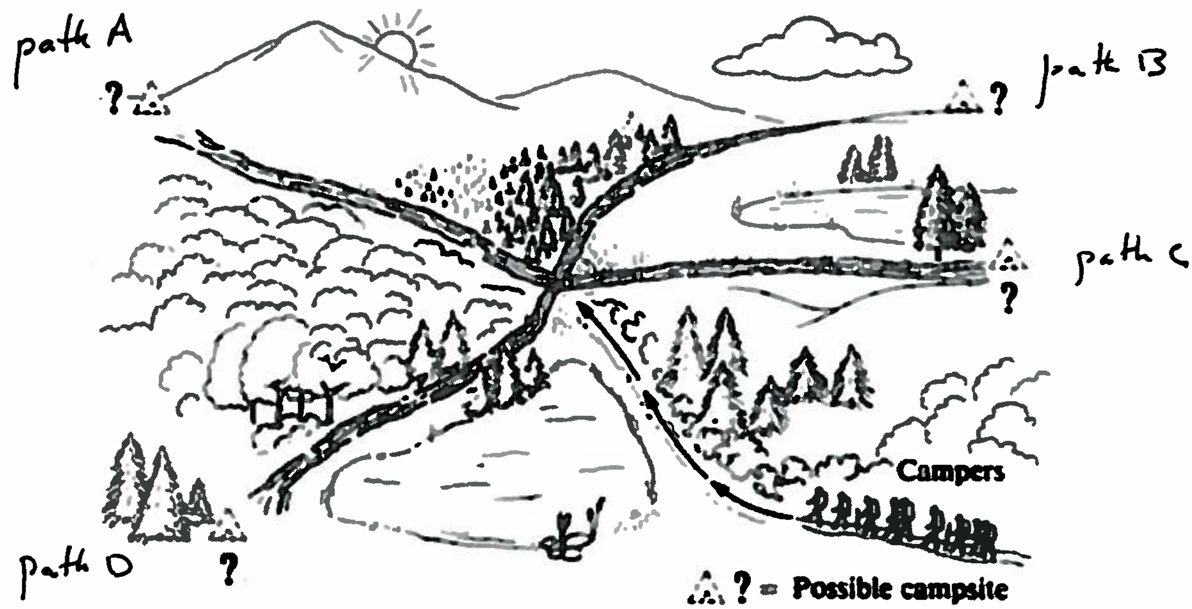
# Lecture 7

Warm up problem: **The Campers' Problem** (Section 3.1 in Ecco)

A counsellor has eight campers with her at a junction in a hiking trail. She knows their camp is twenty minutes down one of four possible paths. It will be dark in one hour and the group must find their camp before dark. Two of the eight campers sometimes lie, and unfortunately the counsellor doesn't know which of the eight they are.




**Example 1**: How can the counselor find the camp? Counselor goes down path D (she never lies) so if finds campsite, problem is solved

- Sends campers 1, 2, 3 down path A
- Sends campers 4, 5, 6 down path B
- Sends campers 7, 8 down path C.

Case 1: There is a disagreement on two of the paths

- Listen to the majority on paths A and B.

Case 2: There is a disagreement on at most one path.

- Listen to the two paths that agree.

In both cases the counselor knows where the campsite is.

- Can also do this problem using coding theory.

**Example 2:** Design a code that will enable the counselor to accurately deduce the location of the camp.

The counselor doesn't lie, so she go down ~~one~~ path, say path D.

She sends camper 1, 2, 3 down path A,

sends campers 4, 5, 6. down path B.

Sends campers 7, 8 down path C.

when they return to the junction (after 40 min)

she asks each camper the following question:

"Is the camp on the path that you went down?"

The only possible answers are (1 ≡ yes, 0 ≡ no)

$$\left.\begin{aligned} a &= 111\ 000\ 00 \\ b &= 000\ 111\ 00 \\ c &= 000\ 000\ 11 \end{aligned}\right\}$$

<u>no</u> lies are told

These are the <u>code words</u>!

<u>Hamming</u> <u>distance</u>: $H(a,b) = 6$, $H(a,c) = 5$, $H(b,c) = 5$

If up to two of the campers lied, then the actual answers would differ from a code word in at most two places. This enables the counselor to see at a glance what the legitimate answer is.

- Each pair of codewords has a Hamming distance of 5 or more between them. This means that the minimum Hamming distance between any two code word is 5.

To find out how many errors we can correct, we want

$$2n+1 = \min\{H(x,y) : x,y \text{ codewords}\} = 5$$

so we can correct up to $n$ errors where

$$2n+1 = 5$$

that is, $n = 2$.

