n,k) = n(n-1)\cdots(n-k+1) = \frac{n!}{(n-k)!},$$

for $0 \leq k \leq n$.

If x = # of ways to choose k objs from n distinct objs. then we can arrange them in $k!$ ways.

Therefore

$$x \cdot k! = P(n,k)$$

So $$x = \frac{P(n,k)}{k!} = \frac{n!}{k!\,(n-k)!}$$

ie $$x = \binom{n}{k}.$$

$\blacksquare$

Theorem 5: (*Pascal's Identity*)
For integers n and k with $1 \le k \le n$, we have

$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$$

proof.

$$\binom{n-1}{k-1} + \binom{n-1}{k} = \frac{(n-1)!}{(k-1)!\,(n-k)!} + \frac{(n-1)!}{k!\,(n-1-k)!}$$

$$= \frac{(n-1)!}{k!\,(n-k)!}\left[k + n - k\right]$$

$$= \frac{(n-1)!}{k!\,(n-k)!}\left[n\right] = \frac{n!}{k!\,(n-k)!}$$

So

$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$$

∎

Example 5: Find a closed form for:

$$\sum_{k=3}^{n-1} \binom{k}{3}$$

Let $a_n = \sum_{k=3}^{n-1} \binom{k}{3}$

Pascal's identity we have $\binom{k+1}{4} = \binom{k}{3} + \binom{k}{4}$

So $\binom{k}{3} = \binom{k+1}{4} - \binom{k}{4}$

$$a_n = \sum_{k=3}^{n-1} \left\{ \binom{k+1}{4} - \binom{k}{4} \right\}$$

get a collapsing or telescoping sum

$$a_n = \cancel{\binom{4}{4}} - \underbrace{\cancel{\binom{3}{4}}}_{0} + \cancel{\binom{5}{4}} - \cancel{\binom{4}{4}} + \cancel{\binom{6}{4}} - \cancel{\binom{5}{4}} + \cdots + \binom{n}{4} - \cancel{\binom{n-1}{4}}$$

$$a_n = \binom{n}{4}$$

Example 6: Show that
$$a_n = \frac{1}{n+1} \cdot \binom{2n}{n}$$
is an integer for $n = 1, 2, 3, \ldots$ The integer a_n is called the n^{th} **Catalan number.**

$$a_n = \frac{1}{n+1}\binom{2n}{n} = \frac{1}{n+1} \cdot \frac{(2n)!}{n!\, n!}$$

$$a_n = \frac{(2n)!}{(n+1)!\, n!} \cdot 1 = \frac{(2n)!}{(n+1)!\, n!} [n+1-n]$$

$$a_n = \frac{(2n)!}{n!\, n!} - \frac{(2n)!}{(n+1)!\,(n-1)!}$$

$$a_n = \binom{2n}{n} - \binom{2n}{n+1}$$

both $\binom{2n}{n}$ and $\binom{2n}{n+1}$ are integers.

So $\frac{1}{n+1}\binom{2n}{n}$ is an integer.

Ex. 7 Counting Subsets

Suppose we have a set of 6 distinct elements

$A = \{a, b, c, d, e, f\}$

Every subset of A can be identified with a 6 bit binary number as follows

Suppose $S \subseteq A$ write down elements of A and underneath them, a "0" or a "1" depending on whether or not the element is in the set

a	b	c	d	e	f
0	0	1	0	1	1

So $S = \{c, e, f\} \longleftrightarrow 001011$

Note that the binary number 000000 corresponds to the empty subset of S, $\emptyset$.

and the binary number 111111 corresponds to the entire set A.

So there is a 1-1 correspondance between the subsets of A and the 6-bit binary numbers.

Ex 7: So the number of subsets of A

is $2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 = 2^6$.

In general, if $A = \{a_1, a_2, \ldots, a_n\}$ is a finite set

containing n elements, can count the number of subsets of

size k in the same way

$$\begin{array}{ccccc} a_1 & a_2 & a_3 & \cdots & a_n \\ 0 & 1 & 0 & \cdots & 1 \end{array}$$

k 1's

The number of subsets of size k is just the # of ways

to choose the positions for the 1's. that is, $\binom{n}{k}$.

So # subsets of A of size k = $\binom{n}{k}$.

The total number of subsets is 2^n.

So we have another proof that the

sum

$$\sum_{k=0}^{n} \binom{n}{k} = 2^n.$$

Previously, used binomial theorem:

$$2^n = (1+1)^n = \sum_{k=0}^{n} \binom{n}{k} 1^k \cdot 1^{n-k} = \sum_{k=0}^{n} \binom{n}{k}$$