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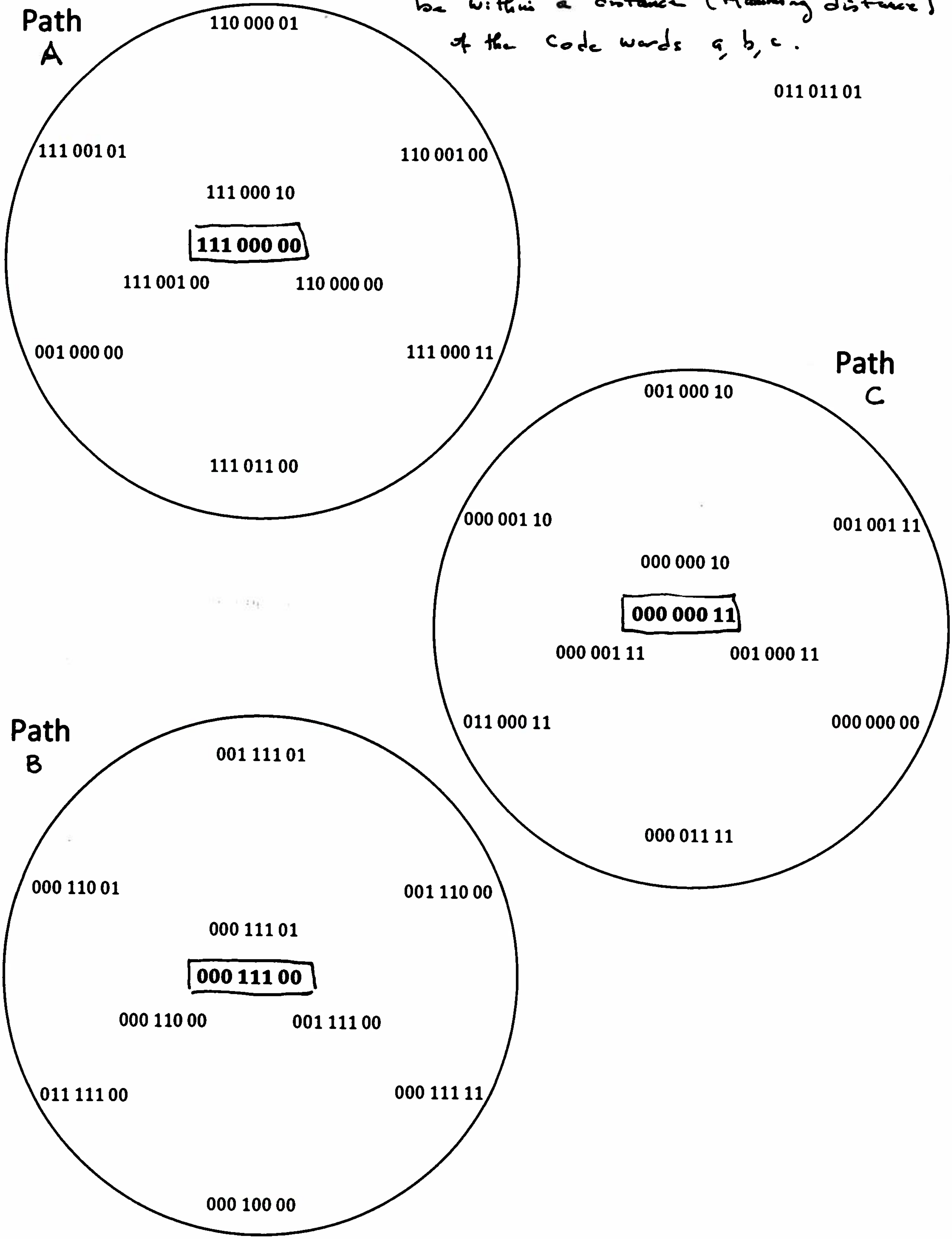
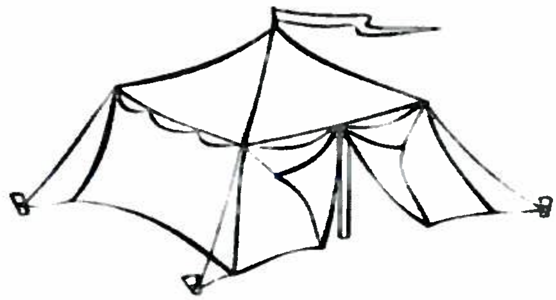
This means the counselor can correct the two campers that are lying, and use the <u>nearest neighbor decoding scheme</u> to find the location of the camp. (See next page).

<u>Ex</u>: Suppose the answers form the binary string 100 001 11. This differs from the $1^{st}$ code word a in 5 places; it differs from the $2^{nd}$ code word b in 5 places, and it differs from the $3^{rd}$ code word in 2 places.

So campsite is down path c.

Any word or string the counselor gets must be within a distance (Hamming distance) 2 of the code words a, b, c.

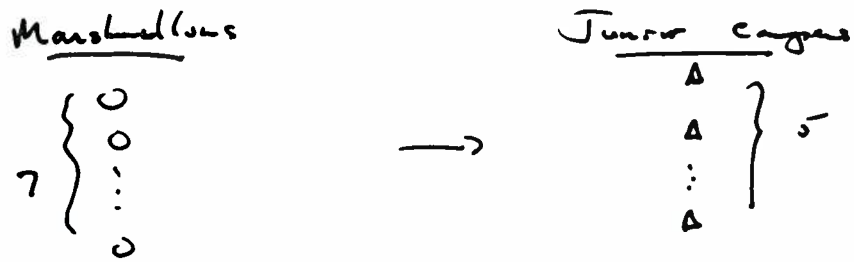




**Example 3:** Back at camp, 7 mini marshmallows have gone missing. The Camp Director found out that the 7 marshmallows had been stuck up 5 junior camper's noses. Show that there must be a group of 3 junior campers with a combined total of 3 or less marshmallows up their noses.


Use the pigeon hole principle: pigeons = marshmallows

pigeonholes = campers noses




By P.H.P. some junior camper has $\lceil \frac{7}{5} \rceil = 2$

($\lceil \frac{7}{5} \rceil$ is the smallest integer $\geq \frac{7}{5}$)

marshmallows in their nose.

The other four junior campers have a total of

$7 - \lceil \frac{7}{5} \rceil = 7 - 2 = 5$ or less marshmallows up their noses.

Marshmallows

$5$ or less $\left\{ \begin{array}{c} \text{O} \\ \text{O} \\ \vdots \\ \text{O} \end{array} \right.$

Junior campers

$\left. \begin{array}{c} \Delta \\ \Delta \\ \vdots \\ \Delta \end{array} \right\} 4$

By the P.H.P. three junior campers have a total of $5 - \lceil \frac{5}{4} \rceil = 5 - 2 = 3$ or less marshmallows up their noses.

**Example 4:** Prove the campers' problem cannot be solved with only 7 campers.

Same as previous campers problem:

- 1 counselor, 7 campers, up to 2 campers may lie.
- Campsite is down one of 4 paths.
- There are many ways to send 7 campers down 3 different paths.
- We consider one possibility.

Counselor send 3 down path A
2 down path B
2 down path C

In this case the codewords are

Campsite

$a = 111\ 00\ 00$ ← down path A

$b = 000\ 11\ 00$ ← down path B

$c = 000\ 00\ 11$ ← down path C

Hamming distances: $d(a,b) = 5$, $d(a,c) = 5$

but $d(b,c) = 4 < 5$.

If the answer is $000\ 1010$, it is within Hamming distance 2 of both codewords b and c so can't tell which is the correct code word.
So this doesn't work.

In general,

| Campers | paths |
|---|---|
| $7 \left\{ \begin{array}{c} \circ \\ \circ \\ \vdots \\ \circ \end{array} \right.$ | $\left. \begin{array}{c} \Delta \\ \Delta \\ \Delta \end{array} \right\} 3$ |

By P.H.P. there is one path that has 3 or more campers.

Therefore 2 paths have a total of 4 campers going down them.

<u>Codewords</u>: If no campers lied.

$$a = 1\cdots1 \quad 000 \quad 000 \quad \leftarrow \text{path } A$$

$$b = 00\cdots0 \quad 1\cdots1 \quad 000 \quad \leftarrow \text{path } B$$

$$c = \underbrace{00\cdots0}_{3 \text{ or more}} \quad \underbrace{0\cdots0 \quad 1\cdots1}_{4 \text{ or less}} \quad \leftarrow \text{path } C.$$

$H(b,c) \le 4$ Since only 4 campers in total go down paths B and C.

$$\min H(x,y) \le H(b,c) \le 4$$

To correct two errors, we need

$$\min H(x,y) \ge 2\cdot 2 + 1 = 5$$

The Counselor can longer correct the 2 errors.